MA341D Answers and solutions to homework assignment 3

1. It is clear that it is a total order. Let us demonstrate that $<_{rp}$ is a well order. Suppose that there is an infinite decreasing sequence

$$m_1 >_{rp} m_2 >_{rp} m_3 >_{rp} \dots$$

The number of occurrences of x_0 in these words must stabilise at some point, since it cannot keep decreasing, so without loss of generality we may assume that they all contain the same number of occurrences of x_0, say l, so that each of these words gets split into $l+1$ parts by the occurrences of x_0. Let us look at the first of those $l+1$ parts in each word. They form a non-increasing sequence which therefore must stabilise at some point, so without loss of generality we may assume that the first parts are all the same. Then we look at the second parts, they also form a non-increasing sequence, and therefore must stabilise, etc.

It remains to show that if $u <_{rp} v$ then $uw <_{rp} vw$ and $wu <_{rp} wv$. If the number of occurrences of x_0 in u is less than the number of occurrences of x_0 in v, then the same holds for uw, vw and wu, wv, so the claim follows. Thus, it is sufficient to look at the case of a tie, so that

$$u = u_0x_0u_1x_0 \cdots u_{l-1}x_0u_l \quad \text{and} \quad v = v_0x_0v_1x_0 \cdots v_{l-1}x_0v_l$$

with u_i, v_i being words that do not contain x_0. Suppose that $w = w_0x_0w_1x_0 \cdots w_{m-1}x_0w_m$, where w_i do not contain x_0. In this case

$$\begin{aligned} uw &= u_0x_0u_1x_0 \cdots u_{l-1}x_0u_lw_0x_0w_1x_0 \cdots w_{m-1}x_0w_m, \\ vw &= v_0x_0v_1x_0 \cdots v_{l-1}x_0v_lw_0x_0w_1x_0 \cdots w_{m-1}x_0w_m, \\ wu &= w_0x_0w_1x_0 \cdots w_{m-1}x_0w_mu_0x_0u_1x_0 \cdots u_{l-1}x_0u_l, \\ wv &= w_0x_0w_1x_0 \cdots w_{m-1}x_0w_mv_0x_0v_1x_0 \cdots v_{l-1}x_0v_l. \end{aligned}$$

To compare uw and vw, we have to look at the sequences

$$(u_0, u_1, \dots, u_{l-1}, u_lw_0, w_1, \dots, w_m) \text{ and } (v_0, v_1, \dots, v_{l-1}, v_lw_0, w_1, \dots, w_m).$$

It follows that if $k = \min\{i\colon u_i \neq v_i\} < l$, then the same comparison proves that $uw <_{rp} vw$, and if $k = l$, then $u_l < v_l$ implies that $u_lw_0 < v_lw_0$ since we assume $<$ to be a monomial ordering, therefore $uw <_{rp} vw$.

To compare wu and wv, we have to look at the sequences

$$(w_0, w_1, \dots, w_{m-1}, w_mu_0, u_1, \dots, u_l) \text{ and } (w_0, w_1, \dots, w_{m-1}, w_mu_0, u_1, \dots, u_l).$$

It follows that if $k = \min\{i\colon u_i \neq v_i\} > 0$, then the same comparison proves that $uw <_{rp} vw$, and if $k = 0$, then $u_0 < v_0$ implies that $w_mu_0 < w_mv_0$ since we assume $<$ to be a monomial ordering, therefore $uw <_{rp} vw$.

2. (a) At the first stage of Buchberger's algorithm, there are two nontrivial small common multiples of leading monomials with each other. The small common multiple y^3 of y^2 with itself leads to a zero S-polynomial, and the small common multiple $xyxyx$ of xyx with itself leads to the S-polynomial

$$(xyx - yxy)yx - xy(xyx - yxy) = xy^2xy - yxy^2x.$$

The remainder of this element after long division by $\{xyx - yxy, y^2 - 1\}$ is, by direct inspection, $x^2y - yx^2$. The leading monomial x^2y of this element leads to the following three new small common multiples: x^2yx, xyx^2y and x^2y^2. The corresponding S-polynomials are

$$(x^2y - yx^2)x - x(xyx - yxy) = xyxy - yx^3$$

(and the remainder of this element after long division by $\{xyx - yxy, y^2 - 1, x^2y - yx^2\}$ is $yx - yx^3$),

$$(xyx - yxy)xy - xy(x^2y - yx^2) = xy^2x^2 - yxyxy$$

(and the remainder of this element after long division by $\{xyx - yxy, y^2 - 1, x^2y - yx^2\}$ is $x^3 - x$), and

$$(x^2y - yx^2)y - x^2(y^2 - 1) = x^2 - yx^2y$$

(and the remainder of this element after long division by $\{xyx - yxy, y^2 - 1, x^2y - yx^2\}$ is 0). Since $yx - yx^3$ is $y(x - x^3)$, the self-reduced set obtained from this is $\{xyx - yxy, y^2 - 1, x^2y - yx^2, x^3 - x\}$. The element $x^3 - x$ forms the following new small common multiples: x^4, x^5 (redundant by Triangle Lemma), x^3yx, xyx^3, x^3y. The corresponding S-polynomials are

