3416: Group representations

Dr. Vladimir Dotsenko (Vlad)

Lecture 3

These notes account for some differences (and misprints) from notes of 2012/13.

Recall that the datum of a representation is a pair (V, ρ), where V is a vector space, and $\rho\colon G \to GL(V)$ is a group homomorphism. In cases when one of these is obvious from the context, we shall use just V or just ρ, but generally we shall list both as a pair.

Homomorphisms of representations

The notion of a homomorphism is something that exists universally for any algebraic structure, and in particular for representations.

Definition 1. Let (V, ρ) and (W, τ) be two representations of the same group G over the same field F. We say that a linear map $T\colon V \to W$ is a homomorphism of representations if $T\rho(g) = \tau(g)T$ for all $g \in G$. In older textbooks such maps are also called intertwining operators.

A homomorphism is said to be an isomorphism if T is invertible.

Example 1. Let us define a homomorphism α fromthe regular representation (V_l, ρ_l) to the trivial representation $(\mathbb{1}, \rho_{triv})$ by sending every basis vector e_g to the basis vector v of the trivial representation. The calculation proving that it is indeed a homomorphism is

$$\alpha\rho_l(h)(e_g) = \alpha(e_{hg}) = v = \rho_{triv}(h)(v) = \rho_{triv}(h)\alpha(e_g).$$

Example 2. Let us define a homomorphism β from the trivial representation $(\mathbb{1}, \rho_{triv})$ to the regular representation (V_l, ρ_l) by sending the basis vector v of the trivial representation to $\sum\limits_{g\in G} e_g$. The calculation proving that it is indeed a homomorphism is

$$\beta\rho_{triv}(h)(v) = \beta(v) = \sum_{g\in G} e_g = \sum_{g'=hg\in G} e_{g'} = \rho_l(h)(\sum_{g\in G} e_g) = \rho_l(h)\beta(v).$$

Example 3. Let us define a homomorphism γ from the left regular representation (V_l, ρ_l) to the right regular representation (V_r, ρ_r) by sending every basis vector e_g to $e_{g^{-1}}$. The calculation proving that it is indeed a homomorphism is

$$\gamma\rho_l(h)(e_g) = \gamma(e_{hg}) = e_{(hg)^{-1}} = e_{g^{-1}h^{-1}} = \rho_r(h)(e_{g^{-1}}) = \rho_r(h)\gamma(e_g).$$

Reducible, irreducible, decomposable and indecomposable

The image of the map β above is a subspace of V_l that is invariant under all operators $\rho_l(g)$. This means that the representation ρ_l is *reducible*, that is has an invariant subspace. In general, the strategy is to study representations that are not reducible, that is are *irreducible*, and then see how all representations are created out of them.

Definition 2. A representation (V, ρ) is said to be irreducible if the only subspaces that are invariant with respect to all operators $\rho(g)$ are $\{0\}$ and V. Otherwise it is said to be reducible.

Let us also try to explain what we mean by "created out of them".

Definition 3. A representation (V, ρ) is said to be decomposable if there is a direct sum decomposition of vector spaces $V = V_1 \oplus V_2$ with $V_1, V_2 \neq \{0\}$ and both V_1 and V_2 invariant with respect to all operators $\rho(g)$ are $\{0\}$ and V. Otherwise it is said to be indecomposable.

Of course, in such situation, the restriction of ρ on V_1 and V_2 makes them representations themselves, and we say that our representation (V, ρ) is decomposed into a direct sum of two smaller representations.

Example 4. Let $G = \mathbb{Z}/2\mathbb{Z}$ and $F = \mathbb{F}_2$. Consider the left regular representation (V_l, ρ_l). We know already that this representation contains an invariant subspace $U = \text{Im}(\beta)$ so it is reducible. Let us see that it is indecomposable. For that, we shall show that in fact U is the only nontrivial invariant subspace of V_l. Indeed, V_l is two-dimensional, so a nontrivial invariant subspace must be one-dimensional. Suppose that it is spanned by $ae_{\bar{0}} + be_{\bar{1}}$. We have

$$\rho_l(\bar{1})(ae_{\bar{0}} + be_{\bar{1}}) = ae_{\bar{1}} + be_{\bar{0}},$$

which is proportional to $ae_{\bar{0}} + be_{\bar{1}}$ if and only if $\det \begin{pmatrix} a & b \\ b & a \end{pmatrix} = 0$, that is $a^2 = b^2$, that is $a = b$ in case of characteristic 2! Thus, an invariant subspace is spanned by $e_{\bar{0}} + e_{\bar{1}}$, so it must coincide with $\text{Im}(\beta)$.

Example 5. Let $G = \mathbb{Z}/2\mathbb{Z}$ and $F = \mathbb{R}$, or indeed any field of characteristic different from 2. Consider the left regular representation (V_l, ρ_l). We know already that this representation contains an invariant subspace $U = \text{Im}(\beta)$ so it is reducible. Let us see that it is also decomposable. In fact, the previous example suggests very clearly what to take as a complement of U, the linear span of $e_{\bar{0}} - e_{\bar{1}}$. Every vector is of course preserved by $\rho(\bar{0}) = \text{Id}$, and we also have

$$\rho_l(\bar{1})(e_{\bar{0}} - e_{\bar{1}}) = e_{\bar{1}} - e_{\bar{0}},$$

which is proportional to $e_{\bar{0}} - e_{\bar{1}}$. Therefore, the span of $e_{\bar{0}} - e_{\bar{1}}$ is an invariant complement of U.

Next time we shall now prove a theorem which basically explains that the only discrepancy between irreducibility and indecomposability comes from examples like the one we considered above, some issues related to the characteristics of the ground field. Namely, we prove the following result.

Theorem 1. *Suppose that* $\#G$ *is invertible in* F. *Then every reducible representation* $V, \rho)$ *of* G *is decomposable, so for every invariant subspace* U *of* V *there exists an invariant subspace* W *such that* $V = U \oplus W$.