

Mathematical Battle

November 2, 2012

The battle will take place in EELT1 at 4 p.m. There will be two teams competing in the battle. If you wish to join contact either Pádraig Condon or Benen Harrington. Rules of the battle are available on the webpage of the problem solving group (via the http URL <http://tinyurl.com/tcdproblemsolving>).

1. What is the last digit of $\left\lfloor \left(\frac{\sqrt{5}+1}{2}\right)^{2012} \right\rfloor$?
2. Is the series $\sum_{n=1}^{\infty} \frac{\sin(n)}{n}$ convergent or divergent?
3. What is the maximal number of vectors v_1, \dots, v_k in \mathbb{R}^n for which $(v_i, v_j) < 0$ for all $i \neq j$?
4. Let $\{a_n\}$ and $\{b_n\}$ be two unbounded from above sequences of real numbers. Show that there exist $k < l \in \mathbb{N}$ such that both $|a_k - a_l| > 1$ and $|b_k - b_l| > 1$.
5. A polynomial with integer coefficients has 1 and 2 as its roots. Prove that it has at least one coefficient less than -1 .
6. Does there exist a probability distribution on integer numbers such that for every prime p the probability that p divides n is equal to $\frac{1}{p}$?
7. Find all real numbers $a > 0$ such that the graph of every continuous function $f: [0, 1] \rightarrow \mathbb{R}$ with $f(0) = f(1) = 0$ has a horizontal chord of length a .
8. Find all integers x such that the number $(3x+4)^2 + x$ is a perfect square.
9. Evaluate

$$\int_0^{\frac{\pi}{2}} (\cos^2(\cos x) + \sin^2(\sin x)) dx .$$

10. A kingdom has 20 millions inhabitants. The king wants to be reelected but he is supported only by the army which constitutes 1% of the population. The elections are organised as follows. The king divides all inhabitants into *equal* groups. Each group is in turn divided into equal subgroups and so on. (Different groups may be divided in different ways.) In each subgroup of the lowest level a representative is elected by voting, i.e. a soldier or a citizen opposed to the king will represent the group depending on whether soldiers or opposing citizens have majority. (If there is equality, the opposition wins.) Then elections are held on the next level and so on. Can the king distribute inhabitants into groups in a way that allows him to win the elections?