AMM problems August-September 2012, due before 31 December

November 14, 2012

11656. Proposed by Valerio De Angelis. The *sign chart* of a polynomial f with real coefficients is the list of successive pairs (ϵ, σ) of signs of (f', f) on the intervals separating real zeroes of ff', together with the signs at the zeroes of ff' themselves, read from left to right. Thus, for $f = x^3 - 3x^2$, the sign chart is $((1,-1),(0,0),(-1,-1),(0,-1),(1,-1),(1,0),(1,1))$. As a function of n, how many distinct sign charts occur for polynomials of degree n?

11657. Proposed by Gregory Galperin and Yury Ionin. Given a set V of n points in $\mathbb{R}^2$, no three of them collinear, let E be the set of $\binom{n}{2}$ line segments joining distinct elements of V.

(a) Prove that if $n \not\equiv 2 \pmod 3$, then E can be partitioned into triples in which the length of each segment is greater than the sum of the other two.

(b) Prove that if $n \equiv 2 \pmod 3$ and e is an element of E, then $E \backslash \{e\}$ can be so partitioned.

11658. Proposed by Greg Oman. Let V be the vector space over $\mathbb{R}$ of all (countably infinite) sequences $(x_1, x_2, \ldots)$ of real numbers, equipped with the usual addition and scalar multiplication. For $v \in V$, say that v is *binary* if $v_k \in \{0,1\}$ for $k \geq 1$, and let B be the set of all binary members of V. Prove that there exists a subset I of B with cardinality $2^{\aleph_0}$ that is linearly independent over $\mathbb{R}$. (An infinite subset of a vector space is linearly independent if all its finite subsets are linearly independent).

11659. Proposed by Albert Stadler. Let x be real with $0 < x < 1$, and consider the sequence $\langle a_n \rangle$ given by $a_0 = 0$, $a_1 = 1$, and for $n > 1$,

$$a_n = \frac{a_{n-1}^2}{x a_{n-2} + (1-x) a_{n-1}}$$

Show that

$$\lim_{n \to \infty} \frac{1}{a_n} = \sum_{k=-\infty}^{\infty} (-1)^k x^{k(3k-1)/2}.$$

11660. Proposed by Stefano Siboni. Consider the following differential equation: $s''(t) = -s(t) - s(t)^2 \text{sgn}(s'(t))$, where $\text{sgn}(u)$ denotes the sign of u. Show that if $s(0) = a$ and $s'(0) = b$ with $ab \neq 0$, then (s, s') tends to $(0,0)$ with $\sqrt{s^2 + s'^2} \leq C/t$ as $t \to \infty$, for some $C > 0$.

11661. Proposed by Giedrius Alkauskas. Find every function f on $\mathbb{R}^+$ that satisfies the functional equation

$$(1-z)f(x) + f\left(\frac{1-z}{z}f(xz)\right)$$

for $x > 0$ and $0 < z < 1$.

11662. Proposed by H. Stephen Morse. Let $ABCD$ be the vertices of a square, in that order. Insert P and Q on AB (in the order $APQB$) so that each of P and Q divides AB 'in extreme and mean ratio' (that is $|AB|/|BQ| = |BQ|/|QA|$ and $|AB|/|AP| = |AP|/|PB|$.) Likewise, place R and S on CD so that $CRSD$ is divided in the same proportions as $APQB$. The four intersection points of $AR, BS, CP,$ and DQ are called the *harmonious quartet* of the square on its *base pair* (AB, CD). They form a rhombus whose long diagonal has length $(\sqrt{5}+1)/2$ times the length of its short diagonal.

Given a cube, create the harmonious quartet for each of its six faces, using each edge as part of a base pair exactly once, according to this scheme: label the vertices on one face of the cube $ABCD$ and the corresponding vertices of the opposite face $A'B'C'D'$. Pair AB with CD, AA' with BB', and BC with $B'C'$. The rest of the pairings are then forced: $A'B'$ with $C'D'$, AD with $A'D'$, and CC' with DD'. This generates 24 points.

(a) Show that these 24 points are a subset of the 32 vertices of a *rhombic triacontahedron* (a convex polyhedron bounded by 30 congruent rhombic faces, meeting three each across their obtuse angles at 20 vertices, and five each across their acute angles at 12 vertices), and find a construction for the remaining eight vertices.

(b) Show, moreover, that the 12 end points of the longer diagonals of the six constructed rhombi are the vertices of an icosahedon I, and these diagonals are edges of the icosahedron.

(c) Show that the 12 end points of the shorter diagonals of the cosntructed rhombi, together with the eight additional vertices of the triacontahedron, are the vertices of a dodecahedron. Show also that these shorter diagonals are edges of that dodecahedron.