

MA 3416: Group representations  
Selected answers/solutions to the assignment due March 23, 2015

**1. (a)** Indeed, scalar matrices do not change under conjugation by anything, in particular by matrices of representation, so if  $g$  is in  $G_\rho$ , then  $hgh^{-1}$  is in  $G_\rho$  as well.

**(b)** Since trace of an operator is equal to the sum of its eigenvalues, and all eigenvalues are roots of unity, our claim is clear.

**(c)** For complex numbers  $\lambda_1, \dots, \lambda_n$  we have  $|\lambda_1 + \dots + \lambda_n| = |\lambda_1| + \dots + |\lambda_n|$  if and only if the vectors representing these numbers in 2d plane have the same direction. Given that all these numbers are roots of unity in our case, we conclude that they are all equal, and so are eigenvalues of a scalar matrix.

**2.** We shall use both intertwining numbers and characters to show how they all work.

**(a)** Clearly, this representation is isomorphic to the standard representation on  $\mathbb{C}^4$ , and hence is isomorphic to the sum of the trivial representation and  $V$ , the constant functions and functions that add up to zero.

**(b)** The intertwining number of this representation with itself is equal to three, since there are three different orbits on pairs of edges: two edges can coincide, have a common vertex, or be disjoint. Also, the intertwining number of this representation with the representation arising from the set of vertices is two: in a pair of a vertex and an edge, the vertex can belong to the edge or not. Therefore, this representation contains a copy of the trivial one, a copy of  $V$ , and something else. That something else can only be  $U$ , as it is the only two-dimensional representation. The three invariant subspaces where those representations are realised are constant functions, functions that take opposite values on pairs of disjoint edges, and functions that take the same values on pairs of disjoint edges and add up to zero.

**(c)** Clearly, this representation is isomorphic to the standard representation on  $\mathbb{C}^4$ , and hence is isomorphic to the sum of the trivial representation and  $V$ , the constant functions and functions that add up to zero.

**(d)** Let us augment the character table of  $S_4$  with the values of character of the corresponding representations:

	$e$	$(ij)$	$(ijk)$	$(ijkl)$	$(ij)(kl)$
triv	1	1	1	1	1
sign	1	-1	1	-1	1
$V$	3	1	0	-1	-1
$V'$	3	-1	0	1	-1
$U$	2	0	-1	0	2
edges	12	2	0	0	0

Computing the scalar products with irreducible characters, we see that this representation is isomorphic to  $\text{triv} \oplus V \oplus V \oplus V' \oplus U$ .

Let us note that the space of all even functions on edges has three natural invariant subspaces: constant functions, functions that are constant on each of the three directions and add up to zero, and functions that take opposite values on each two edges of the same face that are not connected with an edge. It remains to find two invariant subspaces in the space of odd functions to guarantee that we found all the irreducible constituents. Those invariant subspaces consist of functions for which the sum of values on edges in each face is zero and of

functions for which for each face the sum of values of two opposite edges is equal to the sum of edges on two other opposite edges.

3. In class, we proved that  $A_5$  is the group of all rotations of the dodecahedron, and identified the conjugacy classes. Since the only rotations of the dodecahedron preserving some face are rotations about the axis passing through the centre of that face, only cycles of length 5 have fixed points in the set of faces, and we get the following character of CM:

	$1^5$	$2^2 1$	$3 1^2$	$5_1$	$5_2$
#	1	15	20	12	12
CM	12	0	0	2	2

Using the character table

	$1^5$	$2^2 1$	$3 1^2$	$5_1$	$5_2$
#	1	15	20	12	12
triv	1	1	1	1	1
V	4	0	1	-1	-1
W	5	1	-1	0	0
$V_1$	3	-1	0	$\tau$	$1 - \tau$
$V_2$	3	-1	0	$1 - \tau$	$\tau$

from class (here  $\tau = \frac{1+\sqrt{5}}{2}$ ), and computing scalar products, we see that

$$\mathbb{C}M \simeq \text{triv} \oplus V_1 \oplus V_2 \oplus W.$$

4. It is clear that  $W_1 \otimes W_2 \simeq W_2 \otimes W_1$  and  $W \otimes \text{triv} \simeq W$ , so we just need to decompose 10 pairwise products whose characters are

