THREE APPLICATIONS OF EULER'S FORMULA

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1. Introduction

In Graph Theory, a planar graph is a graph that can be embedded in the plane, or in other words, can be drawn in such a way that none of the edges cross each other. Such a given and fixed drawing is referred to as a plane graph.A plane graph then decomposes the plane into a number of regions (including the unbounded outer region). These regions are referred to as "faces". Euler's Formula describes the relationship between the number of edges, vertices and faces in any plane graph. This report shall introduce the formula and the relevant concepts in graph theory, and present its application in three different contexts. The below is mainly composed of material from [1, Chapter 12] and from Peter Phelan's talk which took place in two parts on the 21st November and 28th November.

2. Euler's Formula

A graph is said to be connected if there is a path between any two vertices.

Theorem 2.1 (Euler's Formula)**.** *G=(V,E) , V Vertex set, E Edge set*
If G is a connected plane graph with n vertices, e edges and f faces then

$$n - e + f = 2$$

Proof. Let G be a connected planar graph.
G has a cycle means (informally) that there is a sequence of edges where the first edge is also the last.
The Spanning Tree

$$T \subset E$$

is the minimal spanning subgraph

$$G' \subset G$$

such that T is connected.
The Dual Graph G* is a graph that has a vertex for every face of G, and has an edge wherever any two faces of G are seperated by an edge of G. T* corresponds to edges in

$$E \backslash T.$$

Claim: T* is connected.
Claim: T* does not contain a cycle. Suppose that it does.
For every tree, the number of vertices+ 1 = the number of edges.

$$e_T + e_{T*} = e$$

$$e_T + 1 = n$$
$$e_{T*} + 1 = f$$
$$e_T + e_{T*} + 2 = n + f$$
$$n - e + f = 2$$

□

This is an example of a topological invariant, used in discerning whether topological spaces are not homeomorphic.
The degree of a vertex is the number of edges touching that vertex.

$$n_i$$

= the number of vertices with dergree i

$$n = n_1 + n_2 + ...$$

$$f_k$$

= the number of k-faces in a graph (where a k-face is a face bounded by k edges).

$$f = f_1 + f_2 + ...$$
$$2e = n_1 + 2n_2 + 3n_3 + ...$$
$$2e = f_1 + 2f_3 + 3f_3 + ...$$

$$\delta = 2e/n$$
$$\zeta = 2e/f$$

Let G be a simple (connected) plane graph with n¿2 vertices.
A) G has at most 3n-6 edges.
B) G has a vertex of degree at most 5.
C) If the edges of G are 2 coloured, then there is a vertex in G with at most 2 colour changes in the cyclic order of the edges around that vertex.

A)

$$f = f_1 + f_2 + f_3 + ...$$
$$2e = f_1 + 2f_2 + 3f_3 + ...$$
$$2e - 3f = f_4 + 2f_5 + 3f6 + ... \geq 0$$
$$3(e - f) \geq e, n - e + f = 2$$
$$\Rightarrow n - 2 = e - f$$
$$3(n - 2) \geq e$$
$$\Rightarrow e \leq 3n - 6$$

B)

$$\delta = 2e/n \leq 2(3n - 6)/n$$
$$= (6n - 12)/n$$
$$\Rightarrow \delta \leq (6 - 12/n) < 6$$

C) Let C be the number of corners where a colour change occurs. Suppose the statement is false. Then we have

$$C \geq 4n$$

corners with colour changes, since at every vertex there is an even number of changes. Every face with 2k or 2k +1 sides has at most 2k such corners, so we conclude that

$$4n \leq C \leq 2f_3 + 4f_4 + 4f_5 + 6f_6 + 6f_7 + 8f_8 + ...$$

$$\leq 2f_3 + 4f_4 + 6f_5 + 8f_6 + 10f_7$$

$$= 2(3f_3 + 4f_4 + 5f_5 + 6f_6 + 7f_7 + ...)$$

$$-4(f_3 + f_4 + f_5 + f_6 + f_7 + ...)$$

$$= 4e - 4f$$

So we have

$$e \geq n + f,$$

again contradicting Euler's formula.

3. The Sylvester-Gallai Theorem

Theorem 3.1 (The Sylvester-Gallai Theorem)**.** *Given any set of*

$$n \geq 3$$

points on the plane, not all on one line, then there is always a line containing exactly 2 of the points.

The problem can be restated as follows.
Given any set of

$$n \geq 3$$

great circles on

$$S^2$$

not all intersecting at one point, there is always a point on exactly 2 of the great circles.

Proof. This arrangement of great circles yields a plane graph on

$$S^2,$$

whose vertices are points of intersection of two great circles, which divide the great circles into edges. All the vertex degrees are even, and all are at least 4.

$$B) \Rightarrow \exists v : d(v) \leq 5$$

$$\Rightarrow \exists v : d(v) = 4$$

□

4. Monochromatic Lines

Theorem 4.1. *Given any finite configuration of black and white points on the plane, not all on one line, there is always a monochromatic line.*

Proof. Again this theorem can be restated.
Given any configuration of

$$n \geq 3$$

black/white great circles on

$$S^2,$$

not all intersecting at one point, then there is always an intersecting point which lies only on great circles of one colour.
This is clear through C), since in every vertex where great circles of different colours interact, we always have at least 4 corners with colour changes. □

5. Pick's Theorem

Theorem 5.1. *For any polygon*

$$Q \subset R^2$$

(not necessarily convex) with integral vertices

$$A(Q) = n_{int} + (1/2)n_{bd} - 1$$

where

$$n_{int}$$

= the number of integral points enclosed by Q

$$n_{bd}$$

= the number of integral points on the boundary of Q.

For the following, we call a convex polygon P 'elementary' if its vertices are integral, and contains no further integral points. Every elementary triangle

$$conv(p_0, p_1, p_2)$$

has area 1/2.

Proof. Both the parallelogram P with corners

$$P_0, P_1, P_2, P_1 + P_2 - P_0$$

and the lattice

$$Z^2$$

are symmetric with respect to

$$\sigma : x \mapsto p_1 + p_2 - x$$

Thus P is elementary, and its integral translates tile the plane.
Hence

$$p_1 - p_0, p_2 - p_0$$

is a basis of the lattice, it has determinant

$$\pm 1,$$

P is a parallelogram of area 1, and our triangle has area 1/2. □

We now are equipped to prove Pick's Theorem.

Proof. Every such polygon can be triangulated using all the lattice points on the interior, and on the boundary of Q.
if we interpret this triangulation as a plane graph, the plane is then subdivided into one unbounded face, plus f-1 triangles of area

$$A(Q) = 1/2(f-1).$$

Every triangle has 3 sides, where each of the

$$e_{int}$$

interior edges bounds 2 triangles, while the

$$e_{bd}$$

boundary edges appear in one single traingle each.
So

$$3(f-1) = 2e_{int} + e_{bd}$$

and thus

$$f = 2(e-f) - e_{bd} + 3.$$

Note

$$e_{bd} = n_{bd}$$

So

$$f = 2(e-f) - n_{bd} + 3$$
$$2n_{int} + n_{bd} - 1$$

And so, finally

$$A(Q) = 1/2(f-1) = n_{int} + (1/2)n_{bd} - 1.$$

□

References

[1] Martin Aigner and Gunter M. Ziegler. Proofs from The Book. Springer-Verlag, Berlin, fifth edition, 2014. Including illustrations by Karl H. Hofmann.