# AMM problems February 2014, due before 30 June

TCDmath problem group
Mathematics, Trinity College, Dublin 2, Ireland*

June 20, 2014

**11754.** *Proposed by David Beckwith.* When a fair coin is tossed $n$ times, let $P(n)$ be the probability that the lengths of all runs (maximal constant strings) in the resulting sequence are of the same parity as $n$. Prove that

$$P(n) = \begin{cases} \left(\frac{1}{2}\right)^{n/2} & \text{if } n \text{ is even} \\ \left(\frac{1}{2}\right)^{n-1} F_n & \text{if } n \text{ is odd} \end{cases}$$

where $F_n$ is the $n$-th Fibonacci number, defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

**11755.** *Proposed by Pál Péter Dályay.* Compute

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} \sum_{k=n+1}^{\infty} \frac{(-1)^{k-1}}{2k-1}$$

**11756.** *Proposed by Paolo Perfetti.* Let $f$ be a function from $[-1, 1]$ to $\mathbb{R}$ with continuous derivatives of all orders up to $2n + 2$. Given $f(0) = f'(0) = \ldots = f^{(2n)}(0) = 0$, prove

$$\frac{1}{2}((2n+2)!)^2(4n+5)\left(\int_{-1}^{1} f(x)dx\right)^2 \leq \int_{-1}^{1} (f^{(2n+2)}(x))^2 dx.$$

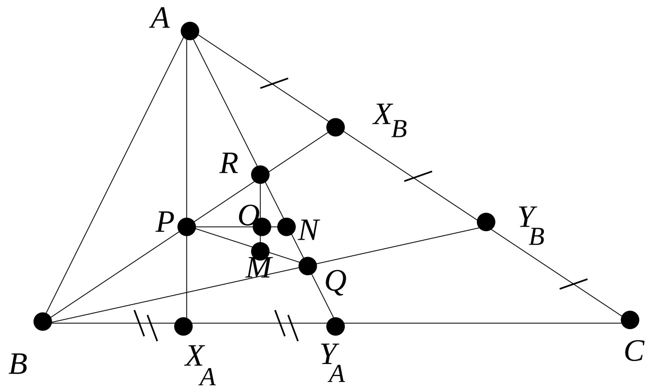
**11757.** *Proposed by Ira Gessel.* Let $[x^a y^b]f(x, y)$ denote the coefficient of $x^a y^b$ in the Taylor's series expansion of $f$. Show that

$$[x^n y^n] \frac{1}{(1-3x)(1-y-3x+3x^2)} = 9^n.$$

**11758.** *Proposed by Bryan Brzycki.* Acute triangle $ABC$ has several ancillary points and properties shown in the figure. Segment $AX_A$ is perpendicular to $BC$, and segments marked with the same

*This group involves students and staff of the Department of Mathematics, Trinity College, Dublin. Please address correspondence either to Timothy Murphy (`tim@maths.tcd.ie`), or Colm Ó Dúnlaing (`odunlain@maths.tcd.ie`).

symbol have the same length. The angle at $C$ is less than the angle at $B$, and lines $PN$ and $RM$ are parallel and perpendicular, respectively, to $BC$.




**(a)** Prove that $|RO|/|OM| = 2$.
**(b)** Prove that $PQ$ is not parallel to $BC$.
**(c)** Letting $D$ be the intersection of $PQ$ and $BC$, show that if $AD$ is perpendicular to $BX_B$, then $P, R, N$, and $M$ lie on a common circle.
**(d)** For fixed $B$ and $C$, describe the set of $A$ such that $AD$ is perpendicular to $BX_B$.

**11759.** *Proposed by Octavian Ganea and Cezar Lupu.* Let $A$ be an $n \times n$ skew-symmetric real matrix. Show that for positive real numbers $x_1, \ldots, x_k$ with $k \geq 2$,

$$\det(A + x_1 I) \cdots \det(A + x_k I) \geq (\det(A + (x_1 \cdots x_k)^{1/k} I))^k.$$

In addition, show that if also all $x_i$ lie on the same side of 1, then

$$\det(A + I)^{k-1} \det(A + x_1 \cdots x_k I) \geq \det(A + x_1 I) \cdots \det(A + x_k I).$$

**11760.** *Proposed by Stefano Siboni.* Let $D$ be the closure of a simply connected, bounded open subset of $\mathbb{R}^2$. Let $W$ be the subset of $[0, 1]^n$ consisting of all points $(w_1, \ldots, w_n)$ such that $w_1 + \ldots + w_n = 1$. Let $g$ be a point in $D$, and let $n$ be an integer, $n > 1$. With $p = (p_1, \ldots, p_n) \in D^n$, let $M$ be the function from $D^n \times W$ to $\mathbb{R}$ given by

$$M(p, w) = \sum_{k=1}^{n} w_k \|p_k - g\|^2,$$

where $\| \cdot \|$ is the Euclidean norm on $\mathbb{R}^2$.
**(a)** Show that if $M(p, w)$ is maximised at $(p', w')$, then all entries of $p'$ lie on the boundary of $D$.
**(b)** Restricting now to the case in which $n = 2$ and the boundary of $D$ is an ellipse, let $(p'_1, p'_2)(w'_1, w'_2))$ be a point at which $M((p_1, p_2), (w_1, w_2))$ is maximized. Show that $p'_1$ and $p'_2$ lie opposite each other on the major axis of the ellipse.
**(c)** Show that if $D$ is a disk of radius $r$ about the origin, then the maximum value of $M$ is $r^2 - \|g\|^2$.