Mathematical Battle 2

December 13th, 2012

The first battle between teams of the Second Year and Riemann Surfers took place in November, and it seemed to be fun. We all, the players, jury and crowd, are looking forward for the next one, which will take place at 4 p.m. in the Geography lecture theatre in the Museum building. (Check `http://tinyurl.com/tcdproblemsolving` for the rules, or just come and see.)

1. Let T be a trivalent tree with n vertices (that is, a tree that only has *internal* vertices with three neighbours each and *leaves* with 1 neighbour each). For each internal vertex v, define its *mass* as the product of the three numbers: the numbers of leaves that can be reached from this vertex through each of its three outgoing edges. Show that the sum of masses of all internal vertices depends only on n, but not on the choice of a tree T.

2. Evaluate
$$\frac{1}{2\pi i}\int_{|z|=1}\log|2z^4-5z^2+2|\,\frac{dz}{z}\,.$$

3. Prove that any k-dimensional affine subspace in $\mathbb{R}^n$ contains a vector with at least k zero coordinates.

4. Let $f_1, f_2 \in \mathbb{Z}[x]$ be two monic polynomials such that $\langle f_1, f_2\rangle = \mathbb{Z}[x]$. Prove that $\prod_{\alpha:f_1(\alpha)=0}\prod_{\beta:f_2(\beta)=0}|\alpha-\beta| = 1$.

5. Prove that for any integer $n > 1$ the number $\sum_{k=1}^{\infty} n^{-k^2}$ is irrational.

6. Let $h:\mathbb{R}\to\mathbb{R}$ be a continuous even function. Denote $a_n = \int_0^1 h(x+n)dx$. Suppose that $\lim_{n\to+\infty} a_n$ exists and equals a. Show that $\lim_{n\to+\infty}\int_{-1}^1 h(nx)(1+e^x)^{-1}dx$ exists and evaluate it.

7. In the first day of the IMC, 49 participants were proposed three problems, each worth 7 points. Show that there exist two participants such that for each of the three problems the first participant received the same number of points as the second participant or less for its solution.

8. Let $f(z) = \sum_{n=1}^{\infty} 5^n z^{n!}$. Show that for any real C there is $0 < r < 1$ such that the minimal value of $|f(z)|$ on the circle $|z| = r$ is greater than C.

9. A hunter is chasing a wolf. The hunter's running speed is always equal to 12 mph, and the wolf's running speed is equal to $1/S$, where S is the distance between the wolf and the hunter: the closer the hunter gets, the faster the wolf runs. Each of them can change the direction of running momentarily. In the beginning of the chase $S > 1/12$. Prove that the wolf can be choosing running directions in such a way that S always remains greater than $1/12$.

10. Let $\mathcal{F}$ be the real vector space of continuous function $f:[0,1]\to\mathbb{R}$. Consider on $\mathcal{F}$ the distance $d(f,g) = \int_0^1 |f(x)-g(x)|dx$. Show that the intersection of any affine line directed by a nowhere zero function with any sphere consists of at most two points.