A THEOREM OF PÓLYA ON POLYNOMIALS

ALDEN MATHIEU

1. Introduction

George Pólya, a Hungarian mathematician, has made numberous contributions to analysis. While the chapter is entitled "a theorem," it would be more accurate to describe it as "twice a theorem" regarding complex monic polynomials. Both statements of the theorem are essentially consequences of Chebyshev's celebrated theorem, but have surprising and elegant results. This paper expands upon [1, Chapter 23] and David Glynn's talk.

2. Projections of sets onto a line

We consider polynomials in $\mathbb{C}$. Suppose that $f(z)$ is a complex monic polynomial of degree $n \geq 1$. We define the set C to be all points in $\mathbb{C}$ mapped under f into the circle of radius 2; that is, $C := \{z \in \mathbb{C} : |f(z)| \leq 2\}$. We permit C to be disconnected.

For any line L in the complex plane, the projection of the set C onto L has a maximum length of 4. While this is clearly true for $f(z)$ when $n = 1$ (in which case, C is a disk of diameter 4 and thus has a maximal projection of length 4), Pólya showed that this holds for any choice of monic $f(z)$ and thus C.

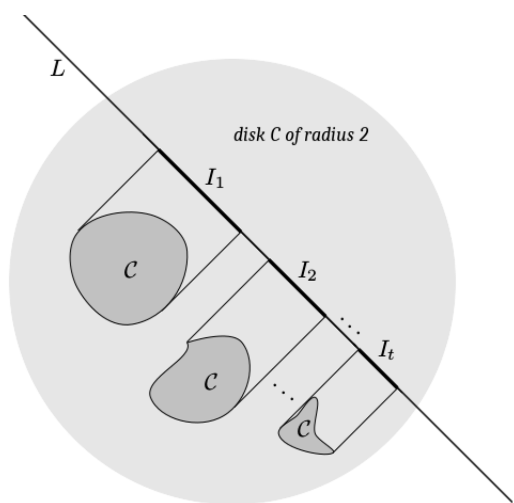

Figure 1

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Remark 2.1. Through rotation, we can always make the line L coincide with the real axis.

Notation 2.2 (Length)**.** We denote the length of an interval I_j (where length is defined in the usual way) as $l(I_j)$.

Theorem 2.3 (Initial formulation)**.** *Let $f(z)$ be a complex monic polynomial of degree $n \geq 1$. Define $C = \{ z \in \mathbb{C} : |f(z)| \leq 2\}$ and let R be the orthogonal projection of C onto the real axis. Then R is covered by intervals $I_1, ..., I_t$ on the real line that satisfy*

$$l(I_1) + ... + l(I_t) \leq 4$$

Proof. This is clearly true for $n = 1$, as mentioned above.

For $n > 1$, we write $f(z)$ as the product of complex factors:

$$f(z) = (z - c_1) \cdots (z - c_n)$$

where $c_k = a_k + ib_k$ and $z = x + iy \in \mathbb{C}$, and compare this to the real polynomial $p(x) \in \mathbb{R}[x]$:

$$p(x) = (x - a_1) \cdots (x - a_n)$$

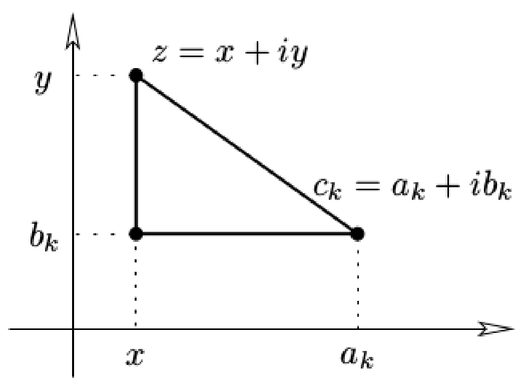

FIGURE 2. The Pythagorean theorem in the complex plane.

By the Pythagorean theorem, as illustrated in Fig. 2, we have

$$|x - a_k|^2 + |y - b_k|^2 = |z - c_k|^2$$

Hence, for all k, we have that every real factor is bounded above by every complex factor:

$$|x - a_k| \leq |z - c_k|$$

and applying this to our complex $f(z)$ and real $p(x)$ polynomials,

$$|p(x)| = |x - a_1| \cdots |x - a_n| \leq |z - c_1| \cdots |z - c_n| = |f(z)| \leq 2$$

We consider the set $P = \{x \in \mathbb{R} : |p(x)| \leq 2\}$. By our choice of C, we know that its orthogonal projection $R \subset P$. If we can show that P can be covered by intervals whose length sums to at most 4, then we are done.

