341D: Gröbner bases

Implicitisation

A curve in (x, y)-plane is given parametrically by

$$x(t) = 2t - 4t^3,$$
$$y(t) = t^2 - 3t^4.$$

Let us use Gröbner bases to find a polynomial $P(x, y)$ such that $P(x(t), y(t)) = 0$.

Implicitisation

To eliminate t from these parametric equations, we consider the `lex` ordering with $t > x > y$, and compute a Gröbner basis of the ideal $I = (x - 2t + 4t^3, y - t^2 + 3t^4)$; it is

$$2t^2 - 3tx + 4y,$$
$$9tx^2 - 8ty - 2t - 12xy + x,$$
$$24txy - 2tx + 3x^2 - 32y^2 - 8y,$$
$$192ty^2 + 32ty - 4t + 27x^3 - 120xy + 2x,$$
$$27x^4 - 144x^2y - 4x^2 + 256y^3 + 128y^2 + 16y$$

so we may take $P(x, y) = 27x^4 - 144x^2y - 4x^2 + 256y^3 + 128y^2 + 16y$.

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so we may take $P(x, y) = 27x^4 - 144x^2y - 4x^2 + 256y^3 + 128y^2 + 16y$. Note that the Gröbner basis contains the element $2t^2 - 3tx + 4y$ where the coefficient of the highest power of t is nonzero, so the Extension Theorem guarantees that over an algebraically closed field any common root (x, y) of the elimination ideal $I_1 = (P(x, y))$ extends to a root (t, x, y) of I, so we can guarantee that every complex solution corresponds to some value of t.

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Suppose that we look for that curve over the real numbers. Is it true that every point corresponds to some value of the parameter t?
The second element in the Gröbner basis is $9tx^2 - 8ty - 2t - 12xy + x$, so if some triple (t_0, x_0, y_0) solves our equations, then

$$t_0(9x_0^2 - 8y_0 - 2) - 12x_0y_0 + x_0 = 0.$$

If $9x_0^2 - 8y_0 - 2 \neq 0$, we have

$$t = \frac{12x_0y_0 - x_0}{9x_0^2 - 8y_0 - 2},$$

so if x_0 and y_0 are real, t_0 is also real.

Implicitisation

If $9x_0^2 - 8y_0 - 2 = 0$, the equation

$$t_0(9x_0^2 - 8y_0 - 2) - 12x_0y_0 + x_0 = 0$$

also forces $12x_0y_0 - x_0 = 0$, so $x_0 = 0$ or $y_0 = \frac{1}{12}$. Substituting each of these into $9x_0^2 - 8y_0 - 2 = 0$, we get three pairs which we have to examine more carefully: $(0, -\frac{1}{4})$ and $(\pm\frac{2\sqrt{2}}{3\sqrt{3}}, \frac{1}{12})$. For those, we look at the first element of the Gröbner basis, $2t^2 - 3tx + 4y$. Substituting these pairs, we obtain the polynomials

$$2t^2 - 1,$$

$$2t^2 + \frac{2\sqrt{2}}{\sqrt{3}}t + \frac{1}{3} = (t\sqrt{2} + \frac{1}{\sqrt{3}})^2,$$

$$2t^2 - \frac{2\sqrt{2}}{\sqrt{3}}t + \frac{1}{3} = (t\sqrt{2} - \frac{1}{\sqrt{3}})^2.$$

All these polynomials have only real roots, so the corresponding values of t are always real.

Conditional extrema

Suppose that we would like to minimize $(a-c)^2+(b-d)^2$ given that $ab=4$ and $c^2+4d^2=4$. In more geometric terms, we would like to find the minimal distance between the hyperbola $xy=4$ and the ellipse $x^2+4y^2=4$ in $\mathbb{R}^2$.

