AMM problems April 2014, due before 31 August

TCDmath problem group
Mathematics, Trinity College, Dublin 2, Ireland*

July 25, 2014

11768. *Proposed by Ovidiu Furdui.* Let f be a bounded continuous function mapping $[0, \infty)$ to itself. Find

$$\lim_{n\to\infty} n\left(\sqrt[n]{\int_0^\infty f^{n+1}(x)e^{-x}dx} - \sqrt[n]{\int_0^\infty f^n(x)e^{-x}dx}\right).$$

11769. *Proposed by Pál Péter Dályay.* Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be positive real numbers. Show that

$$\left(\sum_{j=1}^{n} \frac{a_j}{b_j}\right)^2 - 2\sum_{j,k=1}^{n} \frac{a_j a_k}{(b_j + b_k)^2} \le 2\left(\sum_{j,k=1}^{n} \frac{a_j a_k}{(b_j + b_k)} \sum_{l,m=1}^{n} \frac{a_l a_m}{(b_l + b_m)^3}\right)^{1/2}.$$

11770. *Proposed by Spiros P. Andriopoulos.* Prove, for real numbers a, b, x, y with $a > b > 1$ and $x > y > 1$, that

$$\frac{a^x - b^y}{x - y} > \left(\frac{a+b}{2}\right)^{(x+y)/2} \log\left(\frac{a+b}{2}\right).$$

11771. *Proposed by D. M. Bătineţ-Giurgiu and Neculai Stanciu.* Let $n!! = \prod_{i=0}^{\lfloor (n-1)/2 \rfloor}(n - 2i)$. Find

$$\lim_{n\to\infty}\left(\sqrt[n]{(2n-1)!!}\left(\tan\frac{\pi\sqrt[n+1]{(n+1)!}}{4\sqrt[n]{n!}} - 1\right)\right).$$

11772. *Proposed by Mircea Merca.* Let n be a positive integer. Prove that the number of integer partitions of $2n + 1$ that do not contain 1 as a part is less than or equal to the number of integer partitions of $2n$ that contain at least one odd part.

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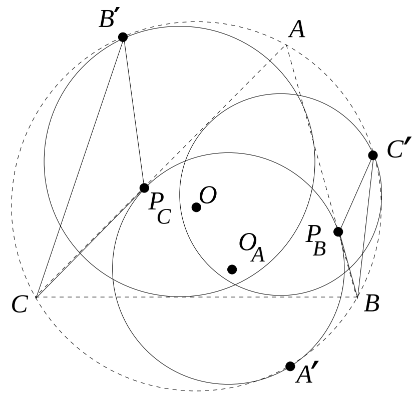
11773. *Proposed by Moubinool Omarjee.* Given a positive real number a_0, let $a_{n+1} = \exp\left(-\sum_{k=0}^{n} a_k\right)$ for $n \geq 0$. For which values of b does $\sum_{n=0}^{\infty}(a_n)^b$ converge?

11774. *Proposed by Yunus Tunçbilek and Danny Lee.* Let ω be the circumscribed circle of triangle ABC. The A-*mixtilinear incircle* of ABC and ω is the circle that is internally tangent to ω, AB, and AC, and similarly for B and C. Let A', P_B, and P_C be the points on ω, AB, and AC, respectively, at which the A-mixtilinear circle touches. Define B' and C' in the same manner that A' was defined. (See figure.)



Prove that triangles $C'P_BB$ and CP_CB' are similar.