MA 3416: Group representations
Selected answers/solutions to the assignment due February 12, 2015

1. (**a**) An irreducible representation of $\mathbb{Z}/5\mathbb{Z}$ must be one-dimensional, so a two-dimensional representation is the sum of two one-dimensional ones, and there are $15 = \binom{6}{2}$ different choices $(V_i, \rho_i) \oplus (V_j, \rho_j)$ that we can use (here V_i corresponds to the i-th 5th root of 1 representing $\overline{1}$).

(**b**) The sum of squares of dimensions of irreducible representations is 8, and there are, as we know, four one-dimensional representations, so there is one two-dimensional irreducible representation; it is the standard representation in 2D by symmetries of the square. Of course, there are also $10 = \binom{5}{2}$ reducible representations.

(**c**) The sum of squares of dimensions of irreducible representations is 10, and there are, as we know, two one-dimensional representations, so there are two two-dimensional irreducible representations; one of them is the standard representation in 2D by symmetries of the square, and the other one represents each rotation through α by a rotation through 2α. Of course, there are also $3 = \binom{3}{2}$ reducible representations.

(**d**) The sum of squares of dimensions of irreducible representations is 8, and there are, as we know, four one-dimensional representations, so there is one two-dimensional irreducible representation; it is the representation in 2D by "Pauli matrices" $I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $K = IJ = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. Of course, there are also $10 = \binom{5}{2}$ reducible representations.

2. (**a**) For the case of D_4, there are 4 one-dimensional representations $\rho_{\pm 1, \pm 1}$, and a two-dimensional representation U. For the latter, traces can be computed directly, from the fact that the character of any reflection is equal to zero, and the trace of any rotation is $2\cos\alpha$, where α is the angle of rotation. The result is

	e	a^2	$\{b, ba^2\}$	$\{a, a^3\}$	$\{ba, ba^3\}$
$\chi_{\rho_{1,1}}$	1	1	1	1	1
$\chi_{\rho_{1,-1}}$	1	1	1	-1	-1
$\chi_{\rho_{-1,1}}$	1	1	-1	1	-1
$\chi_{\rho_{-1,-1}}$	1	1	-1	-1	1
χ_U	2	-2	0	0	0

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(**b**) For the case of D_5, there are 2 one-dimensional representations $\rho_{\pm 1}$, and two two-dimensional representation U_1 and U_2 constructed above. For them, traces can be computed directly (as above). The result is

	e	$\{a, a^4\}$	$\{a^2, a^3\}$	$\{b, ba, ba^2, ba^3, ba^4\}$
χ_{ρ_1}	1	1	1	1
$\chi_{\rho_{-1}}$	1	1	1	-1
χ_{U_1}	2	$2\cos(2\pi/5)$	$2\cos(4\pi/5)$	0
χ_{U_2}	2	$2\cos(4\pi/5)$	$2\cos(2\pi/5)$	0

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(**c**) For the case of Q_8, there are 4 one-dimensional representations $\rho_{\pm 1, \pm 1}$, and a two-dimensional representation U. For the latter, traces can be computed directly, from the explicit formula with Pauli matrices. The result is

	1	-1	$\{i,-i\}$	$\{j,-j\}$	$\{k,-k\}$
$\chi_{\rho_{1,1}}$	1	1	1	1	1
$\chi_{\rho_{1,-1}}$	1	1	1	-1	-1
$\chi_{\rho_{-1,1}}$	1	1	-1	1	-1
$\chi_{\rho_{-1,-1}}$	1	1	-1	-1	1
χ_U	2	-2	0	0	0

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Note that this shows that a group is not determined by its table of characters (character tables for D_4 and Q_8 are the same).

3. If $e_1, \ldots, e_n$ is a basis of eigenvectors for $\rho_V(g)$ with eigenvalues λ_i, then $e_i \cdot e_j$, $i \leqslant j$, form a basis of $S^2(V)$, they are eigenvectors of $S^2(\rho_V(g))$ with eigenvalues $\lambda_i\lambda_j$. Hence our identity reduces to

$$\sum_{i \leqslant j} \lambda_i\lambda_j = \frac{1}{2}((\sum_i \lambda_i)^2 + \sum_i \lambda_i^2),$$

which is obvious.

4. Let us note that if v_1, v_2 form a basis of a two-dimensional vector space V, then the symmetric products $v_1^a v_2^b$, $a+b=k$, form a basis in the k-th symmetric power of V. Therefore, if a matrix A has eigenvalues $\lambda_1 \neq \lambda_2$, then the matrix S^kA has eigenvalues λ_1^k, $\lambda_1^{k-1}\lambda_2$, $\ldots$, λ_2^k on $S^k(V)$, and the trace of that matrix is $\frac{\lambda_1^{k+1}-\lambda_2^{k+1}}{\lambda_1-\lambda_2}$. Let us apply it for our question. The unit element has trace $k+1$, since $\dim S^k(\mathbb{C}^2) = k+1$. Each transposition has eigenvalues 1 and -1, so the trace is

$$\frac{1-(-1)^{k+1}}{2} = \begin{cases} 0, & \text{if } k \text{ is odd,} \\ 1, & \text{if } k \text{ is even.} \end{cases}$$

Each 3-cycle has eigenvalues $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$, so the trace is

$$\begin{cases} 1 \text{ if } k \equiv 0 \pmod 3, \\ -1 \text{ if } k \equiv 1 \pmod 3, \\ 0 \text{ if } k \equiv 2 \pmod 3. \end{cases}$$

Altogether, the character depends on the behaviour modulo 6. We obtain the following table for both characters and multiplicities of irreducibles:

	e	2-cycles	3-cycles	(χ, χ_{triv})	(χ, χ_{sgn})	(χ, χ_V)
$k \equiv 0 \pmod 6$	$k+1$	1	1	$k/6+1$	$k/6$	$k/3$
$k \equiv 1 \pmod 6$	$k+1$	0	-1	$(k-1)/6$	$(k-1)/6$	$(k+2)/3$
$k \equiv 2 \pmod 6$	$k+1$	1	0	$(k+4)/6$	$(k-2)/6$	$(k+1)/3$
$k \equiv 3 \pmod 6$	$k+1$	0	1	$(k+3)/6$	$(k+3)/6$	$k/3$
$k \equiv 4 \pmod 6$	$k+1$	1	-1	$(k+2)/6$	$(k-4)/6$	$(k+2)/3$
$k \equiv 5 \pmod 6$	$k+1$	0	0	$(k+1)/6$	$(k+1)/6$	$(k+1)/3$