AMM problems April 2013, due before 31 August 2013

TCDmath problem group
Mathematics, Trinity College, Dublin 2, Ireland*

July 1, 2013

11698. *Proposed by Timothy Hall.* Provide an algorithm that takes as input a positive integer n and a nonzero constant k and returns polynomials F and G in variables u and v such that when x^n is substituted for u, and $x+k/x$ is substituted for v, $F(u,v)/G(u,v)$ simplifies (disregarding removable singularities) to x. (For example, when $k=1$ and $n=3$, $F=u+v$ and $G=v^2-1$ will do.)

11699. *Proposed by Bakir Farhi.* Let $\langle a_k \rangle$ be a strictly increasing sequence of positive integers such that $\sum_{k=2}^{\infty} \frac{1}{a_k \log a_k}$ diverges. Prove that lcm($a_1, \ldots, a_k$) = lcm($a_1, \ldots, a_{k+1}$) for infinitely many k in $\mathbb{N}$.

11700. *Proposed by Evan O'Dorney (student).* Let n be an integer greater than 1. Let $a, b,$ and c be complex numbers with $a+b+c = a^n + b^n + c^n = 0$. Prove that the absolute values of $a, b,$ and c cannot be distinct.

11701. *Proposed by D.M. Bătineţu-Giurgiu and Neculai Stanciu.*
(a) Let $\langle x_n \rangle$ be the sequence defined by $\sum_{k=1}^{mn} 1/k = \gamma + \log(mn + x_n)$, where γ is the Euler-Mascheroni constant. Find $\lim_{n\to\infty} x_n$.
(b) let $\langle y_n \rangle$ be the sequence defined by $\sum_{k=1}^{mn} = \gamma + \log(m(n+y_n))$. Find $\lim_{n\to\infty} y_n$.

11702. *Proposed by Greg Oman.* Find all nonzero rings R (not assumed to be commutative or to contain a multiplicative identity) with these properties:
(a) There exists $x \in R$ that is neither a left nor a right zero divisor, and
(b) Every map ϕ from R to R that satisfies $\phi(x+y) = \phi(x) + \phi(y)$ also satisfies $\phi(xy) = \phi(x)\phi(y)$. (That is, every additive homomorphism on R is a ring homomorphism.)

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11703. *Proposed by Richard Bagby.* For $\lambda > 0$, let $\Gamma(\lambda) = \{(x, y, z) \in \mathbb{R}^3 : z \geq \lambda\sqrt{x^2 + y^2}\}$, and let $C(\lambda)$ be the (half-cone) boundary of $\Gamma(\lambda)$. Prove that every point in the interior of $\Gamma(\lambda)$ is the focus of at least one ellipse in $C(\lambda)$, and find the largest μ such that every ellipse in $C(\lambda)$ has at least one focus in $\Gamma(\mu)$.

11704. *Proposed by Olivier Bernardi, Thaynara Arielly de Lima, and Richard Stanley.* Let S_{2n} denote the symmetric group of all permutations of $\{1, \ldots, 2n\}$ and let T_{2n} denote the set of all fixed-point-free involutions in S_{2n}. Choose u and v from T_{2n} at random (any element of T_{2n} being as likely as any other) and independently. What is the probability that 1 and 2 will be in the same cycle of the permutation uv? (For example, when $n = 2$, $T_{2n} = \{2143, 3412, 4321\}$, (u, v) can be $(3412, 4321)$ or $(4321, 3412)$, and the required probability is $2/9$.)