BINOMIAL COEFFICIENT ARE (ALMOST) NEVER POWERS

JONATHAN KEOGH

1. Introduction

When is $\binom{n}{k}$ equal to an integer power m^l? Clearly, there are infinitely many solutions: take $k = l = 2$ and consider an n such that $\binom{n}{2} = \frac{n(n-1)}{2} = m^2$ for some m, then $\binom{(2n-1)^1}{2} = \frac{(2n-1)^2((2n-1)^2-1)}{2} = \frac{(2n-1)^2(4n^2-4n)}{2} = (2n-1)^2 4m^2$; hence, beginning with $\binom{9}{2} = 6^2$, this generates an infinite sequence of solutions. This case is an exception, however, to the general phenomenon. We shall prove that for $k \geq 4$ and $l \geq 2$, there are no solutions. The proof is by contradiction.

2. Preliminaries

Since $\binom{n}{k} = \binom{n}{n-k}$, we can implicitly assume $n \geq 2k$ throughout. We shall use two theorems, due to Sylvester and Legendre respectively, which we state without proof.

Theorem. *One of the numbers $n, n-1, \ldots, n-k+1$ is divisible by a prime $p > k$, for $n \geq 2k$. Equivalently, $\binom{n}{k}$ has a always has a prime factor $p > k$, for $n \geq 2k$.*

Theorem. *The number $n!$ contains the prime factor p exactly $\sum_{k\geq 1} \lfloor \frac{n}{p^k} \rfloor$ times*

We first prove a proposition, then a lemma, that will smooth out the main proof.

Proposition. *If $\binom{n}{k}$ is equal to an integer power m^l, then $n > k^2$*

Proof. If $\binom{n}{k} = m^l$ then, by Sylvester's theorem, this would imply that $p^l | n(n-1)\ldots(n-k+1)$ for some $p > k$. It follows that $p^l | n-j$ for some j, implying

$$n \geq p^l > k^l \geq k^2.$$

□

Lemma. *Consider the factors $n-j$ in the numerator of the binomial coefficient $\binom{n}{k} = m^l$; if we write $n - j = a_j m_j^l$ for each j, where a_j is not divisible by any l-th power, then the numbers a_j are simply a permutation of $1, \ldots, k$*

Proof. Assume $a_i = a_j$ for some $i < j$, then $a_i m_i = n - i > n - j = a_j m_j$ implying $m_i \geq m_j + 1$. We have

$$k > (n-i) - (n-j) = a_j(m_i^l - m_j^l) \geq a_j((m_j+1)^l - m_j^l) >$$

$$a_j l m_j^{l-1} \geq l(a_j m_j^l)^{1/2}(n-k+1)^{1/2} \geq l(\frac{n}{2}+1)^{1/2} > n^{1/2},$$

in contradiction to the previous proposition. It remains to show that the integers a_i are actually just $1, \ldots k$; for this end, it suffices to prove

$$a_0 a_1 \ldots a_{k-1} \text{ divides } k!\ .$$

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We rewrite $\binom{n}{k} = m^l$ as

$$a_0 a_1 \dots a_{k-1} (m_0 m_1 \dots m_{k-1})^l = m^l k! \ .$$

In cancelling factors we have

$$a_0 a_1 \dots a_{k-1} u^l = v^l k!,$$

with $\gcd(u, v) = 1$. If we show $v = 1$ then we'll be done; so assume not, and let v have a prime divisor p. It follows that p divides $a_0 a_1 \dots a_{k-1}$ also and is therefore less than or equal to k. We estimate to what exponent it divides it by. Let i be an arbitrary integer, and $b_1 < \cdots < b_s$ be the multiples of p^i among $n, n-1, \dots, n-k+1$ $(s \leq k)$. We have $b_s \geq b_1 + (s-1)p^i$; hence,

$$(s-1)p^i \leq b_s - b_1 \leq n - (n-k+1) = k+1,$$

implying

$$s \leq \lfloor \frac{k-1}{p^i} \rfloor + 1 \leq \lfloor \frac{k}{p^i} \rfloor + 1.$$

Therefore, if we consider this for each i, it implies the exponent of p in $a_0 a_1 \dots a_{k-1}$ is at most

$$\sum_{i=1}^{l-1} (\lfloor \frac{k}{p^i} \rfloor + 1);$$

while at the same time the exponent of p in $k!$ is

$$\sum_{i \geq 1} \lfloor \frac{k}{p^i} \rfloor;$$

both by Legendre's theorem. Finally, then, this implies that p has exponent

$$\sum_{i=1}^{l-1} (\lfloor \frac{k}{p^i} \rfloor + 1) - \sum_{i \geq 1} \lfloor \frac{k}{p^i} \rfloor \leq l - 1,$$

a contradiction to v^l being a l-th power. □

By inspection of this proof, if $l = 2$ then, since $k \geq 4$, one of the a_i must be equal to 4, a square, which can't happen. Hence we assume that $l \geq 3$ for the following proof of the main theorem.

3. The proof

Theorem. *There are no solutions to the equation* $\binom{n}{k} = m^l$ *for* $k \geq 4$ *and* $l \geq 3$

Proof. Since $k \geq 4$ we must have $a_{i_1} = 1, a_{i_2} = 2, a_{i_3} = 4$,for some i_1, i_2, i_3; or, rewritten,

$$n - i_1 = m_1^l, n - i_2 = 2m_2^l, n - i_3 = 4m_3^l.$$

We have then $(n-i_2)^2 \neq (n-i_1)(n-i_3)$; for if not then, putting $b = n - i_2$ and $n - i_1 = b - x$,$n - i_3 = b + y$, with $0 < |x|, |y| < k$, gives

$$(y - x)b = xy;$$

then by our previous proposition, this implies $|xy| = b|y - x| > n - k > (k-1)^2 \geq |xy|$. Now since $m_2^2 \neq m_1 m_3$, we assume without loss of generality the case $m_2^2 > m_1 m_3$; the other case being similar. We have

$$2(k-1)n > n^2 - (n-k+1)^2 > (n-i_2)^2 - (n-i_1)(n-i_3)$$

$$= 4[m_2^{2l} - (m_1m_3)^l] \geq 4[(m_1m_3+1)^l - (m_1m_3)^l] \geq 4lm_1^{l-1}m_3^{l-1}.$$

Note that $n > k^l \geq k^3 > 6k$. Multiplying across by m_1m_3 gives

$$2(k-1)nm_1m_3 > 4lm_1^l m_3^l = l(n-i_1)(n-i_3) > l(n-k+1)^2 > 3(n-\frac{n}{6})^2 > 2n^2.$$

But $m_i \leq n^{\frac{1}{l}} \leq n^{\frac{1}{3}}$; hence

$$kn^{\frac{2}{3}} \geq km_1m_3 > (k-1)m_1m_3 > n,$$

or $k^3 > n$, a contradiction. □

References

[1] Martin Aigner and Gunter M. Ziegler. *Proofs from The Book.* Springer-Verlag, Berlin, fifth edition, 2014. Including illustrations by Karl H. Hofmann.