$$(x^3 - x)x - x(x^3 - x) = 0,$$

$$(x^3 - x)yx - x^2(xyx - yxy) = x^2yxy - xyx$$

(and the remainder of this element after long division by $\{xyx - yxy, y^2 - 1, x^2y - yx^2, x^3 - x\}$ is 0),

$$(xyx - yxy)x^2 - xy(x^3 - x) = xyx - yxyx^2$$

(and the remainder of this element after long division by $\{xyx - yxy, y^2 - 1, x^2y - yx^2, x^3 - x\}$ is 0), and

$$(x^3 - x)y - x(x^2y - yx^2) = xyx^2 - xy$$

(and the remainder of this element after long division by $\{xyx - yxy, y^2 - 1, x^2y - yx^2, x^3 - x\}$ is 0). Thus, the reduced Gröbner basis is $\{xyx - yxy, y^2 - 1, x^2y - yx^2, x^3 - x\}$.

(b) Cosets of normal monomials form a basis in $F\langle x, y\rangle/I$, so we just need to show that there are finitely many normal monomials. The normal monomials of degree at most 2 are 1, x, y, x^2, xy, yx. Appending anything on the right of x^2 or xy we get a leading term of one of the elements of the Gröbner basis. Appending x on the right of yx, we get a normal monomial yx^2, and appending y we get a normal monomial yxy. There is nothing we can append on the right of those, so there are no normal monomials of length 4, and the total number of normal monomials, and hence the dimension, is $1 + 2 + 3 + 2 = 8$.

3. (a) The leading monomials of these relations then are x^2, xy, and xz. There are three small common multiples, x^3, x^2y and x^2z. By a direct computation, the corresponding S-polynomials all have zero remainders, so the generators of our ideal do form a Gröbner basis. (This can also be established with Magma.)

(b) The normal monomials are those not containing x^2, xy, and xz. In other words, a normal monomial of length n has all letters except for the last one equal to y or z, and the last letter may be x, y, or z. Altogether, the dimension d_n in question is $3 \cdot 2^{n-1}$.

4. (a) For ideals generated by monomials, all S-polynomials of generators are always zero, so xyz is a Gröbner basis of (xyz). Every monomial of degree n that is normal with respect to xyz is one of the monomials x^n, y^n, z^n, $x^a y^b$, $x^a z^b$, $y^a z^b$ (with $a + b = n$, $a > 0$, $b > 0$, so $b = n - a$ and $a = 1, \ldots, n-1$). The total number of such is $1 + 1 + 1 + n - 1 + n - 1 + n - 1 = 3n$.

(b) This set of generators is not self-reduced; the reduced form of xyz is zyx. The leading monomials of the self-reduced set of generators are xy, xz, yz, zyx, forming small common multiples xyz, $xzyx$, $zyxy$, $zyxz$. The corresponding S-polynomials are

$$(xy - yx)z - x(yz - zy) = xzy - yzx$$

(and the remainder of this element after long division is zero),

$$(xz - zx)yx - x(zyx) = -zxyx$$

(and the remainder of this element after long division is zero),

$$(zyx)y - zy(xy - yx) = zy^2x$$

(and the remainder of this element after long division is zy^2x),

$$(zyx)z - zy(xz - zx) = zyzx$$

(and the remainder of this element after long division is zero). Adjoining the new element zy^2x, we form new small common multiples, establish that most S-polynomials for those have zero remainders, and the only new nonzero remainder is zy^3x, etc. By induction, it is easy to see that $\{xy - yx, yz - zy, xz - zx, zy^kx \colon k \geq 1\}$ is the reduced Gröbner basis.

(c) Suppose that for some monomial ordering J has a finite Gröbner basis G, and that N is the largest degree of elements in G. We note that J is generated by elements of two shapes: those of the form $m_1 - m_2$ where m_1 and m_2 are monomials, and a single monomial $m = xyz$. This, by induction, implies that all S-polynomials computed by the Buchberger's algorithm will be of the same kind, either differences of monomials, or single monomials. Moreover, another easy inductive argument shows that, with the exception of quadratic relations $xy - yx$, $yz - zy$, $xz - zx$, every element of the reduced Gröbner basis computed via the Buchberger's algorithm would involve all three variables x, y, z. Let us consider the six elements ab^Nc, where a, b, c is some permutation of x, y, z. By a direct inspection of eight different possibilities, one of those elements is not divisible by any of the leading terms of the quadratic elements of J. (Indeed, for the leading terms xy, xz, yz, or xy, zx, yz, we can take the element zy^Nx, for the leading terms yx, xz, yz, or yx, xz, zy, the element zx^Ny, for the leading terms xy, xz, zy, the element yz^Nx, for the leading terms xy, zx, zy, the element yx^Nz, for the leading terms yx, zx, yz the element xz^Ny, and for the leading terms yx, zx, zy the element xy^Nz.) Also, every divisor of degree at most N of these elements invoves at most two different generators, and hence it must be normal, since a non-quadratic leading term involves all three generators. However, these elements vanish in the quotient, so they cannot be normal, a contradiction.