CHEBYSHEV'S BOUNDS FOR $\pi(N)$ (MA2316, FIRST WEEK)

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The next obvious question to ask once we know that $\lim\limits_{N\to+\infty}\pi(N)=+\infty$ is how fast does the function $\pi(N)$ increase. The fact that was already known to Euler is $\lim\limits_{N\to+\infty}\frac{\pi(N)}{N}=0$; the proof of Erdös implies that $\pi(N)\ge\log_2\sqrt{N}$. These two statements do not yet tell much. Legendre, in early 1800s, conjectured the asymptotic formula

$$\pi(N)\approx\frac{N}{\ln N-C}$$

with C a certain constant, which he believed, based on available numeric data, to be approximately equal to 1.08366. (Interestingly enough, later research established that this constant is equal to 1). Gauss, around the same time, proposed an asymptotic formula

$$\pi(N)\approx\int_2^N\frac{dt}{\ln t}.$$

However, the first substantial progress in the proof was obtained by Chebyshev in 1850; he proved that for some positive constants a, b we have

$$a\frac{N}{\ln N}\le\pi(N)\le b\frac{N}{\ln N}.$$

Let us outline a proof of these results (it is different from the proof Chebyshev had, which was a bit more analytic; some ideas of that proof will be discussed later).

Lemma. *For each n, we have*

$$\operatorname{lcm}(1,2,\ldots,2n+1)>4^n.$$

Proof. Consider the polynomial $f_n(x)=x^n(1-x)^n$. Since for all $x\in[0,1]$ except for $x=0,1/2,1$ we have $0<x(1-x)<1/4$, for those x we have $0<f_n(x)\le\frac{1}{4^n}$, so

$$0<\int_0^1 f_n(x)\,dx<\frac{1}{4^n}.$$

Expanding $f_n(x)=a_nx^n+a_{n+1}x^{n+1}+\cdots+a_{2n}x^{2n}$, with $a_i\in\mathbb{Z}$, we see that

$$\int_0^1 f_n(x)\,dx=\frac{a_n}{n+1}+\frac{a_{n+1}}{n+2}+\cdots+\frac{a_{2n}}{2n+1},$$

so

$$\operatorname{lcm}(1,2,\ldots,2n+1)\int_0^1 f_n(x)\,dx\in\mathbb{Z}.$$

This implies that

$$\frac{\operatorname{lcm}(1,2,\ldots,2n+1)}{4^n}>\operatorname{lcm}(1,2,\ldots,2n+1)\int_0^1 f_n(x)\,dx\ge 1,$$

and the lemma is proved. □

Theorem 1. *For all $N\ge 6$ we have $a\frac{N}{\ln N}\le\pi(N)$ for some positive a.*

Proof. Let $2n+1$ be the largest odd number not exceeding N. Consider the prime decomposition of $\operatorname{lcm}(1,2,\ldots,2n+1)$, let it be $p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}$. Note that $k=\pi(2n+1)$, and that $p_i^{a_i}\le 2n+1$ for all i. Therefore,

$$4^n < \operatorname{lcm}(1,2,\ldots,2n+1) \le (2n+1)^{\pi(2n+1)},$$

so

$$\pi(2n+1)\ln(2n+1) > n\ln 4,$$

which implies

$$\pi(N) \ge \pi(2n+1) > \frac{n\ln 4}{\ln(2n+1)} > \frac{\frac{1}{2}(N-3)\ln 4}{\ln N} \ge \frac{\frac{1}{4}N\ln 4}{\ln N},$$

since $2n+3 > N \ge 2n+1$ by construction, and $N-3 \ge \frac{1}{2}N$ for $N>6$. Thus, we can take $a=\frac{1}{4}\ln 4 \approx 0.3465$. □

Lemma. *For each n, we have*

$$\prod_{\text{prime } p\le n} p < 4^n.$$

Proof. Induction on n. The statement is true for $n=2,3$. The step of induction is trivial when we move from an odd number to an even number:

$$\prod_{\text{prime } p\le 2m} p = \prod_{\text{prime } p\le 2m-1} p < 4^{2m-2} < 4^{2m-1}.$$

Let us prove the other case. We have

$$\prod_{\text{prime } p\le 2m+1} p = \prod_{\text{prime } p\le m+1} p \cdot \prod_{\text{prime } m+1<p\le 2m+1} p.$$

Note that the product $\prod_{\text{prime } m+1<p\le 2m+1} p$ divides the binomial coefficient $\binom{2m+1}{m+1}$. Since $\binom{2m+1}{m} = \binom{2m+1}{m+1}$ and $\sum_k \binom{2m+1}{k} = 2^{2m+1}$, we have $2\binom{2m+1}{m+1} < 2^{2m+1}$ and $\binom{2m+1}{m+1} < 4^m$. Hence

$$\prod_{\text{prime } p\le 2m+1} p = \prod_{\text{prime } p\le m+1} p \cdot \prod_{\text{prime } m+1<p\le 2m+1} p < 4^{m+1}\cdot 4^m = 4^{2m+1},$$

which completes the proof. □

Theorem 2. *We have $\pi(N) \le b\frac{N}{\ln N}$ for some positive b.*

Proof. Let us denote $k=\pi(N)$, so that all the primes not exceeding N are $p_1,\ldots,p_k$. Clearly, we have $p_i > i$, so

$$k! < p_1p_2\ldots p_k < 4^N$$

by Lemma we just proved. Since

$$(k!)^2 = (1\cdot k)(2\cdot(k-1))\cdots(k\cdot 1) \ge k^k,$$

we have

$$k^{k/2} \le k! < 4^N.$$

Let us show that this implies

$$k \le 5\ln 2\frac{N}{\ln N}.$$

Note that

$$5\ln 2\frac{N}{\ln N} \ge N^{4/5}$$

since

$$5\ln 2\frac{N^{1/5}}{\ln N} \geq 1,$$

which in turn follows from

$$N^{1/5} \geq \log_2(N^{1/5}),$$

where we recognise a classical inequality $x \geq \log_2 x$. Therefore, if

$$k > 5\ln 2\frac{N}{\ln N},$$

we have

$$k > N^{4/5},$$

so

$$k^{k/2} > (N^{4/5})^{\frac{5}{2}\ln 2\frac{N}{\ln N}} = 4^N,$$

a contradiction. We conclude that

$$\pi(N) = k \leq 5\ln 2\frac{N}{\ln N},$$

so we can take $b = 5\ln 2 \approx 3.4657$. □

Let us conclude with mentioning the following result, which was conjectured by Bertrand in 1845, and since then has the name of *Bertrand's postulate*.

Corollary. *For each n, there exists a prime number p between n and $2n$.*

Chebyshev was able to prove this statement using more precise values $a \approx 0.92129$, $b \approx 1.10555$, but our constants are not quite sufficient to take that path. We shall prove this statement in a different way in our tutorial class next week.