MA 3419: Galois theory
Homework problems due November 28, 2017

Solutions to this are due by the end of the 11am class on Tuesday November 28. Please attach a cover sheet with a declaration `http://tcd-ie.libguides.com/plagiarism/declaration` confirming that you know and understand College rules on plagiarism.

Please put your name and student number on each of the sheets you are handing in. (Or, ideally, staple them together).

1. (**a**) Determine the Galois group of the splitting field of $x^5 - 4x + 2$ over $\mathbb{Q}$.

(**b**) Same question for the Galois group of the splitting field of $x^4 - 4x + 2$ over $\mathbb{Q}$.

2. Prove that a subgroup of A_5 that contains a 3-cycle and acts transitively on $1, 2, 3, 4, 5$ coincides with A_5.

3. (**a**) Show that for a prime n a subgroup of S_n that contains an n-cycle and a transposition coincides with S_n. (**b**) Give an example showing that this statement does not have to hold if n is composite.

4. Let p be a prime number. Show that for large N the polynomial

$$x^p - N^3 p^3 x(x-1)\cdots(x-(p-4)) - p$$

has $p-2$ real roots. Use it to deduce that for such values of N the Galois group of the splitting field of this polynomial over $\mathbb{Q}$ is S_p.

5. (**a**) Let k be a field of characteristic p, $K = k(x, y)$ the field of rational functions in two variables, and $L = K(\sqrt[p]{x}, \sqrt[p]{y})$. Show that there does not exist an element $a \in L$ for which $L = K(a)$.

(**b**) Let k be an infinite field of characteristic p, $K = k(x, y)$ the field of rational functions in two variables, and $L = K(\sqrt[p]{x}, \sqrt[p]{y})$. Show that there are infinitely many intermediate fields between K and L.

6. (**a**) Show that if $\mathbb{F}_{p^n}$ is isomorphic to an extension of $\mathbb{F}_{(p')^{n'}}$ (where p and p' are primes) then $p = p'$ and n is divisible by n'.

(**b**) Explain why $\mathbb{F}_{p^n}$ is a Galois extension of $\mathbb{F}_p$.

(**c**) Show that the Galois group $\mathrm{Gal}(\mathbb{F}_{p^n} : \mathbb{F}_p)$ is the cyclic group $\mathbb{Z}/n\mathbb{Z}$. (*Hint*: show that $x \mapsto x^p$ is an automorphism, and that it is of order n in the Galois group).

(**d**) Show that if n is divisible by n' then $\mathbb{F}_{p^n}$ is isomorphic to a field extension of $\mathbb{F}_{p^{n'}}$.

(**e**) Show that if n is divisible by n' then $\mathbb{F}_{p^n}$ is a Galois extension of $\mathbb{F}_{p^{n'}}$, and describe the Galois group $\mathrm{Gal}(\mathbb{F}_{p^n} : \mathbb{F}_{p^{n'}})$.