TILING RECTANGLES

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1. Introduction

Problems surrounding the tiling of rectangles are easy to dismiss as rather construed recreational mathematics. To do so however is a mistake. Recreational mathematics often has an elegance that is all its own as the contents of this report will showcase. We will present two theorems and four proofs in this area following [1,28] and Daniel Mulcahy's presentation.

2. Tiling Rectangles with Rectangles

Definition 2.1. Given a rectangle R, a *tiling of* R the finite decomposition of R into the disjoint union of smaller rectangle

Given this definition, a natural question is to ask what the side lengths of the smaller rectangles say about the sides of the larger rectangle. A partial, number theoretical, answer to this question is supplied by the following theorem.

Theorem 2.2. *Given a tiling of a rectangle R, if all the smaller rectangles have at least one side of integer length, then R has at least one side of integer length.*

We will provide three proofs of this theorem. The first two are, broadly speaking, equivalent. They both rely on the concept of area, expressed as an integral in the first proof and more concretely as chessboard squares in the second. The proofs are due to Nicolaas de Bruijn and, Richard Rochberg and Sherman Stein respectively.

In what follows we assume that the large rectangle R is placed parallel to the x, y-axes with $(0, 0)$ as the bottom left corner. We can then assume that the smaller rectangles R_i have their sides parallel to the coordinate axes too.

Proof 1. Consider any rectangle S, with its sides extending from a to b along the x-axis and from c to d along the y-axis, and the quantity

$$I(S) = \int\int_S e^{2\pi i(x+y)} dxdy$$

By Fubini's Theorem we can write

$$= \int_a^b e^{2\pi ix} dx \cdot \int_c^d e^{2\pi iy} dy = \frac{1}{4\pi^2}(e^{2\pi ib} - e^{2\pi ia})(e^{2\pi id} - e^{2\pi ic})$$

$I(S) = 0$ iff $(e^{2\pi ib} - e^{2\pi ia})$ or $(e^{2\pi id} - e^{2\pi ic})$ is 0. Now,$e^{2\pi ix}$ is periodic with period 1 and injective within that period. Thus $I(S) = 0$ iff $b - a$ or $d - c$ is an integer multiple of 1 - an integer. In other words $I(S) = 0$ iff S has a side of integer length.

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We have assumed all the smaller rectangles R_i have at least one side of integer length. Therefore $I(R_i) = 0$. Integrals are additive over area so we can write

$$I(R) = \int\int_R e^{2\pi i(x+y)} dxdy = \sum_{j \in J} \int\int_{R_j} e^{2\pi i(x+y)} dxdy = \sum_{j \in J} I(R_j) = 0$$

where the sum is taken over the whole tiling. Therefore R must have a side of integer length □

Proof 2. Cover the plane with ordinary alternating black and white chessboard squares of size $\frac{1}{2} \times \frac{1}{2}$. Start this colouring at the origin with a black square.

Since each small rectangle R_i has an integer side, it must contain an equal area coloured black and white. Therefore the large rectangle R must contain such equal area.

But this means R must have an integer side, else we obtain a contradiction as follows. Dissect R into four pieces by drawing the largest rectangle S with integer sides that can fit inside R, and extend the edges until they meet the sides of R. Then as S and the top and right rectangles have at least one integer sides it means that they contain that equal areas of black and white.

The top-right rectangle has both of its sides smaller than 1. It must always contain more black then white, as it has a black square in its bottom left corner. If it is a $x \times y$ square, then $0 < x, y < 1$. If $x \leq \frac{1}{2}$ and $y \leq \frac{1}{2}$ then it is entirely black. If $x > \frac{1}{2}$ and $y \leq \frac{1}{2}$ then the area $\frac{y}{2}$ is black $> (x - 0.5)y$ which is white. Similarly for $y > \frac{1}{2}$ and $x \leq \frac{1}{2}$. Lastly $x > \frac{1}{2}$ and $y > \frac{1}{2}$ then $0.25 + (x - 0.5)(y - 0.5)$ is black $> \frac{1}{2}(x + y - 1)$ which is white.

