3416: Group representations

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Lecture 14

These notes account for some differences (and misprints) from notes of 2012/13.

Today, we shall talk about the relationship between representations of G and representations of its index two subgroups. Let us assume that G is a finite group, and H is an index two subgroup in G. It is well known from elementary group theory that H is normal (because both left and right cosets are just H and the complement $G \setminus H$).

Note that as a normal subgroup, H is a union of G-conjugacy classes. Let us first investigate the relationship between conjugacy classes of G and H.

Theorem 1. *For each conjugacy class C of G that is contained in H, two situations are possible: (1) C is a conjugacy class in H, and (2) C is a disjoint union of two conjugacy classes of the same cardinality.*

Proof. Recall that the number of elements in a conjugacy class $C^G(x)$ of $x \in G$ is equal to the index of the centraliser $C^G_x = \{g \in G\colon gx = xg\}$. This follows from a more general statement that for a group action, the number of elements in an orbit of a point p is equal to the index of the stabiliser of p.

Therefore, for $x \in H$, $\#C^G(x) = \frac{\#G}{\#C^G_x}$, and $\#C^H(x) = \frac{\#H}{\#C^H_x} = \frac{\#G}{2\#C^H_x}$. Also, either $C^G_x \subset H$, in which case $C^G_x = C^H_x$, or $C^G_x \not\subset H$, in which case multiplication by any element $a \in C^G_x \setminus H$ provides a bijection between C^H_x and $C^G_x \setminus H$, so either $\#C^G_x = \#C^H_x$, or $\#C^G_x = 2\#C^H_x$. We conclude that

$$\#C^H(x) = \begin{cases} \#C^G(x), \text{ if } C^G_x \not\subset H, \\ \frac{1}{2}\#C^G(x), \text{ if } C^G_x \subset H. \end{cases}$$

In the latter case, $C^G(x) = C^H(x) \cup C^H(yxy^{-1})$, where y is any element from $G \setminus (H \cup C^G_x)$. □

This theorem has a counterpart on the level of complex representations.

Theorem 2. *For each irreducible representation (V, ρ) of G, two situations are possible: (1) (V, ρ) remains irreducible as a representation of H, and (2) (V, ρ) splits, as a representation of H, into a direct sum of two non-isomorphic irreducible representations of the same dimension.*

Proof. We have

$$1 = (\chi_V, \chi_V)_G = \frac{1}{\#G}\sum_{g\in G}|\chi_V(g)|^2 \geqslant \frac{1}{2\#H}\sum_{g\in H}|\chi_V(g)|^2,$$

so

$$2 \geqslant (\chi_V, \chi_V)_H =\geqslant \frac{1}{\#H}\sum_{g\in H}|\chi_V(g)|^2,$$

and hence (V, ρ) is isomorphic to a sum of at most two irreducibles, and if it is isomorphic to a sum of two irreducibles, then they are non-isomorphic. Moreover, it remains irreducible if and only if there exist elements of $G \setminus H$ on which χ_ρ takes non-zero values.

Now, suppose that $(V, \rho) = (V_1, \rho_1) \oplus (V_2, \rho_2)$ as representations of H. Fix $y \in G \setminus H$. The subspace $\rho(y)(V_1)$ is H-invariant: we have $\rho(h)\rho(y)(V_1) = \rho(y)\rho(y^{-1}hy)(V_1) \subset \rho(y)(V_1)$ because H is normal, and V_1 is H-invariant. Therefore, either $\rho(y)(V_1) = V_1$ or $\rho(y)(V_1) = V_2$. In the former case, we would conclude that V_1 is G-invariant (since $G = H \cup Hy$), which contradicts irreducibility of V. Hence $\rho(y)(V_1) = V_2$, so the two summands are isomorphic as vector spaces (but not representations). □

Let us now apply this to the example $A_4 \subset S_4$.

The conjugacy classes of even permutations in S_4 are the unit element, the 3-cycles, and pairs of disjoint transpositions. The centraliser of a 3-cycle is the cyclic subgroup generated by it, so it is contained in A_4, and the conjugacy class splits into two parts, $\{(123), (243), (134), (142)\}$ and $\{(132), (124), (143), (234)\}$. The centraliser of a product of two disjoint transpositions contains each of those transpositions, so the conjugacy class does not split. Altogether, A_4 has four conjugacy classes.

The irreducible representations of S_4 are: the trivial representation and the sign representation (they become the same one-dimensional representation of A_4), the three-dimensional representation (V, ρ) and $(V, \rho \otimes \mathrm{sgn})$ (they become the same representation of A_4, it is irreducible because the character χ_ρ takes non-zero values on some odd permutations), and the two-dimensional representation, for which the character vanishes on all odd permutations, so this representation splits as a direct sum of two non-isomorphic one-dimensional representations. Those are representations by cubic roots of unity that you constructed in homework.