3416: Group representations

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Lecture 22-23

These notes account for some differences (and misprints) from notes of 2012/13.

In these lectures, we shall outline the classification of complex representations of the group S_n.

Let us start with some basic combinatorics. By a *partition* λ of n, we mean a decomposition $n = \lambda_1 + \lambda_2 + \cdots + \lambda_k$ into several positive parts. The order of parts does not matter, so we may order them as we please; we shall always assume $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_k > 0$. We shall often append several parts $\lambda_{k+1}, \ldots, \lambda_n$ equal to zero, if we want the total number of parts to be equal to n (in general, the number of parts does not exceed n).

We shall say that a partition λ *dominates* a partition μ, and write $\lambda \trianglerighteq \mu$, if

$$\begin{gathered}\lambda_1 \geqslant \mu_1,\\ \lambda_1 + \lambda_2 \geqslant \mu_1 + \mu_2,\\ \ldots,\\ \lambda_1 + \cdots + \lambda_n \geqslant \mu_1 + \mu_2 + \cdots + \mu_n.\end{gathered}$$

Partitions are usually represented by *Young diagrams*: a Young diagram is an arrangement of unit squares in the quadrant $\{x > 0, y < 0\}$, where for each m the part λ_m of a partition is represented by λ_m unit squares with top left corners $(1-m, 0)$, $(1-m, 1)$, $\ldots (1-m, \lambda_m - 1)$.

Let λ be a partition of n. A *Young tableau* of shape λ is a numbering of unit squares of the Young diagram of λ by distinct integers from 1 to n.

The following result is the core fact of representation theory of symmetric groups.

Lemma 1. *Let λ and μ be partitions of n. Suppose that T is a Young tableau of shape λ, T' is a Young tableau of shape μ, and further suppose that any two numbers from the same row of T' are in different columns of T. Then $\lambda \trianglerighteq \mu$.*

Proof. Consider the numbers marking squares from the first m rows of T'. Our assumption implies that no $m+1$ of these numbers are in the same column of T. Thus, by a permutation of numbers within columns of T, we can ensure that all these numbers mark squares end up in the first m rows of T. Thus, the sum of lengths of the first m rows of T', that is $\mu_1 + \cdots + \mu_k$, does not exceed the sum of lengths of the first m rows of T, that is $\lambda_1 + \cdots + \lambda_k$ for each k, so $\lambda \trianglerighteq \mu$. □

A *Young tabloid* of shape λ is a Young tableau up to row permutations: two tableau are considered equal as tabloids if one is obtained from the other by permuting numbers inside the rows.

Both the set T_λ of all Young tableau of shape λ and the set M_λ of all Young tabloids of shape λ have a natural action of the group S_n permuting the numbers assigned to squares: the number assigned to the square k is sent to $\sigma^{-1}(k)$. These sets give rise to the corresponding permutation representations $(\mathbb{C}T_\lambda, \tilde{\rho}_\lambda)$ and $(\mathbb{C}M_\lambda, \rho_\lambda)$. We shall denote basis elements of $\mathbb{C}T_\lambda$ by Latin letters, e.g. T, and basis elements of $\mathbb{C}M_\lambda$ by Latin letters in square brackets, e.g. $[T]$. There is a homomorphism of representations $\mathbb{C}T_\lambda \to \mathbb{C}M_\lambda$ sending each tableau T to the corresponding tabloid $[T]$.

To each Young tableau T, one can associate its *column group* C_T; it is a subgroup of the symmetric group that permutes numbers within columns of T, or more formally

$$C_T = \{\sigma \in S_n\colon \text{ for each } i, \text{ the numbers } i \text{ and } \sigma(i) \text{ are in the same column of } T\}.$$

For instance, if T is any tableau of shape $\lambda = (n)$, then $C_T = \{e\}$, and if T is any tableau of shape (1^n), then $C_T = S_n$. In general the column group depends on T, not just on its shape. However, column groups for two different tableau of the same shape are related in a very simple way.

Lemma 2. *We have* $C_{\sigma.T} = \sigma C_T \sigma^{-1}$.

