Problem set 23/10/2012. Masha Vlasenko

- Solving linear reccurences.
  - Fibonacci numbers are defined by $f_0 = 0$, $f_1 = 1$, $f_{n+1} = f_n + f_{n-1}$. Prove the formula

$$f_n \;=\; \frac{1}{\sqrt{5}}\Big((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n\Big)\,.$$

  - For given $n \geq 1$ find all real numbers a for which the $n \times n$ determinant

$$\det \begin{pmatrix} a & 1 & 0 & 0 & 0 & \dots \\ 1 & a & 1 & 0 & 0 & \dots \\ 0 & 1 & a & 1 & 0 & \dots \\ \dots & & & & & \end{pmatrix}$$

    vanishes.

- Positive and non-negative definite matrices. Gram matrices.
  - Let G be a connected graph (non-oriented, multiple edges and loops are allowed). We denote by V and E the sets of vertices and edges of G respectively. The incidence matrix $A_G = (a_{ij})_{i,j \in V}$ of G is defined as follows. For $i \neq j$ one has that $a_{ij} = a_{ji}$ is the number of edges between i and j, and a_{ii} is the degree of the vertex i (the number of edges entering i). Prove that $A_G \geq 0$ and $A_G > 0$ if and only if G is bipartite.

    The idea: One can identify the space of functions $\{f : E \to \mathbb{R}\}$ with $\mathbb{R}^{|E|}$ and consider the inner product $\langle f, g\rangle = \sum_{e \in E} f(e)g(e)$. It is possible to construct functions $f_i : E \to \mathbb{R}$ for all $i \in V$ such that $a_{ij} = \langle f_i, f_j\rangle$. Then A_G is a Gram matrix.