Mathematical battle

Date: February 20, 2014 (time and venue to be confirmed)

Rules of the battle are available on the webpage of TCD problem solving group (via the http URL `http://www.maths.tcd.ie/~vdots/teaching/problemsolving.html`). There will be two teams competing in the battle, one from TCD and one from UCD.

1. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(f(x)) + f(x) = 2x + 9 \quad \text{for all } x \in \mathbb{R}.$$

2. Let $0 < x < 1$. Compute the sum of the infinite series
$$\sum_{n=1}^{\infty} \frac{(-1)^{\lfloor 2^n x \rfloor}}{2^n}.$$

3. Suppose for a smooth function f on $\mathbb{R}^2$ it is true that for each rectangle $ABCD \subset \mathbb{R}^2$ we have
$$f(A) + f(C) = f(B) + f(D).$$
Show that f is a polynomial of degree at most 2.

4. Suppose that A and B are two square matrices of the same size 2013×2013 such that $AB = 0$. Show that one of the matrices $A + A^T$ and $B + B^T$ is not invertible.

5. Suppose that for two positive integers $a, b \geq 2$, it is known that $a^n - 1$ is divisible by $b^n - 1$ for all $n \in \mathbb{N}$. Show that $a = b^k$ for some $k \in \mathbb{N}$.

6. Show that every function $f\colon \mathbb{Z} \to \mathbb{Z}$ can be represented as a sum of three bijections: for each $x \in \mathbb{Z}$,
$$f(x) = f_1(x) + f_2(x) + f_3(x),$$
where $f_i\colon \mathbb{Z} \to \mathbb{Z}$ is a one-to-one correspondence.

7. Let T be a tree, that is, a union of finitely many intervals in the plane which is connected and has no cycles. Let $f : T \to T$ be continuous. Prove that f has a fixed point.

8. Let $Q(x)$ be a polynomial with integer coefficients. Define the sequence $\{a_n\}_{n\geq 0}$ by $a_0 = 0$ and $a_{n+1} = Q(a_n)$, for all $n \geq 0$. Prove that if there exists $m \geq 1$ such that $a_m = 0$, then either $a_1 = 0$ or $a_2 = 0$.

9. A *sawtooth* permutation $b_1, \dots, b_n$ of $1, 2, \dots, n$ is a permutation such that $(b_i - b_{i-1})(b_i - b_{i+1}) > 0$ for all $i = 2, \dots, n-1$. Let B_n be the number of sawtooth permutations of n elements. Prove that for every $\varepsilon > 0$ we have
$$B_n < \frac{n!}{\left(\frac{\pi}{2} - \varepsilon\right)^n}$$
for sufficiently large n.

10. Let $P_1, P_2, \dots, P_{4n} \in \mathbb{R}^3$ be in generic position (no 4 points are coplanar). Prove that one can partition $P_1, P_2, \dots, P_{4n}$ into vertices of mutually non-intersecting tetrahedra.