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$$F(a,b,c,d,\alpha,\beta)=(a-c)^2+(b-d)^2-\alpha(ab-4)-\beta(c^2+4d^2-4).$$

We have

$$\begin{aligned}\partial_a F &= 2(a-c)-\alpha b,\\ \partial_b F &= 2(b-d)-\alpha a,\\ \partial_c F &= 2(c-a)-2\beta c,\\ \partial_d F &= 2(d-b)-8\beta d,\\ \partial_\alpha F &= ab-4,\\ \partial_\beta F &= c^2+4d^2-4.\end{aligned}$$

Vanishing of these is precisely the system of equations we should solve.

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and, on the next slide,

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$$107360640\alpha + 1679218371d^{14} - 6538493421d^{12} + 67910186537d^{10} - 112888398095d^8 + 270453063428d^6 - 157077454960d^4 + 32607369964d^2 - 2435055472.$$

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We note that the first equation

$$81d^{16} - 324d^{14} + 3310d^{12} - 5796d^{10} + \\ 13653d^{8} - 9004d^{6} + 2484d^{4} - 324d^{2} + 16 = 0,$$

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First of all, this equation factors as

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The first factor has no real roots, the second has roots (approximately)

$$-.5814060238, -.3478340245, .3478340245, .5814060238.$$

This leads to the solutions (a, b, c, d) being

$$(-2.39097848, -1.67295522, -1.62722710, -.5814060238),$$
$$(1.722790175, 2.321814961, -1.87511226, -.3478340245),$$
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and to the values of $(a-c)^2 + (b-d)^2$ being 1.774795818, 20.07192764, 20.07192764, 1.774795818.

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At the oral entrance exam to the maths department of the Moscow State University in 1978, some Jewish students were offered the following problem:

> *Given that* $ab = 4$ *and* $c^2 + 4d^2 = 4$, *show that* $(a-c)^2 + (b-d)^2 \geq 1.6$.

A particular system of equations

Classification of "nonsymmetric operads" leads to looking for common zeros of the following 19 polynomials expressed on 2 pages:

$$x_5x_1 + x_4x_1 + 2x_3x_1 - x_2x_1 - 2x_1^2 - x_6 - x_4 + x_2 + x_1,$$

$$2x_6x_1 + x_4x_1 - 2x_3x_1 - x_2x_1 + 2x_1^2 + 2x_6 + x_4 - x_2,$$

$$x_2^2 + x_4x_1 + x_2x_1 + x_1^2 + x_2 + x_1,$$

$$2x_3x_2 + x_4x_1 + 2x_3x_1 - x_2x_1 - x_4 + x_2 + 2x_1,$$

$$x_4x_2 + x_4x_1 + x_2,$$

$$x_5x_2 - x_3x_1 + x_1^2 + x_4,$$

$$2x_6x_2 + x_4x_1 + 2x_3x_1 + x_2x_1 - x_4 + x_2 + 2x_1,$$

$$2x_4x_3 - x_4x_1 + 2x_3x_1 - x_2x_1 - 2x_1^2 - x_4 + x_2,$$

$$2x_5x_3 + x_4x_1 + 2x_3x_1 - x_2x_1 - 2x_1^2 - 2x_6 - x_4 + x_2 + 2x_1,$$

$$x_6x_3 + x_4x_1 + x_1^2 + x_6 + x_1,$$

$$x_4^2 - x_4x_1 - 2x_3x_1 + x_2x_1 + x_1^2 + x_4 - x_1,$$

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$$x_5x_4 + 2x_4x_1 + x_3x_1 - x_2x_1 + x_2 + x_1,$$
$$2x_6x_4 + x_4x_1 + 2x_3x_1 - x_2x_1 - 2x_1^2 - x_4 + x_2,$$
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$$2x_3x_1^2 - 2x_1^3 - x_4x_1 - 2x_3x_1 + 3x_2x_1 + 4x_1^2 - x_4 + x_2 + 2x_1,$$
$$x_4x_1^2 + x_1^3 - x_2x_1 + x_4 - x_2 - x_1,$$
$$2x_3^2x_1 - 2x_1^3 - x_4x_1 + 5x_2x_1 + 4x_1^2 - 3x_4 + 3x_2 + 4x_1.$$

