AMM problems December 2013, due before 30 April 2014

TCDmath problem group
Mathematics, Trinity College, Dublin 2, Ireland*

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11740. *Proposed by Cosmin Pohoata.* Let $\mathcal{P}, \mathcal{Q}$ be prime ideals in a commutative Noetheran ring R with unity. Suppose that $\mathcal{P} \subset \mathcal{Q}$. Let I be the set of all prime ideals $\mathcal{J}$ in R such that $\mathcal{P} \subset \mathcal{J} \subset \mathcal{Q}$. Prove that I is either empty or infinite.

11741 *Proposed by Chindea Filip-Andrei.* Given a ring A, let $Z(A)$ denote the *center* of A, which is the set of all z in A that commute with every element of A. Prove or disprove: For every ring A, there is a map $f : A \to Z(A)$ such that $f(1) = 1$ and $f(a+b) = f(a) + f(b)$ for all $a, b \in A$.

11742 *Proposed by Alexandr Gromeko.* For $0 \leq p < q < 1$, find all zeros in $\mathbb{C}$ of the function f given by

$$f(z) = \sum_{n=-\infty}^{\infty} (aq^n, p/(aq^n); p)(-z)^n q^{n(n-1)/2},$$

where $(u, v; w) = \prod_{m=0}^{\infty}(1 - uw^m)(1 - vw^m)$.

11743 *Proposed by François Capacès.* Let n be a positive integer, let x be a real number, and let B be the n-by-n matrix with $2x$ in all diagonal entries, 1 in all sub- and super-diagonal entries, and 0 in all other entries. Compute the inverse, when it exists, of B as a function of x.

11744 *Proposed by C.P. Cholkar and M.N. Deshpande.* Flip a fair coin until the start of the rth run. (For instance, if $r = 3$ then $TTTHHT$ is one possible outcome.) Let Y be the number of runs consisting of one head. Find the expected value and variance of Y.

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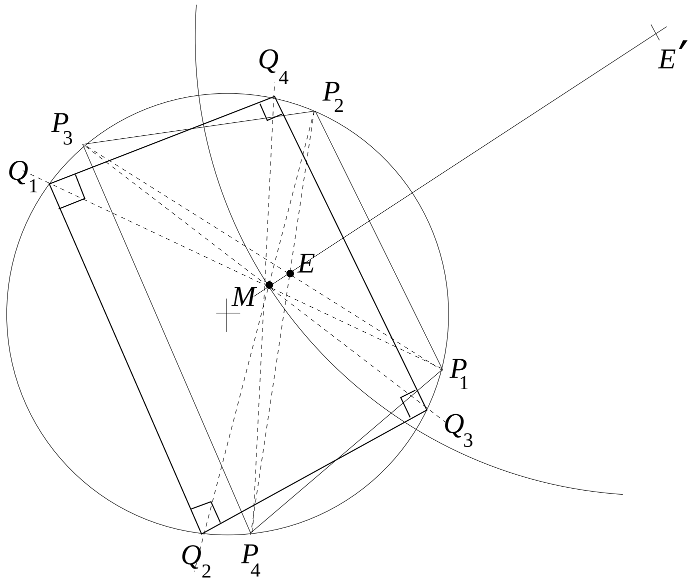
11745 *Proposed by Robin Oakapple.* Let P_1, P_2, P_3, and P_4 be points on a circle K, in the order listed, that do not form a rectangle. Let the diagonals of the convex quadrilateral $P_1P_2P_3P_4$ cross at E. Let E' be the image of E under inversion about K. Let $K^{\perp}$ be the circle with center E' that intersects K orthogonally. Let M and M' be the points at which $K^{\perp}$ meets the line EE', with M inside K and M' outside. For $1 \leq i \leq 4$, let Q_i be the second intersection of MP_i with K, and let Q_i' be the second intersection of $M'P_i$ with K.



(The figure is inaccurate.)

Prove that $Q_1, \ldots, Q_4$ form the vertices of a rectangle, as do $Q_1', \ldots, Q_4'$, and that the two rectangles are mirror images across the line EE'. (The figure omits $Q_1', \ldots, Q_4'$ and some of $K^{\perp}$).

11746 *Proposed by Pál Péter Dályay.* Let F be a continuous function from $[0, \infty)$ to $\mathbb{R}$ such that the following integrals converge: $S = \int_0^\infty f^2(x)dx$, $T = \int_0^\infty x^2 f^2(x)dx$, and $U = \int_0^\infty x^4 f^2(x)dx$. Let $V = T + \sqrt{T^2 + 3SU}$. Given that f is not identically 0, show that

$$\left(\int_0^\infty |f(x)|dx\right)^4 \leq \frac{\pi^2 S(T+V)^2}{9V}.$$