MA2317: Introduction to Number Theory
Homework problems due November 19, 2010

The last two questions are optional. Other questions are assessed.

1. Among 18 maths students, 7 took the number theory course, 10 took the algebraic geometry course, and 10 took quantum mechanics. Also, 3 students took number theory and algebraic geometry, 4 took number theory and quantum mechanics, and 5 — algebraic geometry and quantum mechanics. Finally, one of the students took all three courses. How many students did not take any of the three courses?

2. How many permutations σ of $1, 2, \ldots, n$ move any number from its place (that is, $\sigma(i) \neq i$ for all i)? (*Hint*: let U_k be the set of all permutations σ with $\sigma(k) = k$ [and no other restrictions]; apply the inclusion–exclusion formula to these sets.)

3. Describe all integers n such that (**a**) $\varphi(n) = 6$; (**b**) $\varphi(\varphi(n)) = 6$; (**c**) $\varphi(n) = 2n/3$.

4. (**a**) Compute

$$r(x) = \gcd(x^5-4x^4-11x^3+45x^2-46x+55, x^6-7x^5+12x^4-9x^3+22x^2-x-33).$$

(**b**) Find some polynomials $p(x)$ and $q(x)$ such that $r(x)$ is equal to the combination

$$p(x)(x^5-4x^4-11x^3+45x^2-46x+55)+q(x)(x^6-7x^5+12x^4-9x^3+22x^2-x-33).$$

5. Describe all polynomials $p(x)$ such that $p(3) = -7$ and the remainder of $p(x)$ modulo $x^2 - x - 1$ is equal to $3x + 2$.

6. Show that the polynomial $x^4 - 8x^3 + 12x^2 - 6x - 42$ is irreducible over integers.

7. Show that the polynomial $x^{105} - 9$ is irreducible over integers.

8. Assume that two polynomials $f(x)$ and $g(x)$ with integer coefficients have no common divisors of positive degree. Show that if we consider them over $\mathbb{Z}/p\mathbb{Z}$, it is possible that they are not relatively prime anymore, but that they remain relatively prime over $\mathbb{Z}/p\mathbb{Z}$ for infinitely many primes p.

9. Let $f(x)$ be a polynomial of positive degree with integer coefficients. Show that the congruence $f(x) \equiv 0 \pmod{p}$ has solutions for infinitely many prime numbers p.

10. Let $a_1, \ldots, a_n$ be distinct integers. Show that the polynomial $f(x) = (x-a_1)^2(x-a_2)^2 \cdots (x-a_n)^2 + 1$ is irreducible over integers. (*Hint*: for real x, $f(x)$ only assumes positive values, so if $g \mid f$, then g assumes either only positive values or only negative values.)