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$$2x_6x_5 - x_4x_1 - 2x_3x_1 + x_2x_1 + 2x_1^2 + x_4 - x_2,$$
$$x_6^2 + x_4x_1 + x_3x_1 + x_6 + x_1,$$
$$x_2x_1^2 + x_1^3 - x_4x_1 - 2x_3x_1 + 2x_1^2 + x_4 - x_2 - x_1,$$
$$2x_3x_1^2 - 2x_1^3 - x_4x_1 - 2x_3x_1 + 3x_2x_1 + 4x_1^2 - x_4 + x_2 + 2x_1,$$
$$x_4x_1^2 + x_1^3 - x_2x_1 + x_4 - x_2 - x_1,$$
$$2x_3^2x_1 - 2x_1^3 - x_4x_1 + 5x_2x_1 + 4x_1^2 - 3x_4 + 3x_2 + 4x_1.$$

This is a Gröbner basis for the `glex` order with $x_6 > x_5 > x_4 > x_3 > x_2 > x_1$; the `lex` Gröbner basis is quite disastrous, and consists of polynomials with much larger coefficients.

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First, we note that powers of x_1 do not appear as leading terms, so we should expect infinitely many solutions.

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Note that the sum of the elements

$$2x_6x_4 + x_4x_1 + 2x_3x_1 - x_2x_1 - 2x_1^2 - x_4 + x_2,$$
$$2x_6x_5 - x_4x_1 - 2x_3x_1 + x_2x_1 + 2x_1^2 + x_4 - x_2$$

(elements 13 and 14 of the Gröbner basis above) is $2x_6(x_5 + x_4)$, and so we can split the proof into two parts: first, set $x_6 = 0$; second, set $x_5 = -x_4$. In both cases we reduce the number of variables by one.

Case 1: We set $x_6 = 0$ in the 19 polynomials, and compute the Gröbner basis of the ideal generated by the resulting polynomials in $x_1, \ldots, x_5$. This basis has only 8 elements:

$$\left.\begin{array}{l} x_4 - x_2, \qquad x_1(x_1 + x_2 + 1), \qquad x_1(x_3 - x_1), \\ x_1(x_5 + 1), \qquad (x_2 + x_1 + 1)(x_2 - x_1), \\ x_1^2 + x_2x_3 + x_1, \qquad x_2(x_5 + 1), \qquad x_3x_5 + x_1. \end{array}\right\} \tag{1}$$

The second, third, and fourth elements have x_1 as a factor, and so we can split again into two cases: either $x_1 = 0$, or $x_2 = -x_1 - 1$ and $x_3 = x_1$ and $x_5 = -1$.

Case 1.1: We set $x_1 = 0$ in the polynomials (1) and recompute the Gröbner basis in $x_2, \ldots, x_5$ which consists of these five polynomials:

$$x_4 - x_2, \qquad x_2(1 + x_2), \qquad x_3 x_2, \qquad x_2(x_5 + 1), \qquad x_5 x_3.$$

From this we see that either $x_2 = 0$ or $x_2 = -1$; in the former case, $x_4 = 0$ and either x_3 is free or x_5 is free but not both and the other is zero; in the latter case, $x_3 = 0$, $x_4 = -1$, $x_5 = -1$. This produces three solutions:

$$[x_1, \ldots, x_6] = \begin{cases} [0, 0, X, 0, 0, 0] & (X \in \mathbb{F}) \\ [0, 0, 0, 0, X, 0] & (X \in \mathbb{F}) \\ [0, -1, 0, -1, -1, 0] \end{cases} \tag{2}$$

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Case 1.2: We set $x_2 = -x_1 - 1$, $x_3 = x_1$, $x_5 = -1$ in (1) and recompute the Gröbner basis in x_1, x_4; the ideal is principal with generator $x_4 + x_1 + 1$. We obtain this solution

$$[x_1, \dots, x_6] = [X, -X - 1, X, -X - 1, -1, 0] \; (X \in \mathbb{F}). \tag{3}$$

Note that for $X = 0$ we obtain the previous solution $[0, -1, 0, -1, -1, 0]$.