Proof. Conjugation in the symmetric group corresponds to re-numbering: if τ permutes some elements $i_1, \dots, i_m$ in a certain way, then $\sigma\tau\sigma^{-1}$ permutes $\sigma(i_1), \dots, \sigma(i_m)$ in the same way. So, if $\tau \in C_T$, then $\sigma\tau\sigma^{-1} \in C_{\sigma.T}$, and the other way round, if $\tau \in C_{\sigma.T}$, then $\sigma^{-1}\tau\sigma \in C_T$. □

Let λ be a partition of n, and let T be a Young tableau of shape λ. Let us define, for each representation (V, ρ), a linear transformation $b_{T,\rho}$ by the formula

$$b_{T,\rho} = \sum_{\sigma \in C_T} \operatorname{sign}(\sigma)\rho(\sigma).$$

Let λ be a partition of n, and let T be a Young tableau of shape λ. A *Young polytabloid* e_T associated to a Young tableau T is the element of $\mathbb{C}M_\lambda$ defined by the formula

$$e_T = b_{T,\rho_\lambda}([T]).$$

Let λ be a partition of n. The *Specht module* S_λ is the subspace of $\mathbb{C}M_\lambda$ spanned by all polytabloids e_T associated to all Young tableau of shape λ.

Note that $\rho_\lambda(\sigma)(e_T) = e_{\sigma.T}$:

$$\rho_\lambda(\sigma)(e_T) = \rho_\lambda(\sigma) \sum_{\tau \in C_T} \operatorname{sign}(\tau)\rho_\lambda(\tau)([T]) =$$
$$= \sum_{\tau \in C_T} \operatorname{sign}(\tau)\rho_\lambda(\sigma\tau\sigma^{-1})\rho_\lambda(\sigma)([T]) = \sum_{\tau \in C_{\sigma.T}} \operatorname{sign}(\tau)\rho_\lambda(\tau)([\sigma.T]) = e_{\sigma.T},$$

and therefore S_λ is an invariant subspace of $\mathbb{C}M_\lambda$; moreover, this computation shows that an invariant subspace of $\mathbb{C}M_\lambda$ that contains one of the polytabloids e_T must contain S_λ. We shall use this later.

Lemma 3. *Let λ and μ be partitions of n, let T be a Young tableau of shape λ, and let T' be a Young tableau of shape μ. Suppose that*

$$b_{T,\rho_\mu}([T']) \neq 0.$$

Then $\lambda \trianglerighteq \mu$, and if $\lambda = \mu$, then $b_{T,\rho_\lambda}([T']) = \pm e_T$.

Proof. Suppose that a and b are in the same row of T', or, in other words, that $(ab).[T'] = [T']$. Let us show that under our assumptions these two numbers must be in different columns of T. Assume the contrary, so that $(ab) \in C_T$. Let $x_1, \ldots, x_l$ be representatives of the right cosets $C_T/\langle(ab)\rangle$. We have $C_T = \{x_1, \ldots, x_l, x_1(ab), \ldots, x_l(ab)\}$, and this shows that

$$b_{T,\rho_\mu}([T']) = \sum_{i=1}^{l} (\text{sign}(x_l)\rho_\mu(x_l)([T']) + \sum_{i=1}^{l} (\text{sign}(x_l(ab))\rho_\mu(x_l)\rho_\mu((ab))([T']) = 0,$$

a contradiction. Thus, $\lambda \trianglerighteq \mu$ by Lemma 1. Moreover, if $\lambda = \mu$, the proof of Lemma 1 shows that T' is obtained from T by action of a permutation $\tau \in C_T$. Therefore,

$$b_{T,\rho_\lambda}([T']) = \sum_{\sigma \in C_T} \text{sign}(\sigma)\rho(\sigma)\rho_\lambda(\tau)([T]) = \text{sign}(\tau) \sum_{\sigma \in C_T} \text{sign}(\sigma\tau)\rho_\lambda(\sigma\tau)([T]) = \text{sign}(\tau)e_T.$$

□

Since every vector in $\mathbb{C}M_\lambda$ is a combination of Young tabloids of shape λ, we conclude that

Corollary 1. *Suppose that T is a Young tableau of shape λ, and that $v \in \mathbb{C}M_\lambda$. Then $b_{T,\rho_\lambda}(v)$ is a scalar multiple of e_T.*

We are quite close to obtain a full classification of irreducible representations. Let us state the last auxiliary result.

