Assignment 8

MA341C — Seminar on *Proofs from THE BOOK*

Trinity College Dublin

NAME AND SURNAME: ....

STUDENT NUMBER: .... NUMBER OF PAGES: ....

Note: solutions to this assignment are due by 11am on Wednesday, November 14th. Please attach a cover sheet with a declaration (http://tcd-ie.libguides.com/plagiarism/declaration) confirming that you know and understand College rules on plagiarism. All exercises are weighed equally unless otherwise stated.

In questions below, c_n are the Catalan numbers, $c_n = \frac{1}{n+1}\binom{2n}{n}$. You may use without proof the well known fact that c_n is equal to the number of lattice paths from $(0,0)$ to (n,n) that go only north or east *and never go below the line* $y = x$. (In fact, we strongly recommend to use this fact in the first two questions.)

Exercise 1. Prove that $\det \begin{pmatrix} c_{2n-1} & c_{2n-2} & c_{2n-3} & \cdots & c_n \\ c_{2n-2} & c_{2n-3} & c_{2n-4} & \cdots & c_{n-1} \\ c_{2n-3} & c_{2n-4} & \cdots & \cdots & c_{n-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ c_n & c_{n-1} & c_{n-2} & \cdots & c_1 \end{pmatrix} = 1.$

Exercise 2. Prove that $\det \begin{pmatrix} c_{2n} & c_{2n-1} & c_{2n-2} & \cdots & c_{n+1} \\ c_{2n-1} & c_{2n-2} & c_{2n-3} & \cdots & c_n \\ c_{2n-2} & c_{2n-3} & \cdots & \cdots & c_{n-1} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ c_{n+1} & c_n & c_{n-1} & \cdots & c_2 \end{pmatrix} = n+1.$

Exercise 3. It is well known (and explained in chapter 13 of the book) that the complete graph K_n (with n vertices, all pairwise connected by edges) is not a plane graph for $n \geq 5$. Show that:

1. K_5 can be drawn in the Möbius band without self-intersections;
2. K_5 can be drawn in the surface of the torus without self-intersections;
3. K_6 can be drawn in the Möbius band without self-intersections;
4. K_6 can be drawn in the surface of the torus without self-intersections.

Exercise 4.

1. Each point of the plane is coloured in one of three colours. Show that there exist two points at the unit distance that are of the same colour.
2. Show that it is possible to colour each point of the plane in one of seven colours so that any two points at the unit distance are coloured differently.