MA3413: Group representations I
Homework problems due on October 4, 2012

1. A function in n variables x_1, x_2, ..., x_n is called (fully) symmetric, if

$$f(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}) = f(x_1, x_2, \dots, x_n)$$

for all permutations $\sigma \in S_n$, and (fully) skew-symmetric, if

$$f(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}) = \mathrm{sgn}(\sigma) f(x_1, x_2, \dots, x_n)$$

for all permutations $\sigma \in S_n$.

(**a**) Prove that any function of 2 variables can be represented as a sum of a symmetric and a skew-symmetric function.

(**b**) Give an example showing that the previous statement is not true for functions in 3 variables. (*Hint*: take $f(x_1, x_2, x_3) = x_1$.)

2. (**a**) Let $\omega = \frac{1+\sqrt{-3}}{2}$ be a primitive cubic root of unity. Check that

$$\det \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix} = (a+b+c)(a+\omega b+\omega^2 c)(a+\omega^2 b+\omega c).$$

(*Hint*: show that $(a+b+c)$, $(a+\omega b+\omega^2 c)$, and $(a+\omega^2 b+\omega c)$ are the eigenvalues of this matrix.)

(**b**) The numbers a, b, c are put in the vertices of a regular triangle. Consider a transformation that replaces each of the numbers by the arithmetic mean of the numbers in neighbour vertices (simultaneously; for example, after the first operation one gets $\frac{b+c}{2}$, $\frac{a+c}{2}$, $\frac{a+b}{2}$). Let a_n be the number at the first vertex after n iterations; thus, $a_0 = a$, $a_1 = \frac{b+c}{2}$, $a_2 = \frac{2a+b+c}{4}$ etc. Does the sequence $\{a_n\}$ converge to a limit as $n \to \infty$?

4. For two elements a and b of a group G, the element $aba^{-1}b^{-1}$ is called the *commutator* of a and b. The group generated by all commutators is called the *derived group*, or the *commutator subgroup* of G, and is denoted by $[G, G]$.

(**a**) Prove that the commutator subgroup is normal, and that the quotient $G/[G, G]$ is abelian.

(**b**) Prove that the commutator subgroup is contained in the kernel of any 1-dimensional representation.

(**c**) Prove that the number of (equivalence classes of) 1-dimensional representations of any group G is equal to the same number for the group $G/[G, G]$.

5. For each positive integer n give an example of a group that has an irreducible complex representation of dimension n. (*Hint*: try S_{n+1}.)