Lemma 4. *Let λ and μ be partitions of n, and suppose that there exists homomorphism of representations $\theta \in \text{Hom}_{S_n}(\mathbb{C}M_\lambda, \mathbb{C}M_\mu)$ for which $S_\lambda \not\subset \text{Ker}\,\theta$. Then $\lambda \trianglerighteq \mu$. If $\lambda = \mu$, then the restriction of θ on S_λ is a multiplication by a scalar.*

Proof. Let T be a Young tableau of shape λ. Clearly, $e_T \notin \text{Ker}\,\theta$ (the kernel of a homomorphism of representations is an invariant subspace, and we know that an invariant subspace of $\mathbb{C}M_\lambda$ containing one polytabloid must contain S_λ). We have

$$0 \neq \theta(e_T) = \theta(b_{T,\rho_\lambda}([T])) = b_{T,\rho_\mu}(\theta([T])).$$

But $\theta([T])$ is a linear combination of Young tabloids of shape μ, so at least one of them is not annihilated by b_{T,ρ_μ}. By Lemma 3, we have $\lambda \trianglerighteq \mu$. Furthermore, if $\lambda = \mu$, then by Corollary 1, $\theta(e_T) = b_{T,\rho_\lambda}(\theta([T]))$ is a scalar multiple of e_T, and since $\rho_\lambda(\sigma)(e_T) = e_{\sigma.T}$, we conclude that θ acts by that very scalar on S_λ, the span of all polytabloids. □

Corollary 2. *Let λ and μ be partitions of n, and suppose that there exists a nonzero linear map $\theta \in \text{Hom}_{S_n}(S_\lambda, \mathbb{C}M_\mu)$. Then $\lambda \trianglerighteq \mu$. If $\lambda = \mu$, then θ is a multiplication by a scalar.*

Proof. As an invariant subspace of $\mathbb{C}M_\lambda$, the Specht module S_λ admits an invariant complement, $\mathbb{C}M_\lambda = S_\lambda \oplus U$. Every homomorphism $\theta \in \text{Hom}_{S_n}(S_\lambda, \mathbb{C}M_\mu)$ can be extended to a homomorphism $\tilde{\theta} \in \text{Hom}_{S_n}(\mathbb{C}M_\lambda, \mathbb{C}M_\mu)$ by letting it be equal to zero on U, and Lemma 4 applies. □

Theorem 1. *1. Each Specht module S_λ is an irreducible representation of S_n.*

2. If $S_\lambda \simeq S_\mu$, then $\lambda = \mu$.

3. Each complex irreducible representation of S_n is isomorphic to one of the Specht modules S_λ.

4. *The irreducible constituents of* $\mathbb{C}M_\mu$ *are Specht modules* S_λ *with* $\lambda \trianglerighteq \mu$ *(with some multiplicities), and* S_μ *appears in* $\mathbb{C}M_\mu$ *with multiplicity one.*

Proof. 1. Every homomorphism $\alpha \in \mathrm{Hom}_{S_n}(S_\lambda, S_\lambda)$ can be viewed as a homomorphism from S_λ to $\mathbb{C}M_\lambda$, and Corollary 2 implies that this homomorphism is a multiplication by a scalar. This implies that the intertwining number $c(S_\lambda, S_\lambda)1$ is equal to 1, so S_λ is irreducible.

2. If $S_\lambda \simeq S_\mu$, then, recalling that Specht modules are subspaces of the spaces spanned by all Young tabloids, we can construct nonzero homomorphisms $\theta_\lambda \in \mathrm{Hom}_{S_n}(S_\lambda, \mathbb{C}M_\mu)$ and $\theta_\mu \in \mathrm{Hom}_{S_n}(S_\mu, \mathbb{C}M_\lambda)$. This implies that $\lambda \trianglerighteq \mu$ and $\mu \trianglerighteq \lambda$, so $\lambda = \mu$.
3. We know that the number of complex irreducible representations of a finite group is equal to the number of conjugacy classes. Since the Specht modules are irreducible and pairwise non-isomorphic, and the number of conjugacy classes of S_n is equal to the number of partitions of n, we already exhausted all irreducible representations.
4. This follows immediately from Corollary 2: if S_λ is a constituent of $\mathbb{C}M_\mu$, then there exists a nonzero linear map $\theta \in \mathrm{Hom}_{S_n}(S_\lambda, \mathbb{C}M_\mu)$, and if $\lambda = \mu$, this same corollary states that $c(S_\lambda, \mathbb{C}M_\mu) = 1$.

□

Remark 1. In fact, the previous theorem is valid over any field F of characteristic zero; the proof needs to be slightly modified if the field is not algebraically closed, but essentially it all boils down to the fact that the Specht modules are "defined over $\mathbb{Q}$": all the coefficients of all the polytabloids written in the basis of tabloids are integers, so the matrices representing all permutations have rational entries.