Olympiad problem solving: Freshmen sessions
Problem set 2, November 23,2010

Rules: none. Attempt any problems you like (including those from previous hand-outs), work individually or in groups (in the latter case, be quiet to not disturb others). At every problem session there will be several members of staff to discuss your solutions, give hints etc. Good luck!

1. The numbers $1, 2, \ldots, n$ are written (in this order) at the vertices of a regular n-gon. You are allowed to add 1 to two adjacent numbers, and to repeat this operation several times. For which n is it possible to end up with the configuration where all numbers are the same?

2. All entries of a $n \times n$-matrix are equal to either 1 or -1. You are allowed to multiply all elements of one of the rows by -1, or to do the same to all elements of one of the columns, and to repeat these operations several times. Find out how to check, given a matrix, whether it is possible to make all the matrix entries the same.

3. In a group of 1981 people, every person knows at least 45 other people. Show that it is possible to pick four people from that group and sit them at the four sides of a square table so that every one of them sits next to people he or she knows.

4. The triple (a, b, n) is called a Ramsey triple if in each group of n there exist either a mutual acquaintances, or b mutual strangers. In the previous handout, you were asked to prove that $(3, 3, 6)$ is a Ramsey triple, and that $(3, 4, 10)$ is a Ramsey triple.

(a) Show that $(3, 3, 5)$ is not a Ramsey triple.

(b) Show that $(3, 4, 9)$ is a Ramsey triple, and $(3, 4, 8)$ is not.

5. Prove that if $(a, b - 1, n)$ is a Ramsey triple, and $(a - 1, b, m)$ is a Ramsey triple, then $(a, b, n + m)$ is also a Ramsey triple.

6. Find all integer solutions to the equation $x^2 + y^2 = 1000006$.

7. Find all integer solutions to the equation $x^2 + y^2 + z^2 = 2xyz$.

8. Show that the equation $x^{\lfloor x \rfloor} = 9/2$ has no positive rational solutions.

9. Assume that for a sequence of real numbers a_n we have

$$\lim_{n \to \infty} (a_n + a_{n^2}) = 0.$$

Does that imply that $\lim_{n \to \infty} a_n = 0$?

10. Show that it is impossible to place two triangles, each of area greater than 1, in a circle of radius 1 without overlaps.

11. Given that $ab = 4$ and $c^2 + 4d^2 = 4$, show that $(a - c)^2 + (b - d)^2 \geqslant 1.6$.

12. Prove the identity (for a, b, c not equal to one another):

$$\frac{(x - a)(x - b)}{(c - a)(c - b)} + \frac{(x - b)(x - c)}{(a - b)(a - c)} + \frac{(x - c)(x - a)}{(b - c)(b - a)} = 1$$

13. Suppose $p(x) = a_0 + a_1 x + \ldots + a_n x^n$ is a polynomial with integer coefficients such that $p(0) = p(1) = 2009$. Show that $p(x)$ has no integer zeros.

14. Show that if p is a prime number, then the polynomial $x^n - p$ cannot be decomposed into a product of two polynomials of positive degrees with integer coefficients