FIVE-COLOURING PLANE GRAPHS

DAVID GLYNN

1. Introduction

The colouring of plane graphs has been an area of great interest to mathematicians since the beginnings of graph theory. This is mainly due to its connection to one of the most famous and elusive problems of the subject: the four colour problem. The problem was first posed by Francis Guthrie in 1852 and asked whether it was always possible to colour the regions of a plane map with four colours such that the regions which share a common boundary receive different colours.

This question remained unanswered for over a century and withstood numerous attempts at a solution. In 1976, Kenneth Appel and Wolfgang Haken finally succeeding in proving the problem using a method of attack unavailable to the mathematicians of the 19th century: it was the first proof of a major theorem using a computer. While the proof was initially met with scepticism, it is now generally accepted to be correct. A simpler proof was given in 1997 by Robertson, Sanders, Seymour and Thomas to further dispel any doubts.

This report will tackle the much more manageable problem of whether the regions of a map can be coloured with only 5 colours. This question had been answered by the turn of the 20th century by English mathematician Percy John Heawood. We shall follow the proof from [1, Chapter 38], with the inclusion of some additional notes presented in Daniel Matthew's talk.

2. Plane Maps to Plane Graphs

It may not be immediately apparent how our discussion above about the colouring of maps relates to the colouring of plane graphs. It is easy to demonstrate that these two tasks are the same.

A colouring of a plane graph is an assignment of colours to each vertex such that no two vertices of the same colour are joined by an edge. For a given map M, we can construct its *dual graph* as follows: place a vertex in the interior of each region (including the outer region) and connect two such vertices belonging to neighbouring regions by an edge through the common boundary. The resulting graph G is a plane graph, and colouring vertices of G corresponds to colouring regions of M.

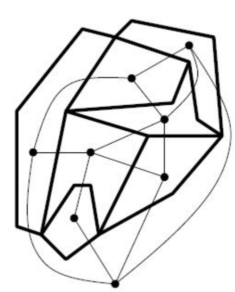


Figure 1. The dual graph of a map

3. Every plane graph is 6-colourable

We shall start with a warm-up problem: showing that every plane graph is 6-colourable. Any plane graph divides the plane into a finite number of connected regions (including the outer region), which are referred to as faces. The relation between the number of vertices, edges and faces of a plane graph is captured by Euler's Formula, a proof of which can be found in [1, Chapter 13].

Euler's Formula. If G is a connected plane graph with n vertices, e edges and f faces then

$$n - e + f = 2$$

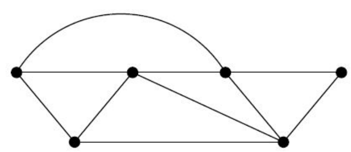


Figure 2. This plane graph has 6 vertices, 10 edges and 6 faces

We will need the following result in the proof that every plane graph is 6-colourable.

Proposition. *Let G be any simple plane graph with $n > 2$ vertices then G has a vertex of degree at most 5.*

Proof. We can count the number of faces of G as follows: let f_k denote the number of faces that are bounded by k edges, then

$$f = f_1 + f_2 + f_3 + f_4 + \dots \tag{1}$$

Since every edge is a side of two faces, we see that

$$2e = f_1 + 2f_2 + 3f_3 + 4f_4 + \dots \tag{2}$$

G is simple, so every face has at least 3 sides. Using (1) and (3) we get

$$f = f_3 + f_4 + f_5 + \dots$$

and

$$2e = 3f_3 + 4f_4 + 5f_5 + \dots$$

Thus

$$2e - 3f = (3f_3 + 4f_4 + 5f_5 + \dots) - 3(f_3 + f_4 + f_5 \dots) \geq 0$$

Using Euler's Formula we get

$$3n - 6 = 3(e - f) = e + (2e - 3f) \geq e$$

We can use this bound on the number of edges to get a bound on the average degree $\overline{d}$ of G

$$\overline{d} = \frac{2e}{n} \leq \frac{6n - 12}{n} < 6$$

If the lowest degree of any vertex in G is 6, then the average degree cannot be less than 6. Therefore there must be a vertex of degree at most 5. $\square$

Define the *chromatic number* of G, denoted $\chi(G)$, to be the smallest number of colours with which we can find a colouring of G. The problem can be stated succinctly as follows:

