Gradual buildup towards solving some Olympiad problems

November 16, 2010

1. In a row of railings, every k-th is white and the rest are black. Show that in every group of k consecutive railings, there is exactly one white railing.

2. Show that every product of k consecutive integers is divisible by k.

3. Define
$$\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k!}, \quad n \geq k \geq 0$$
This covers the cases where $k = 0$:
$$\binom{n}{0} = 1.$$
Show (directly, by simplifying both sides)
$$\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$$
and deduce the binomial coefficients are all integers, and hence every product of k consecutive integers is divisible by $k!$.

4. Show that if $x_i \leq \alpha$, $1 \leq i \leq n$, then the average is $\leq \alpha$.

5. Show that in a class of 25 or more students, there are at least 3 whose birthday is in the same month.

6. Suppose that 200 people answer a survey with 6 questions, and each question is answered by at least 101 people. Show that, for any pair (there are 15 pairs, not that that matters) of questions, at least 2 people answer both of the questions.

   Show that somebody answers at least 4 questions.

   Show there exist at least two different people who between them answer all the questions.

7. Suppose n numbers x_i are given, and $m \leq n$, and the average of any m of the numbers is $< \beta$. Show that the $n - m$ smallest numbers are $< \beta$.

8. Show that from a group of seven people whose (integer) ages add up to 332, one can select 3 with a total age of at least 142.

9. In how many ways can a cube be rotated around its centre so that every corner is mapped to a corner, possibly itself? (Include the identity map.)

10. A graph (V, E) has a finite set V of *vertices* and a set E of edges. Each edge is an unordered pair of distinct vertices, $\{u, v\}$ or uv (the curly braces don't help), *incident* to u and to v (and vice-versa).

    The *degree* of each vertex is the number of incident edges. Show that the number of odd-degree vertices is even.

11. A *path* in a graph is a sequence $u_0, \ldots, u_k$ of vertices such that for $0 \leq j \leq k-1$, $u_j u_{j+1}$ is an edge.

    A graph is *acyclic* if there exists at most one path joining any two vertices.

    A *connected component* of a graph is a maximal nonempty set of nodes which are pairwise connected by paths. (A connected acyclic graph, that is, an acyclic graph with one component, is called a *free tree.*)

    Show (by induction?) that if G is an acyclic graph with n vertices, then G has at most $n-1$ edges.

12. A set S of points in the plane is *convex* if for any two points $x, y \in S$, the line-segment $[x, y]$ is contained in S.

    More long-windedly, for any straight line L, $L \cap S$ is either (i) $\emptyset$ (ii) a single point (L tangent) (iii) a bounded line-segment, possibly including one or both endpoints (iv) a semi-infinite line (possibly including its endpoint), or (v) all of L.

    Show that the set

    $$\{(x, y) \in \mathbb{R}^2 : \; y \geq x^2\}$$

    is convex. More specifically, the only possibilities for $L \cap S$ are (i) $\emptyset$ (ii) a single point (iii) a closed line-segment (including both endpoints), or (iv) a closed semi-infinite line (including its endpoint).

13. Show that in any graph (with 2 or more vertices) at least 2 vertices have the same degree.

14. Show that in any graph with at least 1 edge at least 2 vertices have the same degree where the degree is nonzero.