MA341D Homework assignment 2

Due on February 13, 2018

In each of the questions below, using computer algebra to compute Gröbner bases is permitted, but using computer solvers for system of equations is not allowed; you are supposed to analyse the Gröbner bases you computed and derive the final answer yourselves. Whenever you use a computer algebra system (probably the Magma online calculator `http://magma.maths.usyd.edu.au/calc/` will be sufficient), please attach a printout or a screenshot of the commands you executed.

1. Use Gröbner bases to describe the solution set to the system of equations

$$\begin{cases} xy = z^2 + z, \\ yz = x^2 + x, \\ xz = y^2 - y. \end{cases}$$

over complex numbers .

2. Use Gröbner bases to describe the solution set to the system of equations

$$\begin{cases} xy = z^2 + z, \\ yz = x^2 + x, \\ xz = y^2 + y. \end{cases}$$

over complex numbers.

3. (a) Explain how to use Lagrange multipliers for the conditional extrema problem for the values of $x^3 + y^3 + z^3$ on the circle which is the intersection of the plane $x + y + z = 0$ with the sphere $x^2 + y^2 + z^2 = 1/2$.

   (b) Use computer software to compute the reduced Gröbner basis for the ideal generated by the corresponding equations.

   (c) Determine all solutions to the corresponding system of polynomial equations using the Gröbner basis you computed.

4. A *graph* is defined as a set of vertices, some of which are connected with edges. A colouring of vertices of a graph in k colours is said to be *regular* if every two vertices connected with an edge are coloured in different colours.

   (a) Consider a graph Γ with n vertices, and fix a positive integer k. Let us associate to it a collection C_Γ of commutative polynomials in n variables consisting of all polynomials $x_i^k - 1$, $i = 1, \ldots, n$, and all polynomials $x_i^{k-1} + x_i^{k-2}x_j + \cdots + x_i x_j^{k-2} + x_j^{k-1}$ for each pair (i, j) such that the vertices i and j are connected with an edge in Γ. Prove that the polynomials in C_Γ have a common zero (over complex numbers) if and only if Γ admits a regular colouring of vertices in k colours.

   (b) For the graph with 8 vertices plotted in the picture, establish if there is a regular colouring of vertices of Γ in three colours, and if yes, describe all such colourings.