3416: Group representations

Dr. Vladimir Dotsenko (Vlad)

Lecture 5

These notes account for some differences (and misprints) from notes of 2012/13.

The main point today is to start studying representations under the hypothesis of complete reducibility from the previous class.

Schur's lemma

The following result holds in a much more general context than just representations of finite groups (as it should, as we remarked that the definition of a homomorphism is very general).

Theorem 1 (Schur's Lemma)**.** *Let (V, ρ) and (W, τ) be two irreducible representations of G.*

1. *A homomorphism of representations $T\colon V \to W$ is either zero or an isomorphism.*
2. *Over an algebraically closed field F, e.g. $\mathbb{C}$ a homomorphism of representations $T\colon V \to V$ is a scalar multiple of the identity map, $T = \lambda\,\mathrm{Id}$.*

Proof. For the first part, let us note that $\ker(T)$ is an invariant subspace of V and $\mathrm{Im}(T)$ is an invariant subspace of W. Indeed, if $v \in \ker(T)$, then $T\rho(g)v = \tau(g)Tv = 0$, so $\rho(g)v \in \ker T$, and similarly, if $w \in \mathrm{Im}(T)$, then $w = Tv$ for some v, and $\tau(g)w = \tau(g)Tv = T\rho(g)v \in \mathrm{Im}(T)$.

Because V is irreducible, we conclude that $\ker(T) = \{0\}$ or $\ker(T) = V$. In the former case, T is an injective map, in the second case $T = 0$. Because W is irreducible, we conclude that $\mathrm{Im}(T) = \{0\}$ or $\mathrm{Im}(T) = W$. In the former case, $T = 0$, in the latter case, T is a surjective map. We conclude that if $T \neq 0$, then T is both injective and surjective, therefore bijective. A bijective map is invertible, so is an isomorphism.

For the second part, let us note that over an algebraically closed field, every linear transformation has an eigenvalue. Let λ be an eigenvalue of T. Then $T - \lambda\,\mathrm{Id}$ is not invertible, and hence must be equal to zero. □

Intertwining numbers

A very useful invariant of representation theory is the *intertwining number* of two representations.

Definition 1. Let (V, ρ) and (W, τ) be two representations of G. The set $\mathrm{Hom}_G(V, W)$ of all homomorphisms of representations $T\colon V \to W$ is clearly a subspace in the vector space of all linear maps $\mathrm{Hom}(V, W)$. The dimension of this subspace is called the intertwining number of V and W and is denoted by $c(V, W)$.

Example 1. Schur's Lemma essentially states that for two irreducible representations over an algebraically closed field we have $c(V, W) = 0$ if these representations are not isomorphic, and $c(V, W) = 1$ if these representations are isomorphic.

Theorem 2. *Suppose that the ground field is algebraically closed. Let $V_1, \ldots, V_k$ be several pairwise non-isomorphic representations of G, and let $V \simeq \bigoplus_{i=1}^k V_i^{\oplus a_i}$, $W \simeq \bigoplus_{i=1}^k V_i^{\oplus b_i}$. Then*

$$c(V, W) = a_1 b_1 + a_2 b_2 + \cdots + a_k b_k.$$

Proof. A linear map $T\colon V \to W$ is made of "blocks", T_{pq} corresponding to inclusion of the p-th of the direct summands into V, applying the map T on that summand, and projecting onto the q-th of the direct summands in W. Each of these blocks is a homomorphism of representations, since it is made of homomorphisms (the original map T, inclusions and projections). Each of the summands is irreducible, so by Schur's Lemma, each block is zero or a scalar, depending on whether the corresponding summands are isomorphic. For each i, there are $a_i b_i$ pairs of summands which both are isomorphic to V_i, and each such pair contributes 1 to the total dimension, since there is just an arbitrary scalar to choose for the corresponding block. Altogether the dimension of the vector space is equal to $a_1 b_1 + a_2 b_2 + \cdots + a_k b_k$, as required. □

Remark 1. In particular, if $V = W$, we get $c(V, V) = a_1^2 + a_2^2 + \cdots + a_k^2$. This means that if $c(V, V) = 1$, then V is irreducible, if $c(V, V) = 2$, then V is isomorphic to a direct sum of two non-isomorphic irreducible representations, and if $c(V, V) = 3$, then V is isomorphic to a direct sum of three non-isomorphic irreducible representations. Indeed, the equation $1 = a_1^2 + a_2^2 + \cdots + a_k^2$ has the only positive solution $k = 1$, $a_1 = 1$, the equation $2 = a_1^2 + a_2^2 + \cdots + a_k^2$ has the only positive solution $k = 2$, $a_1 = a_2 = 1$, the equation $3 = a_1^2 + a_2^2 + \cdots + a_k^2$ has the only positive solution $k = 3$, $a_1 = a_2 = a_3 = 1$.

Proposition 1. *For every group G, we have*

$$c(V_l, V_l) = \#G$$

Proof. In general, a linear map $T\colon V_l \to V_l$ is given by a $\#G \times \#G$-matrix A with entries a_{g_1, g_2} such that

$$T(e_h) = \sum_{g \in G} a_{g,h} e_g.$$

Let us examine the condition $T\rho_l(k) = \rho_l(k)T$. We have

$$T\rho_l(k)(e_h) = T(e_{kh}) = \sum_{g \in G} a_{g,kh} e_g$$

and

$$\rho_l(k)T(e_h) = \rho_l(k) \sum_{g \in G} a_{g,h} e_g = \sum_{g \in G} a_{g,h} e_{kg} = \sum_{g' = kg \in G} a_{k^{-1}g', h} e_{g'}.$$

We conclude that in order for T to be a homomorphism, we must have

$$a_{g,kh} = a_{k^{-1}g,h}$$

for all g, h, k, or replacing g by kg,

$$a_{kg,kh} = a_{g,h},$$

or in other words $a_{g,h}$ only depends on $g^{-1}h$, since putting $k = g^{-1}$, we see that $a_{e,g^{-1}h} = a_{g,h}$. This means that the intertwining number is equal to the number of choices for $g^{-1}h$, that is $\#G$. □

Example 2. Let us apply the previous result for $G = S_3$ and complex representations. We have $c(V_l, V_l) = 6$, and the only two ways to represent 6 as a sum of squares is either have $6 = 1+1+1+1+1+1$ or $6 = 1+1+4$. In the former case, V_l must be a sum of six non-isomorphic representations, and since V_l is six-dimensional, we conclude that all those representations are one-dimensional. But we already know that there are just two different one-dimensional representations, so it is impossible. We conclude that $V_l \simeq V_1 \oplus V_2 \oplus V_3 \oplus V_3$, where V_1, V_2, and V_3 are three non-isomophic representations.

Remark 2. In fact, the space $\mathrm{Hom}_G(V_l, V_l)$ has a basis $\rho_r(g)$, $g \in G$: these linear maps are clearly homomorphisms of representations, and linearly independent by inspections.

We shall see in one of the next classes that every irreducible representation (V, ρ) of G is isomorphic to a direct summand of (V_l, ρ_l), and the multiplicity of such a summand is $\dim(V)$. This implies that $\dim(V) < \sqrt{\#G}$, which already makes irreducible representations much more manageable than the regular representation.