MA341D Homework assignment 3

Due on March 9, 2018

1. Suppose that $<$ is a monomial ordering on all noncommutative monomials in $x_1, \ldots, x_n$. Let us define the *recursive path ordering* $<_{rp}$ on all noncommutative monomials in x_0, x_1, $\ldots$, x_n as follows:

   - If the number of occurrences of x_0 in u is less than the number of occurrences of x_0 in v, then $u <_{rp} v$.
   - If there is a tie, that is u and v both have m occurrences of x_0, we have factorisations
   $$u = u_0 x_0 u_1 x_0 \cdots u_{m-1} x_0 u_m \quad \text{and} \quad v = v_0 x_0 v_1 x_0 \cdots v_{m-1} x_0 v_m$$
   where u_i, v_i are noncommutative monomials in $x_1, \ldots, x_n$. Let $k = \min\{i \colon u_i \neq v_i\}$; we put $u <_{rp} v$ if $u_k < v_k$.

   Show that $<_{rp}$ is a monomial ordering.

2. Let us consider the ideal $I = (xyx - yxy, y^2 - 1) \subset F\langle x, y\rangle$.

   (a) Compute (without using any software) the reduced Gröbner basis of I for the `glex` ordering with $x > y$.

   (b) Use the Gröbner basis you computed to show that the quotient $F\langle x, y\rangle / I$ is finite dimensional, and to compute the dimension of the quotient.

3. Let us consider the ideal $I = (x^2 + yz + zy, y^2 + xz + zx, z^2 + xy + yx) \subset F\langle x, y, z\rangle$.

   (a) Compute (with or without computer software) the reduced Gröbner basis of I for the `glex` ordering with $x > y > z$.

   (b) Describe the normal monomials with respect to the Gröbner basis you computed. How many normal monomials of degree n are there?

4. Let us consider the ideal $I = (xyz) \subset F[x, y, z]$, and the ideal $J = (xy - yx, yz - zy, xz - zx, xyz) \subset F\langle x, y, z\rangle$. Clearly, $F[x, y, z]/I \cong F\langle x, y, z\rangle / J$.

   (a) Show that the element xyz forms a Gröbner basis of I. How many normal (commutative) monomials of degree n with respect to xyz are there?

   (b) Compute (with or without computer software) the (infinite) reduced Gröbner basis of J for the `glex` ordering with $x > y > z$.

   (c) Show that there does not exist a monomial ordering of noncommutative monomials for which the ideal J has a finite Gröbner basis.