In any case, R must contain more black than white, our desired contradiction.

□

The third proof is very different in flavour from the first two. It links quite nicely with Pedro Tamaroff's talk on 12/09 as it involves counting in two ways on a graph. It is attributed to Mike Paterson.

Proof 3. Let T be the set of all smaller rectangles R_i. Let C be the set of all corners with both coordinates integers. Form a bipartite graph connecting $T \to C$ in the obvious manner - connect each rectangle in T to any of its corners which lie in C.

We observe that each rectangle has $0, 2, 4$ of its vertices in C corresponding first with whether the small rectangle R_i has any corners lying on the integer lattice and then with whether R_i has either one or two integer sides. So each vertex in T has even degree.

Each $c \in C$ will be connected with four $t \in T$ if c lies in the interior of R. It will be linked with two such t if c is on the edge of R but not an endpoint. Finally, the origin is connected to only one $t \in T$. The total degree of the vertices in T must be the same as in C. Therefore, to make the total parities the same, at least one of the $c \in C$ we have not discussed must have nonzero degree. The only candidates left are the remaining corners of R. But one of these having integer coordinates means that R has a side of integer length. □

It is worth noting that the theorem, and in fact all three proofs, can easily be generalised to higher dimensions.

Theorem 2.3. *Whenever an n-orthotope R is tiled by orthotopes all of which have at least one side of integer length, then R has a side of integer length.*

The proof is left as an exercise to the interested reader.

3. Tiling Rectangles with Squares

As we have seen, the big question when talking about tilings is relating properties of the smaller rectangles to that of the large rectangle. Sometimes special cases have special properties. And, what rectangle is more special than a square?

Theorem 3.1. *A rectangle R can be tiled by squares iff the sides of the rectangle have a rational ratio.*

The proof we will give was found by Max Dehn, and involves a seemingly out-of-place use of linear algebra.

Proof. If the sides have a rational ratio then there exists a such that both sides are integer multiples of a. Then R can be tiled with $a \times a$ squares.

Conversely assume a rectangle R can be tiled by squares. Rescale R such that one of its sides is of length 1. Let a be the ratio of sides - then R is a $a \times 1$ rectangle. Assume $a \notin \mathbb{Q}$. Extend every side of every square in the rectangle in both directions until they connect with the edges of R. R is now tiled with rectangles. Let $\mathbb{A} = \{a_1, a_2, a_3 \ldots a_n\}$ be the side lengths of these rectangles in any order.

Let $V(a)$ be the vector space spanned by $\mathbb{A}$ over $\mathbb{Q}$. 1 and a are then in $V(a)$ and can be extended to a basis $\{1, a, v_1, v_2 \ldots v_k\}$ of $V(a)$ over $\mathbb{Q}$.

Define $f : V(a) \to \mathbb{R}$ by $f(1) = 1$, $f(a) = -1$, $f(v_i) = 0$ for all i, extended to $V(a)$ by linearity. Let T be the set of all rectangles within R. Given any $c \times d$ rectangle K define $g : T \to \mathbb{R}$ to be $g(K) = f(a)f(b)$.

Then for any two rectangles K_1, K_2 sharing a common side, by the linearity of f, $g(K_1 \cup K_2) = g(K_1) + g(K_2)$. Considering the original tiling by squares T_i and taking the sum we have

$$g(R) = \sum g(T_i).$$

Now, $g(T_i) = f(t)f(t) \geq 0$ where t is the side length of T. Therefore $\sum g(T_i) \geq 0$. But $g(R) = f(1)f(a) = -1 < 0$. This is a contradiction and we therefore conclude $a \in \mathbb{Q}$ as required. $\square$

References

[1] Martin Aigner and Gunter M. Ziegler. Proofs from The Book. Springer-Verlag, Berlin, fifth edition, 2014. Including illustrations by Karl H. Hofmann