MA 3416: Group representations
Selected answers/solutions to the assignment due March 4, 2015

1. Clearly, $\mathrm{triv} \otimes W \simeq W$ for all W. Also, $\mathrm{sgn} \otimes \mathrm{sgn} \simeq \mathrm{triv}$, $\mathrm{sgn} \otimes U \simeq U$, $\mathrm{sgn} \otimes V \simeq V'$, $\mathrm{sgn} \otimes V' \simeq V$, as it is obvious from examining characters of these. It remains to compute $U \otimes U$, $U \otimes V$, $V \otimes V$, and $V \otimes V'$. Note that $U \otimes V' \simeq U \otimes \mathrm{sgn} \otimes V \simeq U \otimes V$, and $V' \otimes V' \simeq V \otimes \mathrm{sgn} \otimes V \otimes \mathrm{sgn} \simeq V \otimes V$ because of the product formulas for the sign representation above. Computing the scalar products of characters, we find $U \otimes U \simeq U \oplus \mathrm{triv} \oplus \mathrm{sgn}$, $U \otimes V \simeq V \oplus V'$, $V \otimes V \simeq V \oplus V' \oplus \mathrm{triv} \oplus U$, $V \otimes V' \simeq V \otimes V \otimes \mathrm{sgn} \simeq V \oplus V' \oplus \mathrm{sgn} \oplus U$.

2. (**a**) The character is computed by direct inspection: we must figure out, for each type of rotation, how many vertices that rotation keeps intact. The result is written below under the character table of S_4.

	e	(ij)	(ijk)	$(ijkl)$	$(ij)(kl)$
triv	1	1	1	1	1
sign	1	-1	1	-1	1
V	3	1	0	-1	-1
V'	3	-1	0	1	-1
U	2	0	-1	0	2
vertices	8	0	2	0	0

Computing the scalar products with irreducible characters, we see that this representation is isomorphic to $\mathrm{triv} \oplus \mathrm{sign} \oplus V \oplus V'$.

Finally, note that the set of all vertices of the cube is the union of the sets of vertices of two regular tetrahedra, each formed by four vertices of the cube that have no edges of the cube between them. The invariant subspaces are: constant functions, functions which assume the same value on one of the tetrahedra and the same opposite value on the other, even functions which add up to zero on each of the tetrahedra, odd functions that add up to zero on each of the tetrahedra.

(**b**) Since T is an intertwining operator, and since all irreducible constituents of our representation are different, T acts by a scalar on each irreducible constituent, and we should just compute each of these scalars to examine the dynamics of T. Clearly, on the constant functions T acts as multiplication by 1, on functions that assume the same value on one of the tetrahedra and the same opposite value on the other T acts as multiplication by -1 (each a gets replaced by $(-a-a-a)/3 = -a$, on even functions that add up to zero on each of the tetrahedra T acts as multiplication by $-1/3$ (each a gets replaced by $(b+c+d)/3 = (-a)/3)$, and on odd functions that add up to zero on each of the tetrahedra T acts as multiplication by $1/3$ (each a gets replaced by $(b+c+d)/3 = (-(-a))/3)$. Therefore, the limiting behaviour of T applied to a certain vector is determined by the projections of that vector on the trivial representation and the sign representation. Note that for the configuration in question the sums of values on both tetrahedra are the same: $1+3+6+8 = 18 = 2+4+5+7$. Therefore, the projection on the sign representation, where the opposite values are proportional to the difference of values on the two tetrahedra, is equal to zero, and the projection on the trivial representation, the space of constant functions, is the function whose value on each vertex is equal to 4.5. That projection is the limit $\lim\limits_{n\to\infty} T^n(w)$.

3. Note that if $e_1, \ldots, e_n$ is a basis of V, then the wedge products $e_{i_1} \wedge \cdots \wedge e_{i_k}$ with $i_1 < \cdots < i_k$ form a basis of $\Lambda^k(V)$. Assume that A can be diagonalised, and that $f_1, \ldots, f_n$

is a basis of V consisting of eigenvectors of A with eigenvalues $\lambda_1, \dots, \lambda_n$. Then the wedge products $f_{i_1} \wedge \cdots \wedge f_{i_k}$ with $i_1 < \cdots < i_k$ are eigenvectors of $\Lambda^k(A)$ with the respective eigenvalues being $\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}$, and the trace of $\Lambda^k(A)$ is equal to the sum of all of these. Note that the characteristic polynomial of A, that is $\det(A - tI)$, is equal to $\prod_{i=1}^n(\lambda_i - t)$. Therefore, the trace of $\Lambda^k(A)$ is equal, up to a sign $(-1)^{n-k}$, to the coefficient of t^{n-k} in that polynomial.

4. Note that for a finite group G, the transformation $\rho(g)$ can be diagonalised. Assume that it has eigenvalues $\lambda_1, \dots, \lambda_n$. Then $\rho(g^2) = \rho(g)^2$ has eigenvalues $\lambda_1^2, \dots, \lambda_n^2$. Therefore,

$$\frac{1}{2}(\chi_V(g)^2 - \chi_V(g^2)) = \frac{1}{2}((\lambda_1 + \cdots + \lambda_n)^2 - (\lambda_1^2 + \cdots + \lambda_n^2)) = \sum_{i<j} \lambda_i\lambda_j,$$

which is precisely $\chi_{\Lambda^2(V)}(g)$, as we know from the previous question.

5. The characters of the corresponding representations are written below the character table of S_4. Note that U is two-dimensional, so $\Lambda^2(U)$ is one-dimensional, and the corresponding character is given by the determinant of the corresponding matrices acting on U, and $\Lambda^3(U) = 0$. Similarly, V and V' are three-dimensional, so the exterior square $\Lambda^2(V)$ can be handled by the previous question, the exterior square $\Lambda^2(V')$ is isomorphic to the exterior square $\Lambda^2(V)$, because we multiply by the sign twice, the action on the exterior cube $\Lambda^3(V)$ is given by the determinant, and the action on the exterior cube $\Lambda^3(V')$ is given by the determinant multiplied by the sign.

	e	(ij)	(ijk)	$(ijkl)$	$(ij)(kl)$
triv	1	1	1	1	1
sign	1	-1	1	-1	1
V	3	1	0	-1	-1
V'	3	-1	0	1	-1
U	2	0	-1	0	2
$\Lambda^2(V)$	3	-1	0	1	-1
$\Lambda^2(V')$	3	-1	0	1	-1
$\Lambda^2(U)$	1	-1	1	-1	1
$\Lambda^3(V)$	1	-1	1	-1	1
$\Lambda^3(V')$	1	1	1	1	1
$\Lambda^3(U)$	0	0	0	0	0

Thus, $\Lambda^2(V) \simeq \Lambda^2(V') \simeq V'$, $\Lambda^2(U) \simeq \Lambda^3(V) \simeq$ sign, and $\Lambda^3(V') \simeq$ triv.