Case 2: We set $x_5 = -x_4$ in the 19 polynomials, and compute the Gröbner basis of the ideal generated by the resulting polynomials in $x_1, \ldots, x_4, x_6$. This Gröbner basis consists of the following 14 elements:

$$2x_3x_1 - x_2x_1 - 2x_1^2 - x_6 - x_4 + x_2 + x_1,$$
$$2x_4x_1 - x_2x_1 - x_6 + x_4 + 3x_2 + x_1,\ 4x_6x_1 - 3x_2x_1 + 3x_6 - x_4 - 3x_2 + x_1,$$
$$2x_2^2 + 3x_2x_1 + 2x_1^2 + x_6 - x_4 - x_2 + x_1,$$
$$4x_3x_2 + x_2x_1 + 4x_1^2 + 3x_6 - x_4 - 3x_2 + x_1,$$
$$2x_4x_2 + x_2x_1 + x_6 - x_4 - x_2 - x_1,$$
$$4x_6x_2 + 5x_2x_1 + 4x_1^2 + 3x_6 - x_4 - 3x_2 + x_1,$$
$$4x_4x_3 - x_2x_1 + x_6 + x_4 + 3x_2 - x_1,$$
$$2x_6x_3 + x_2x_1 + 2x_1^2 + 3x_6 - x_4 - 3x_2 + x_1,$$
$$2x_4^2 - x_2x_1 - 2x_1^2 - 3x_6 + x_4 + 5x_2 + x_1,$$
$$4x_6x_4 + x_2x_1 + 3x_6 - x_4 - 3x_2 - 3x_1, \quad x_6^2 + x_2x_1 + x_1^2 + 2x_6 - 2x_2,$$
$$4x_1^3 - 13x_2x_1 - 4x_1^2 + x_6 + 9x_4 - 5x_2 - 9x_1,$$
$$4x_2x_1^2 + 7x_2x_1 + 4x_1^2 - 7x_6 - 7x_4 + 11x_2 + 11x_1.$$

Analyzing this Gröbner basis we see that there are equations where the variables x_4 and x_6 appear among linear terms with some scalar coefficients, and do not appear in other terms. This suggests that we should eliminate these variables from our equations, so that only variables x_1, x_2, and x_3 remain. This leads to a system of the nine polynomials

$$x_6 - x_1x_3 + x_2^2 + 2x_1x_2 + 2x_1^2 - x_2,$$

$$x_4 - x_2^2 - x_3x_1 - x_2x_1 - x_1,$$

$$(2x_2 + x_1)(x_3 - x_2 - x_1),$$

$$2x_1^3 + 4x_2^2 + 5x_3x_1 - 3x_2x_1 - 3x_1^2 - 2x_2,$$

$$2x_2x_1^2 - 7x_3x_1 + 7x_2x_1 + 9x_1^2 + 2x_2 + 2x_1,$$

$$x_3x_1^2 + 2x_2^2 + x_3x_1 + x_1^2 + x_1,$$

$$2x_2^2x_1 - 2x_2^2 + 5x_3x_1 - 5x_2x_1 - 7x_1^2 - 2x_1,$$

$$x_3^2x_1 + x_2^2 + x_3x_1 + x_1^2 + x_2 + x_1,$$

$$2x_2^3 + 2x_2^2 - x_3x_1 + x_2x_1 + x_1^2 - 2x_2.$$

Examining the polynomial $(2x_2 + x_1)(x_3 - x_2 - x_1)$, we see that either $x_1 = -2x_2$ or $x_1 = x_3 - x_2$, so we again split into two cases.

Case 2.1: We substitute $x_1 = -2x_2$ in the last six polynomials of the set above and compute the reduced Gröbner basis, obtaining $\{\, x_2(x_2 - 1),\ x_2(x_3 + 2) \,\}$, for which the zero set is

$$[x_2, x_3] \in \{\, [0, X] \mid X \in \mathbb{F} \,\} \cup \{\, [1, -2] \,\}.$$

Solving backwards for the values of the other variables, we obtain

$$[x_1, \dots, x_6] = [0, 0, X, 0, 0, 0],\ [-2, 1, -2, 1, -1, 0].$$

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$$[x_1, \ldots, x_6] = [0, 0, X, 0, 0, 0], \ [-2, 1, -2, 1, -1, 0].$$

The first has already appeared in (2), and the second is the special case $X = -2$ of (3), so there are no new solutions.