We restate the theorem and we show that it is a consequence of Chebyshev's theorem:

Theorem 2.4 (Revised formulation)**.** *Let $p(x) \in \mathbb{R}[x]$ be monic with all roots $\in \mathbb{R}$. Then the set $P = \{x \in \mathbb{R} : |p(x)| \leq 2\}$ can be covered by intervals of total length at most 4.*

Pólya allegedly demonstrates in [2] that this restated theorem is a consequence of Chebyshev's Theorem. We'll have to take his word for it, because my German isn't good enough to check and the editor only offers addenda in English.

3. Chebyshev's Theorem

Theorem 3.1 (Chebyshev's Theorem)**.** *Let $p(x) \in \mathbb{R}[x]$ be monic with degree $n \geq 1$. Then*

$$\max_{-1 \leq x \leq 1} |p(x)| \geq \frac{1}{2^{n-1}}$$

We omit the proof, in keeping with David Glynn's presentation; the result has been thoroughly covered in Junior and Senior Freshman analysis.

Corollary 3.2. *Let $p(x) \in \mathbb{R}[x]$ be monic with degree $n \geq 1$. Suppose $|p(x)| \leq 2, \forall x \in [a, b]$. Then $b - a \leq 4$.*

Proof. Let $y = \frac{2}{b-a}(x - a) - 1$. This is a map from $[a, b] \to [-1, 1]$.

Consider $q(y) = p(\frac{b-a}{2}(y + 1) + a)$. These polynomials have the same maximum over their respective intervals:

$$\max_{-1 \leq y \leq 1} |q(y)| = \max_{a \leq x \leq b} |p(x)|$$

and are bounded above by 2, by Chebyshev:

$$2 \geq \max_{-1 \leq y \leq 1} |q(y)| = \max_{a \leq x \leq b} |p(x)| \geq \left(\frac{b-a}{2}\right)^n \frac{1}{2^{n-1}} = 2\left(\frac{b-a}{4}\right)^n$$

Thus, $(b - a) \leq 4$. □

We're creeping closer to the desired result. We now have that if P is a single interval, then it is of length less than or equal to 4, which is what we're aiming for.

Question. *What if P is several intervals?*

For example, given the illustrative polynomial $p(x) = x^3 - 3x^2$, the set P is the union of real intervals, P= $[1 - \sqrt{3}, 1] \cup [1 + \sqrt{3}, \approx 3.20]$.

We know by continuity of $p(x)$ that P is the union of disjoint closed intervals I_j for $j \geq 1$. Since $p(x) = \pm 2$ at each endpoint of an interval, and since $p(x)$ can only assume any value finitely often, we know that there must be a finite number of intervals: $I_1, ..., I_t$.

Therefore, we construct a new monic polynomial $\tilde{p}(x) \in \mathbb{R}[x]$ of degree $n \geq 1$ such that $\tilde{P} = \{x \in \mathbb{R} : |\tilde{p}(x)| \leq 2\}$ is an interval, and the length of $\tilde{P} \geq l(I_1) + ... + l(I_t)$. The corollary proves that this interval $\tilde{P}$ has length at most 4 and is at least as long as the sum of the lengths of the constituent intervals.

Now we have that the length of P is bounded above by 4, whether it's a single interval or a (finite) sum of closed intervals, we can state a few useful facts and return to our proof of the revised formulation of Pólya's theorem 2.4.

4. Two facts about polynomials with real roots

Lemma 4.1. *If b is a multiple root of $p'(x)$, then b is also a root of $p(x)$.*

Proof. Let $b_1 < \cdots < b_r$ be roots of $p(x)$, with multiplicities $s_1, ..., s_r$ that sum to n. Then $p(x) = (x - b_j)^{s_j} h(x)$ and if $s_j > 1$, $p'(x)$ has it as a root with multiplicity $s_j - 1$. Also, there is a root of $p'(x)$ between each of the roots b_1 and b_2, ..., up to b_{r-1} and b_r. There roots are all single roots, because

$$\sum_{j=1}^{r}(s_j - 1) + (r - 1)$$

counts the roots up to the degree $n - 1$ of $p'(x)$, thus any multiple roots of $p'(x)$ can only occur in the roots of $p(x)$. □

Lemma 4.2. $p'(x)^2 \geq p(x)p''(x)$, $\forall x \in \mathbb{R}$.

