Problem solving methods 1: numbers and polynomials (16/10/2012)

1. Compute the sums $\sum\limits_{k=0}^{n}\binom{n}{k}$, $\sum\limits_{k=0}^{n}k\binom{n}{k}$, $\sum\limits_{k=0}^{n}k^2\binom{n}{k}$, $\sum\limits_{k=0}^{n}2^k$, $\sum\limits_{k=0}^{n}k\cdot 2^k$, and $\sum\limits_{k=0}^{n}\frac{k}{2^k}$.

2. Show that $\sum_{k=0}^{n}\binom{n}{k}^2=\binom{2n}{n}$.

3. Prove that
$$f_0+f_1t+f_2t^2+\ldots=\frac{t}{1-t-t^2},$$
where $f_0, f_1, f_2, \ldots$ are Fibonacci numbers: $f_0=0$, $f_1=1$, $f_n=f_{n-1}+f_{n-2}$ for $n\geq 2$.

4. Prove the identity (for a, b, c not equal to one another):
$$\frac{(x-a)(x-b)}{(c-a)(c-b)}+\frac{(x-b)(x-c)}{(a-b)(a-c)}+\frac{(x-c)(x-a)}{(b-c)(b-a)}=1$$

5. Show that if p is a prime number, then the polynomial x^n-p cannot be decomposed into a product of two polynomials of positive degrees with integer coefficients

6. Suppose $p(x)$ is a polynomial with integer coefficients such that $p(0)=p(1)=2009$. Show that $p(x)$ has no integer roots.

7. Show that the polynomial $(x-1)(x-2)\ldots(x-100)-1$ cannot be decomposed into a product of two polynomials of positive degrees with integer coefficients

8. (Moscow Maths Olympiad 1997) For real numbers $a_1\leq a_2\leq a_3$ and $b_1\leq b_2\leq b_3$, we have $a_1+a_2+a_3=b_1+b_2+b_3$ and $a_1a_2+a_1a_3+a_2a_3=b_1b_2+b_1b_3+b_2b_3$. Prove that if $a_1\leq b_1$, then $a_3\leq b_3$.

9. Let us define polynomials $f_n(x)$ recursively: $f_0(x)=1$, $f_1(x)=x$, $f_{n+1}(x)=2xf_n(x)-f_{n-1}(x)$. Show that for each $a\in[-1,1]$ all roots of the polynomial $f_n(x)-a$ are real.

10. Let $f(t)$ and $g(t)$ be two polynomials in one variable. Show that there exists a polynomial $P(x,y)$ in two variables for which $P(f(t),g(t))=0$.

11. (IMC 2007) Let f be a polynomial of degree 2 with integer coefficients. Suppose that $f(k)$ is divisible by 5 for every integer k. Prove that all coefficients of f are divisible by 5.

12. (IMC 2008) Let n and k be positive integers, and suppose that the polynomial $x^{2k}-x^k+1$ divides the polynomial $x^{2n}+x^n+1$. Then the polynomial $x^{2k}+x^k+1$ also divides $x^{2n}+x^n+1$.

13. (IMC 2012) Consider a polynomial $f(x)=x^{2012}+a_{2011}x^{2011}+\ldots+a_1x+a_0$. Albert Einstein and Homer Simpson are playing the following game. In turn, they choose one of the coefficients $a_0,\ldots,a_{2011}$, and assign a real value to it. Albert has the first move. Once a value is assigned to a coefficient, it cannot be changed anymore. The game ends after each player made 1006 moves, so all the coefficients have been assigned values. Homer's goal is to make $f(x)$ divisible by a certain polynomial $m(x)$, and Albert's goal is to prevent this.

    (1) which of the players has a winning strategy for $m(x)=x-2012$?

    (2) which of the players has a winning strategy for $m(x)=x^2+1$?

14. For a polynomial $f(x)$ with integer coefficients, all values at integer points are powers of 2. Describe all such polynomials.

15. For a polynomial $f(x)$, all values at integer points are integers. Show that all the coefficients of $f(x)$ are rational numbers. Is it true that all coefficients of $f(x)$ always are integers?

16. * Is $\frac{3+4i}{5}$ a root of unity? In other words, does there exist an integer $n > 0$ such that $\left(\frac{3+4i}{5}\right)^n = 1$?

17. ** For a polynomial $f(x)$, all values at integer points are perfect squares of integers. Show that $f(x) = (g(x))^2$ for some polynomial $g(x)$.

Some catchphrases:

- When proving identities for a finite (or infinite) sequence $\{a_n\}$, check out what they might mean for the polynomial (or power series) $\sum_n a_n x^n$. You might be pleasantly surprised. In particular,
  - when proving an identity of the form $\sum_k a_k = b$, it sometimes is easier to prove an identity of the form $\sum_k a_k x^k = b(x)$, and then check $b(1) = b$;
  - sums of the form $s_n = \sum_k a_k b_{n-k}$ beg to consider $A(x) = \sum_k a_k x^k$, $B(x) = \sum_k b_k x^k$, and $S(x) = \sum_k s_k x^k$; for these three we have $S(x) = A(x)B(x)$;
  - if $\sum_k a_k x^k = b(x)$, then obviously $\sum_k k a_k x^k = x b'(x)$;
- For each polynomial $f(x)$ and each a, the polynomial $f(x) - f(a)$ is divisible by $x - a$. Moreover, if $f(x) \in \mathbb{Z}[x]$ and $a \in \mathbb{Z}$, then $\frac{f(x)-f(a)}{x-a} \in \mathbb{Z}[x]$. In particular, if $f(x) \in \mathbb{Z}[x]$ and $a, b \in \mathbb{Z}$, then $(b - a)$ is a divisor of $f(b) - f(a)$.
- When dealing with polynomials with integer coefficients, it is most helpful to reduce them modulo p for a prime p; this way you end up with polynomials with coefficients in a field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, and life becomes easier.
- A polynomial of degree at most n has at most m roots. As a consequence, there is at most one polynomial of degree at most n taking the given values at given $n + 1$ points. (In fact, such a polynomial always exists and is unique.)
- If a polynomial $f(x) = a_n x^n + \ldots + a_1 x + a_0 \in \mathbb{Z}[x]$ has a rational root b/a, we have $b \mid a_0$ and $a \mid a_n$.