	$1^5$	$2^2 1$	$3 1^2$	$5_1$	$5_2$
#	1	15	20	12	12
$V_1 \otimes V_1$	9	1	0	$\tau + 1$	$2 - \tau$
$V_1 \otimes V_2$	9	1	0	-1	-1
$V_1 \otimes V$	12	0	0	$-\tau$	$-1 + \tau$
$V_1 \otimes W$	15	-1	0	0	0
$V_2 \otimes V_2$	9	1	0	$2 - \tau$	$\tau + 1$
$V_2 \otimes V$	12	-1	0	$-1 + \tau$	$-\tau$
$V_2 \otimes W$	15	-1	0	0	0
$V \otimes V$	16	0	1	1	1
$V \otimes W$	20	0	-1	0	0
$W \otimes W$	25	1	1	0	0

Computing scalar products of characters, we obtain

$$\begin{aligned}
V_1 \otimes V_1 &\simeq \text{triv} \oplus V_1 \oplus W \\
V_1 \otimes V_2 &\simeq V \oplus W \\
V_1 \otimes V &\simeq V_2 \oplus V \oplus W \\
V_1 \otimes W &\simeq V_1 \oplus V_2 \oplus V \oplus W \\
V_2 \otimes V_2 &\simeq \text{triv} \oplus V_2 \oplus W \\
V_2 \otimes V &\simeq V_1 \oplus V \oplus W \\
V_2 \otimes W &\simeq V_1 \oplus V_2 \oplus V \oplus W \\
V \otimes V &\simeq \text{triv} \oplus V_1 \oplus V_2 \oplus V \oplus W \\
V \otimes W &\simeq V_1 \oplus V_2 \oplus V \oplus W \oplus W \\
W \otimes W &\simeq \text{triv} \oplus V_1 \oplus V_2 \oplus V \oplus V \oplus W \oplus W
\end{aligned}$$

Note that since under conjugation by a transposition  $V_1$  and  $V_2$  get swapped, and other representations remain unchanged,  $V_1 \otimes V \simeq V_2 \oplus V \oplus W$  implies  $V_2 \otimes V \simeq V_1 \oplus V \oplus W$  etc.

**5. (a)** Clearly  $(g_1, g_2)(h_1, h_2)(g_1, g_2)^{-1} = (g_1 h_1 g_1^{-1}, g_2 h_2 g_2^{-1})$ , from which it is obvious that conjugacy classes of  $G_1 \times G_2$  are  $C_1 \times C_2$ , where  $C_i$  is a conjugacy class of  $G_i$ .

**(b)** Let us compute the scalar product of such a character with itself:

$$\begin{aligned}
(\chi, \chi) &= \frac{1}{\#G_1 \#G_2} \sum_{(g_1, g_2)} |\chi((g_1, g_2))|^2 = \\
&= \frac{1}{\#G_1 \#G_2} \sum_{(g_1, g_2)} |\text{tr}(\rho_1(g_1) \otimes \rho_2(g_2))|^2 = \\
&= \frac{1}{\#G_1 \#G_2} \sum_{(g_1, g_2)} |\text{tr}(\rho_1(g_1))|^2 |\text{tr}(\rho_2(g_2))|^2 = \\
&= \frac{1}{\#G_1} \sum_{g_1} |\text{tr}(\rho_1(g_1))|^2 \frac{1}{\#G_2} \sum_{g_2} |\text{tr}(\rho_2(g_2))|^2 = (\chi_1, \chi_1)(\chi_2, \chi_2) = 1,
\end{aligned}$$

so  $\chi$  is an irreducible character. A similar computation shows that these characters are pairwise orthogonal. Also, their number is equal to the product of numbers of conjugacy classes in our groups which by (a) is the number of conjugacy classes in the product. Thus, these are all irreducible representations.