MA 3416: Group representations
Homework problems due February 12, 2015

In all questions below the ground field is $\mathbb{C}$.

1. Find, up to isomorphism, all 2-dimensional representations of (**a**) $\mathbb{Z}/5\mathbb{Z}$; (**b**) D_4; (**c**) D_5; (**d**) Q_8. (*Note*: those representations are not required to be irreducible).

2. Write down all characters of irreducible representation of (**a**) D_4; (**b**) D_5; (**c**) Q_8. Check directly the orthonormality property for these characters.

Recall that the n^{th} symmetric power of a vector space W (which is denoted by $S^n(W)$) is a subspace in its n^{th} tensor power $W^{\otimes n}$ which is spanned by all symmetric products

$$w_1 \cdot w_2 \cdot \ldots \cdot w_n = \frac{1}{n!} \sum_{\sigma \in S_n} w_{\sigma(1)} \otimes w_{\sigma(2)} \otimes \ldots \otimes w_{\sigma(n)}$$

for all $w_1, \ldots, w_n \in W$. Moreover, if $e_1, \ldots, e_k$ is a basis of W, then the symmetric products $e_{i_1} \cdot e_{i_2} \cdot \ldots \cdot e_{i_n}$ with $1 \leqslant i_1 \leqslant i_2 \leqslant \ldots \leqslant i_n \leqslant k$ form a basis of the space $S^n(W)$. The n^{th} symmetric power $S^n(A)$ of an operator $A\colon W \to W$ is defined by

$$S^n(A)(w_1 \cdot w_2 \cdot \ldots \cdot w_n) = (Aw_1) \cdot (Aw_2) \cdot \ldots \cdot (Aw_n).$$

If (W, ρ) is a representation of a finite group G, $S^n(W)$ is an invariant subspace of all the operators $S^n(\rho(g))$ acting on $W^{\otimes n}$; this subspace is called the n^{th} symmetric power of the representation W.

3. Prove that $\chi_{S^2(V)}(g) = \frac{1}{2}(\chi_V(g)^2 + \chi_V(g^2))$. (*Hint*: recall that each individual matrix $\rho_V(g)$ can be diagonalised, use a basis of V consisting of eigenvectors for $\rho_V(g)$).

4. For each k, compute the multiplicities of irreducible representation in the representations of S_3 arising in k-th symmetric power of its two-dimensional irreducible representation.