Theorem. *$\chi(G) \leq 6$ for any plane graph G.*

Proof. We carry out the proof by induction on the number n of vertices of G. For $n \leq 6$ the truth of the statement is obvious since we can colour every vertex of G with a different colour.
From the proposition above, we know that G has a vertex v of degree at most 5. Remove v and all edges connected to it from G. The resulting graph $G' = G \backslash \{v\}$ is a plane graph with $n-1$ vertices. By our induction hypothesis, G is 6-colourable. Since v has at most 5 neighbours, at most 5 colours are used for these neighbours in the colouring of G'. Therefore we can extend our 6-colouring of G' to a 6-colouring of G by assigning a colour to v that is different to any of the colours of it neighbours in G'. Thus G is 6-colourable. □

4. Every plane graph is 5-colourable

We now move on to the main result of the report. In fact, we will go even further: we shall prove that every plane graph is 5-*list* colourable.

Definition. Suppose in the graph $G = (V, E)$ we are given a set $C(v)$ of colours for each $v \in V$. A *list colouring* is a colouring of G such that every vertex v is assigned a colour from its corresponding colour set $C(v)$.
We define the *list chromatic number* $\chi_l(G)$ to be the smallest number k such that for any list of colour sets with $|C(v)| = k$ for all $v \in V$ a list colouring exists.

Ordinary colouring of graphs is just a special case of list colouring, namely when all the colour sets are the same. Therefore,

$$\chi(G) \leq \chi_l(G)$$

For an example of a graph G with $\chi(G) < \chi(G)$, consider the complete bipartite graph $K_{2,4}$. The chromatic number of any bipartite graph is 2, but suppose we have the colour sets as shown below.

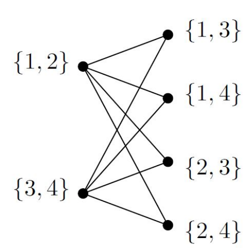


FIGURE 3. Bipartite graph $K_{2,4}$ with colour sets of size 2

All of the four possibilities for colouring the left vertices appears as a colour set on the right-had side. Therefore no list colouring is possible. The reader can check that $\chi_l(K_{2,4}) = 3$.

If we prove the below theorem we will have required result:

Theorem. *All planar graphs G can be 5-list coloured:*

$$\chi_l(G) \leq 5$$

Before moving on to the proof, we note that adding edges to a graph can only increase the chromatic number. Therefore we can assume that G is *near-triangulated*, that is, all bounded faces have triangles as boundaries. Proving the theorem for near triangulated graphs will establish the result for all plane graphs.

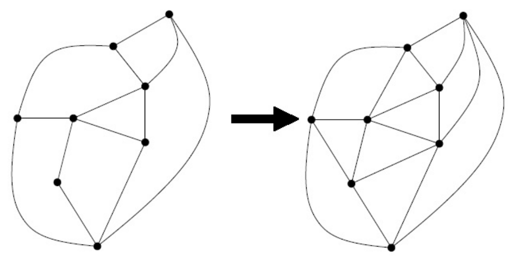


FIGURE 4. Near triangulating a plane graph

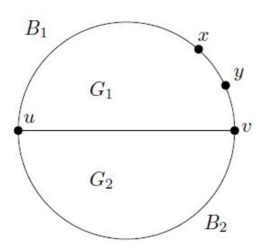
If we show the following stronger result, the theorem is proved:

Proposition. *Let $G = (V, E)$ be a near-triangulated graph, and let B be the cycle bounding the outer region. We make the following assumptions on the colour sets $C(v)$, $v \in V$:*
(1) Two adjacent vertices x, y of B are already coloured with different colours α and β.
(2) $|C(v)| \geq 3$ for all other vertices v of B.
(3) $|C(v)| \geq 5$ for all vertices v in the interior.

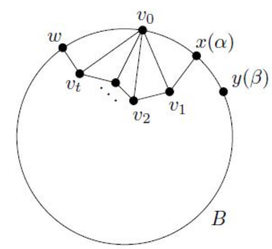
Then the colouring of x, y can be extended to a list colouring of G. In particular, $\chi_l(G) \leq 5$.

Proof. We proceed by induction on the number n of vertices of G. For $|V| = 3$ the result is obvious since we can colour every vertex with a different colour. We must consider two cases to complete the proof.

