HOW TO GUARD A MUSEUM

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1. INTRODUCTION

This report will delve into the lighthearted topic of how many guards are needed to guard a museum depending on the shape of the floor of the museum, as outlined in Daniel Mulcahy's talk. We impose that the guards must be stationary, however, they may look around and every part of the floor must be visible to at least one guard. We will represent the floor of the museum as a polygon and provide an upper bound on the number of guards neccesary that depends on the number of walls in the museum, or equivalently the nuber of edges of the associated polygon.

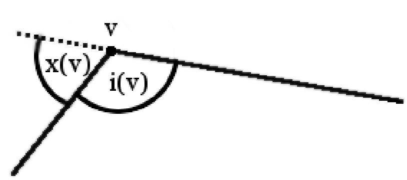
2. SOME RESULTS CONCERNING POLYGONS

We develop a handful of somewhat elementary results concerning polygons that will help us prove our main theorem.

Theorem 2.1. *The sum of the interior angles of a polygon with n vertices is equal to* $180(n-2)^\circ$

Proof. Let P be a polygon with n vertices. We consider an ant circumnavigating the perimeter of our polygon. At each vertex v of P, the ant must turn a certain angle $x(v)^\circ$ to remain on the perimeter. We call $x(v)^\circ$ the exterior angle at v. Now for the ant to walk around the whole perimeter and arrive back to where it started, it must turn 360° in total. Therefore the sum of exterior angles of the vertices of P is equal to 360°. Now note that at each vertex v of P, $i(v) + x(v) = 180^\circ$ where $i(v)$ is the interior angle at v. Therefore the sum of all exterior angles plus the sum of all interior angles of the vertices of P is equal to $180n^\circ$. Therefore the sum of all interior angles is $180n^\circ - 360^\circ = 180n^\circ - 2(180)^\circ = 180(n-2)^\circ$

□



$i(v) + x(v) = 180^\circ$

Proposition 2.2. *For every polygon P, there exists a triangulation of P*

Proof. The statement is obvious for convex P. Assume P is nonconvex, we proceed by induction on the number of vertices n. The statement is obvious for $n = 3$, this will be our base case. Now suppose $n > 3$, if we may draw a diagonal between two vertices of P that are not connected by an edge, then we would have split the polygon into two polygons with less vertices, so by induction we would be done. Therefore it remains to prove that there exists such a diagonal.

We will refer to a vertex v of P as convex if $i(v) < 180°$. There must exist at least one convex vertex of P, else the sum of interior angles of P would be $\geq 180n°$ which contradicts Theorem 2.1. Let A be some convex vertex of P. Let B and C denote the neighbouring vertices of A respectively. Now consider the line segment joining B and C, if this line segment lies completely within P then we are done. Else, there exists at least one vertex of P in the triangle ABC. Let Z denote the vertex within ABC that is closest to A. Now joining A to Z, we have found the desired diagonal. □

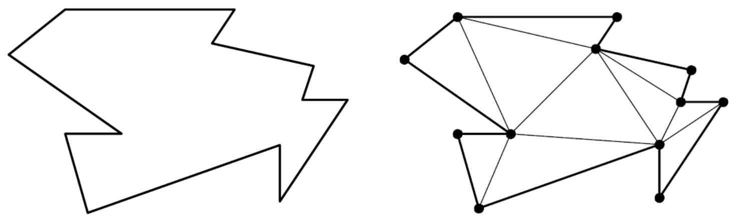
We now move on to the central theorem of this report.

3. AN UPPER BOUND ON THE NUMBER OF GUARDS

Theorem 3.1. *A museum with n walls requires at most $\lfloor \frac{n}{3} \rfloor$ guards to be guarded*

Proof. Let P be the polygon with n vertices that is the shape of the museum floor. From Proposition 2.2, we may triangulate P. Let G be the graph obtained by triangulating P. We claim that G is 3-colourable. We will proceed by induction on the number of vertices n. For $n = 3$, this is obvious and we will use this as our base case. Now for $n > 3$, pick any two vertices u and v of G that are joined by an edge such that they are not joined by a side of P. Then the line segment joining u and v splits G into two graphs with less vertices than G, say G_1 and G_2. Now by induction, there exists a 3-colouring of G_1 and G_2 respectively. Up to a permutation of the colours, we may assume u and v regarded as vertices of G_1 are coloured the same as u and v regarded as vertices of G_2, respectively. Therefore we may append the 3-colouring of G_1 to the 3-colouring of G_2 by sticking them together along the edge connecting u and v to produce a 3-colouring of G. Therefore G is indeed 3-colourable.

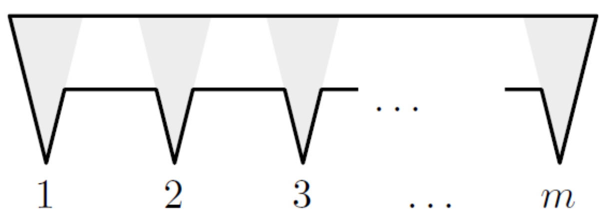
Now picking the colour that appears the least number of times in the colouring, we place our guards on the vertices coloured by this colour. Therefore we have at most $\lfloor \frac{n}{3} \rfloor$ guards. Note that each triangle in the 3-colouring of G must contain each colour as one of it's vertices, else two vertices of the same colour would be connected by an edge which contradicts the colouring. Therefore every triangle in the triangulation of P contains a guard on one of it's vertices. So every triangle is visible to a guard and we have that the whole of P is guarded. □



An example of a museum and it's triangulation

The next example shows us that the above bound is sharp.

Example 3.2. Consider a museum with the layout illustrated below. This museum has $3m$ walls. Note that for a guard to be able to see the corner in one of the inlets, the guard must be standing in the corresponding shaded region as in the illustration. It is easy to see that we therefore need at least $m = \left\lfloor \frac{3m}{3} \right\rfloor$ guards to guard the museum.



A museum that requires $\left\lfloor \frac{n}{3} \right\rfloor$ guards

References

[1] Martin Aigner and Gunter M. Ziegler. *Proofs from The Book*. Springer-Verlag, Berlin, fifth edition, 2014. Including illustrations by Karl H. Hofmann.