Proof. This is straightforward computation. We assume x is not a root, to avoid triviality.

$$p(x) = \sum_{k=1}^{n} \frac{p(x)}{x - a_k}$$

$$\implies \frac{p'(x)}{p(x)} = \sum_{k=1}^{n} \frac{1}{x - a_k}$$

$$\implies \frac{p''(x)p(x) - (p'(x))^2}{p(x)^2} = -\sum_{k=1}^{n} \frac{1}{(x - a_k)^2} < 0$$

□

5. Proof of the revised formulation of Pólya's theorem

Finally we are fully prepared to finish the proof of Thm. 2.4.

We number the finite intervals in the set $P = \{x \in \mathbb{R} : |p(x)| \leq 2\}$ from I_1 at the left to I_t at the right. Without loss of generality, we assume $p(x) = 2$ at both endpoints of P.

Let $p(b) = \min(p(x) \text{ in } I_j)$. This implies $p'(b) = 0$ and $p''(b) \geq 0$. In the first case, $p''(b) = 0$, b is then a multiple root of $p'(x)$ and hence a root of $p(x)$. In the second case, $p''(b) > 0$, we use Lemma 4.2 and conclude $(p'(b)^2 \geq p''(b)p(b)$ and hence it has a root in the interval from b to an endpoint of I_j.

We now construct our polynomial $p(x)$. We number the intervals as before, $I_1, ..., I_t$. We assume I_t has m roots of $p(x)$, $m < n$, which is justified by our earlier work. We let $b_1, ..., b_m$ be roots in I_t and $c_1, ..., c_{m-n}$ be roots in the union of the remaining intervals, $I_1 \cup \cdots \cup I_{t-1}$. Write $p(x) = q(x)r(x)$ where we define $q(x) = (x - b_1) \cdots (x - b_m)$ and $r(x) = (x - c_1) \cdots (x - c_{m-n})$. We define d to be the distance between the rightmost interval and the next rightmost interval as shown below.

We set $p_1(x) = q(x + d)r(x)$; it is again monic with degree n. Let P_1= {x $\in \mathbb{R} : |p_1(x)| \leq 2\}$. We will show that $\cup_{i=1}^{t-1} I_i$ is contained in P_1.

If $x \in \cup_{i=1}^{t-1} I_i$, then $|x + d - b_i| \leq |x - b_i|$ because all the b_i are in the rightmost interval I_t. So $|q(x + d)| \leq |q(x)| \implies |p_1(x)| \leq |p(x)| \leq 2$. Therefore the union of

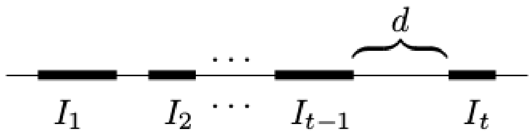

FIGURE 3

intervals are contained in P_1. Similarly, if $x \in I_t$, we have $|r(x-d)| \leq |r(x)| \implies |p_1(x-d)| = |q(x)||r(x-d)| \leq |p(x)| \leq 2$. Therefore $I_t - d \subseteq P_1$.

Now we consider merging the interval leftward: from $p(x)$ to $p_1(x)$, the intervals I_{t-1} and $I_t - d$ merge into a single interval. We infer that we can therefore construct a polynomial to merge all the intervals together. After a maximum of $t-1$ such repetitions, we have constructed a polynomial $\tilde{p}(x)$ representing the single interval $\tilde{P} = \{x \in \mathbb{R} : |\tilde{p}(x) \leq 2|\}$. As a single interval, we can then apply the result

$$4 \geq l(\tilde{P}) \geq l(P)$$

□

REFERENCES

[1] Martin Aigner and Günter M. Ziegler. *Proofs from The Book*. Springer-Verlag, Berlin, fifth edition, 2014. Including illustrations by Karl H. Hofmann.

[2] George. Pólya. *Beitrag zur Verallgemeinerung des Verzurrungssatzes auf mehrfach zusammenhängenden Gebieten*, volume 1: Singularities of Analytic Functions, chapter 111, pages 347–354. MIT Press, 1974.