Case 2.2: We substitute $x_1 = x_3 - x_2$ in the last six polynomials of the set above and compute the reduced Gröbner basis, obtaining these four polynomials:

$$x_2(x_2^2 + x_2 - 1),$$
$$x_3x_2^2 - x_3^2 + 2x_3x_2 - x_2^2 - x_3,$$
$$x_3^2x_2 - x_3^2 + 2x_3x_2 - 2x_2^2 - x_3 + x_2,$$
$$x_3^3 + x_3^2 - x_3x_2 + x_2.$$

The first of these implies that either $x_2 = 0$ or $x_2^2 + x_2 - 1 = 0$.

Case 2.2.1: If $x_2 = 0$, we find that the resulting polynomials generate the principal ideal of multiples of $x_3(x_3 + 1)$. Working backward from $x_3 = 0$ we obtain only the solution $[0, 0, 0, 0, 0, 0]$. Working backward from $x_3 = -1$ we obtain a new solution:

$$[x_1, \dots, x_6] = [-1, 0, -1, 0, 0, -1]. \tag{4}$$

Case 2.2.2: If $x_2^2 + x_2 - 1 = 0$, we see that $x_2 = -\phi$, where $\phi = \frac{-1\pm\sqrt{5}}{2}$ is a root of the polynomial $t^2 - t - 1$. Substituting this value of x_2 in elements above, we obtain the following polynomials:

$$
\begin{aligned}
&-x_3^2 - \phi x_3 + (-\phi - 1), \\
&(-\phi - 1)x_3^2 + (-2\phi - 1)x_3 + (-3\phi - 2), \\
&x_3^3 + x_3^2 + \phi x_3 - \phi.
\end{aligned}
$$

By a direct computation,

$$
\begin{aligned}
(-\phi - 1)x_3^2 + (-2\phi - 1)x_3 + (-3\phi - 2) &= (\phi + 1)(-x_3^2 - \phi x_3 + (-\phi - 1)), \\
x_3^3 + x_3^2 + \phi x_3 - \phi &= (-x_3^2 - \phi x_3 - (\phi + 1))(-x_3 - 1 + \phi),
\end{aligned}
$$

so the corresponding ideal is generated by the polynomial

$$F = x_3^2 + \phi x_3 + (\phi + 1).$$

In fact, we can rewrite $F = x_3^2 + \phi x_3 + (\phi + 1)$ as

$$F = x_3^2 + \phi x_3 + \phi^2,$$

which instantly shows that $\frac{x_3}{\phi} = \omega$ is a root of the polynomial $t^2 + t + 1$, a primitive cube root of unity. Furthermore, we recall that throughout Case 2.2 we have $x_1 = x_3 - x_2$, so $x_1 = \omega\phi + \phi = \phi(\omega + 1) = -\omega^2\phi$, Next, we substitute the values that we found in the first two polynomials of the set above, obtaining

$$x_6 = -\phi^2\omega^3 - \phi^2 - 2\phi^2\omega^2 - 2\phi^2\omega^4 - \phi = -\phi, \quad (5)$$

$$x_4 = \phi^2 - \phi^2\omega^3 + \phi^2\omega^2 - \phi\omega^2 = (\phi^2 - \phi)\omega^2 = \omega^2. \quad (6)$$

Finally, since throughout Case 2 we have $x_5 = -x_4$, we conclude that

$$x_5 = -\omega^2.$$

Overall, in this case we end up with the four points

$$[\omega^2\phi, -\phi, \omega\phi, \omega^2, -\omega^2, -\phi],$$

where ω is a root of the polynomial $t^2 + t + 1$ and ϕ is a root of the polynomial $t^2 - t - 1$.