Case 1: Suppose that there is an edge which is not in B that joins two vertices $u, v \in B$, as shown in the figure across. The subgraph G_1 satifies all of the conditions in the proposition and has fewer vertices than G. Therefore it has a 5-list colouring by induction. If we fix this colouring of G_1, then we can take u and v as being pre-coloured vertices in the lower subgraph G_2. Viewed like this, G_2 also satisfies the conditions in



the proposition. Therefore G_2 can be 5-list coloured, and the same is true for G.



Case 2: Suppose that no such edge as in Case 1 exists. Let v_0 be the other vertex which is adjacent x on B, and let $x, v_1, \ldots, v_t, w$ be all vertices adjacent to v_0, as shown in the figure across. Construct the near-triangulated graph $G' = G \backslash \{v_0\}$ by removing the vertex v_0 and all edges connected to it from G. The outer boundary of G is $B' = (B \backslash \{v_0\}) \cup \{v_1, \ldots v_t\}$. By assumption (2), $|C(v_0)| \geq 3$, so there are two colours γ, δ in $C(v_0)$ different from α. Now replace every colour set $C(v_i)$ by $C(v_i) \backslash \{\gamma, \delta\}$, leaving the other colour sets in G' unchanged. Then we have that $|C(v)| \geq 3$ for $v \in B'$ and $|C(v)| \geq 5$ for all others vertices in G'. Therefore G' satisfies the conditions of the proposition and has fewer vertices than G, so it has a 5-list colouring by induction. By adding v_0 back in and colouring it either γ or δ, we can extend the list colouring of G' to a list colouring of G.

$\square$

5. A Further Conjecture

A stronger conjecture than the 5-list colour theorem claimed that the list-chromatic number of a plane graph G is at most 1 more than the ordinary chromatic number. In light of the 4-colour theorem, we have three cases to consider:

$$\text{Case 1: } \chi(G) = 2 \Rightarrow \chi_l(G) \leq 3$$
$$\text{Case 2: } \chi(G) = 3 \Rightarrow \chi_l(G) \leq 4$$
$$\text{Case 3: } \chi(G) = 4 \Rightarrow \chi_l(G) \leq 5$$

The result that we have just proved deals with Case 3, and Case 1 was shown to be true by Alon and Tarsi.

This leaves us with Case 2, which actually turns out to be *false*. An example of a graph which is 3-colourable but not 4-list colourable was first demonstrated by Margit Voigt. The graph on 130 vertices was originally constructed by Shai Gutner as follows:

Consider the "double octahedron" as shown across. It is easily seen to be 3-colourable. For the lists given in the figure, let $\alpha \in \{5, 6, 7, 8\}$ and $\beta \in \{9, 10, 11, 12\}$. The reader can check that a list colouring with these lists is not possible. We can construct a new graph by taking 16 copies of this graph and identifying all top vertices and all bottom vertices (it is perhaps easier visualise the graph as being on the surface of a sphere). This new graph has $16 \cdot 8 + 2 = 130$ vertices, and is still a 3-colourable plane graph. Now, assign

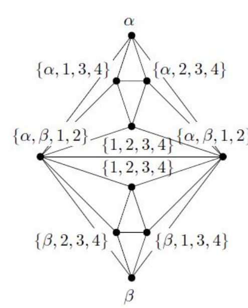


Figure 5. "Double octahedon"

the colour list $\{5,6,7,8\}$ to the top vertex and $\{9,10,11,12\}$ to the bottom vertex. For the remaining colour lists, let all 16 permutations of (α,β), $\alpha \in \{5,6,7,8\}$, $\beta \in \{9,10,11,12\}$ appear across all of the 16 double octahedron subgraphs. With these lists, given any choice of colours for the top vertex and bottom vertex, there will be a double octahedron subgraph for which a list-colouring is not possible. Therefore a list colouring of the entire graph is not possible. Thus we have constructed a graph that is 3-colourable but not 4-list colourable.

Since this first discovery, plane graphs with fewer vertices have been found that have a chromatic number of 3 and list chromatic number of 5. At present, the current record for such a graph contains 63 vertices.

References

[1] Martin Aigner and Gunter M. Ziegler. Proofs from The Book. Springer-Verlag, Berlin, fifth edition, 2014. Including illustrations by Karl H. Hofmann.