Assignment 4

MA341C — Seminar on *Proofs from THE BOOK*

Trinity College Dublin

NAME AND SURNAME: ....

STUDENT NUMBER: .... NUMBER OF PAGES: ....

Note: solutions to this assignment are due by 11am on Wednesday, October 10th. Please attach a cover sheet with a declaration (http://tcd-ie.libguides.com/plagiarism/declaration) confirming that you know and understand College rules on plagiarism. All exercises are weighed equally unless otherwise stated.

Exercise 1. Let p be a prime number. Determine the number of solutions to the equation $x^2 + y^2 = 1$ in the prime field $\mathbb{Z}_p$. (*Hint*: use the fact that the number of solutions to $x^2 = a$ is expressed via the Legendre symbol as $1+\left(\frac{a}{p}\right)$, and the observation that $1-y^2 = \frac{1-y}{1+y}(1+y)^2$ for $y \neq -1$.)

Exercise 2. Let p be a prime number. Show that for an integer a coprime to p, the Legendre symbol $\left(\frac{a}{p}\right)$ coincides with the sign of the permutation $x \mapsto ax$ of $\mathbb{Z}_p$. (*Hint:* applying the permutation σ of distinct numbers $x_1, \ldots, x_n$ to the product of all pairwise differences $x_i - x_j$ multiplies that product by the sign of σ.)

Exercise 3. Let p be a prime number. Consider the ring $\mathbb{H}_p$ of "quaternions modulo p", that is expressions $a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ where $a, b, c, d \in \mathbb{Z}_p$ and $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are quaternionic units that multiply in the usual way. By the theorem from class, $\mathbb{H}_p$ cannot be a division ring. Show directly that for any p this ring has zero divisors.

Exercise 4. (a) Let D_n denote the group of symmetries of the n-gon in the 2D plane. Let us position the n-gon in a way that its centre is at the origin of a rectangular coordinate system, and one of its vertices is $(1,0)$. The group D_n then has n elements that are rotations σ_k through the angle $2\pi k/n$ for $0 \leq k < n$, and n elements that are reflections ρ_k with respect to lines at the angle $\pi(2k+1)/n$ with the x axis for $0 \leq k < n$. Write down the formula for the result of conjugating σ_a with ρ_b. (This formula defines the "dihedral quandle", related to colourings of the Borromean link in the talk.)
(b) Counting Fox n-labellings for appropriately chosen n, show that the trefoil knot is not equivalent to the unknotted circle.