Problem solving: linear algebra methods (29/01/2014)

The main goal today is to progress with Q3 and Q5, and/or other questions if possible.

1. A cube in 3d space is positioned in such a way that all its vertices have integer coordinates. Prove that the side of the cube is of integral length.
2. Show that for all $0 < a < b < c$ and all $x < y < z$ we have $\det \begin{pmatrix} a^x & a^y & a^z \\ b^x & b^y & b^z \\ c^x & c^y & c^z \end{pmatrix} \neq 0$.
3. For a set of $2n + 1$ coins, it is known that if we remove one arbitrary coin from it, the remaining ones can be divided into two set of n coins of the same total weight. Show that all coins have the same weight.
4. Suppose that it is possible to mark several distinct points of $[0, 1]$ in such a way that each marked point is either the midpoint of a segment connecting two other marked points (not necessarily its neighbours) or the midpoint of a segment connecting another marked point with one of the endpoints of $[0, 1]$. Show that all marked points have rational coordinates.
5. Show that for each way to put real numbers in all the $2m + 2n - 4$ positions on the border of a $m \times n$ grid, there exists exactly one way to put real numbers in the remaining positions on the grid so that each of those numbers is equal to the average of its neighbours.
6. For each matrix A with real entries, show that $\mathrm{rk}(A) = \mathrm{rk}(A^T A)$. ($A^T$ denotes the transpose matrix.)
7. Show that for every two polynomials $f(t)$ and $g(t)$ there exists a non-zero polynomial $R(x, y)$ in two variables such that $R(f(t), g(t)) = 0$.
8. For an $n \times n$ matrix A such that $\mathrm{tr}(A^k) = 0$ for all $k = 1, \ldots, n$, show that $A^n = 0$.
9. For two $n \times n$-matrices A and B we have $A^2 + B^2 = AB$. Show that if $\det(AB - BA) \neq 0$, then n is divisible by 3.

Some background info and catchphrases:

- Fredholm's alternative: each system of linear equations $Ax = b$ has exactly one solution if and only if the homogeneous system $Ax = 0$ has exactly one solution; otherwise some of systems $Ax = b$ have infinitely many solutions, and some have no solutions at all.
- Solving linear equations does not change the ground field, e.g. if all coefficients are rational, there exists a rational solution, etc.
- The determinant of a matrix vanishes if and only if its rows are linearly independent, which in turn happens if and only if its columns are linearly independent.
- Over complex numbers, "most matrices are similar to diagonal matrices", and each matrix is similar to a triangular matrix (the latter is enough for computing traces, determinants etc.)