WEAK TOPOLOGIES FOR MODULES OVER RINGS
OF BOUNDED RANDOM VARIABLES

KARL-THEODOR EISELE† AND SONIA TAIEB‡

Abstract

In order to establish a functional analytic basis for representation theorems for conditional and multi-period risk measures, we study locally convex modules over the ring \( \lambda = L^\infty (\mathcal{G}) \). Their topology is determined by \( \lambda \)-seminorms. As expected, central mathematical tools of the analysis are Hahn-Banach type and separation theorems which however have to be treated more carefully in the module case. Once a dual \( \lambda \)-module is introduced, one can establish a module version of the Bipolar theorem. We also prove the Krein-Šmulian as well as the Alaoglu-Bourbaki theorem for \( \lambda \)-modules. For Banach \( \lambda \)-modules their reflexivity is characterized by a compactness criterium in a (very) weak-* topology.

Key words and phrases: Locally convex module over \( L^\infty \), extended Hahn-Banach and separation theorems, bipolar theorem, general Krein-Šmulian and Alaoglu-Bourbaki theorem for modules.


1. Introduction

The present contribution stands in a close context of the widespread research activities in the last 15 years about risk measures (or their negative values, called monetary utility functions or risk assessments). This movement had its starting point in the lighthouse paper [3] about coherent risk measures and their robust representation. Since then, a huge number of subsequent papers have appeared first extending the representation theorem to convex risk measures, and afterwards treating topics like law-invariant, conditional, and multi-period risk measures. A first list of references can be found in [1].

Already in [3] and [7] it became clear that the analysis of risk measures needs a substantial amount of non-trivial functional analysis since weak and weak-* topologies of vector spaces play an important role. The basic representation theorem for risk measures or assessments on \( L^\infty \) as an infimum over “test” probability measures, whose proof is due to F. Delbaen, requires versions of the bipolar theorem and of the Krein-Šmulian theorem (see [9] or [14] as an introduction). First generalizations of the robust representation of convex risk measures to conditional ones are done by the “scalarization” method (see [10] and [20]), thus reducing representations for conditional risk measures to known results of scalar risk measures. Alternatively, one needs more general theorems of functional analysis which extend known results from standard vector spaces to modules over rings of random variables. For example, the connection of the Fatou property of risk measures with the order (or Dedekind) semi-continuity of convex functions on Frechet lattices has been treated in [4].

Extensions from the classical algebraic Hahn-Banach theorem which first appeared in 1927 and 1929 for real valued linear functionals on real vector spaces, to the case of ring-linear functionals on modules over Banach algebras or complete lattice rings with values in complete lattice modules over the same ring are well known (see [19], [25], [6], and [26]).
Independently of the present interest in functional analytic results for modules over rings of random variables, a theory of such modules, also called random spaces, has been already developed several years ago (see for example [23], [24], and [18]).

A part of the generalizations we are interested in, was dealt with in the papers [12], [22], and [13], as well as in [15]. These papers used as an underlying ring, the space \( L^0(\mathcal{G}) \) of all \( \mathcal{G} \)-measurable random variables, with an order bound topology (or ring topology) which is not a vector space topology. This leads naturally to study risk measures on \( L^0(\mathcal{F}) \) where the \( \sigma \)-algebra \( \mathcal{F} \) contains \( \mathcal{G} \) (see [8] and [21]).

The topologies of the modules themselves can be characterized as those of uniform convergence of the associated \( L^0(\mathcal{G}) \)-seminorms.

Another topology for \( L^0(\mathcal{G}) \)-modules is the one of stochastic convergence with respect to the \( L^0(\mathcal{G}) \)-seminorms, also called \((\varepsilon, \lambda)\)-topology (see [16] and the references cited in [18]). A comparison between the two topologies on \( L^0(\mathcal{G}) \)-modules is given in [18]; in particular, the relations between different separation theorems of convex subsets in the two topologies are studied. Thereby, a crucial technical role is played by different countable concatenation properties. We like to emphasize that in the present paper we do not use any kind of countable concatenation property.

Nevertheless, these investigations study risk measures no longer on bounded random variables or processes but also on unbounded ones, they lead to a somewhat different direction than the one we are looking for. They obtain interesting generalizations for example of the Namioka-Klee theorem and thereby the existence of subgradients to convex functionals on theses modules, but they do not investigate a generalization of the polar theory, as is done for example in [5]. Such an extension is needed for more general Krein-Šmulian theorems.

So, let us study functional analytic problems for modules with \( L^\infty \) spaces as rings. As it appears, this has advantages and disadvantages with respect to the \( L^0 \)-approach mentioned before. Of course the results and the proofs are close to one another, but as often, the devil is in the details. For completeness and the convenience of the reader we give complete proofs most of the time.

After the definition of \( \lambda \)-modules (section 3), we start with a version of the Hahn-Banach theorem for partially defined \( \lambda \)-linear functions bounded by a \( \lambda \)-sublinear function in section 4. This essential tool serves for characterizing locally convex \( \lambda \)-modules by a family of \( \lambda \)-seminorms (section 5 and 6). More important is the consequence of Hahn-Banach type theorems for different kinds of separation theorems between disjoint \( \lambda \)-convex subsets (section 7). The case where a closed \( \lambda \)-convex subset can be separated from an external point with a positive uniform distance is essential for later results.

The core of this paper starts in section 8 with the dual \( \lambda \)-module \( E' \), i.e. the space of all continuous \( \lambda \)-linear functions on a locally convex \( \lambda \)-module \( E \). The completeness of \( E' \) is proved. For the dual pair \( (E, E') \), one defines as usual the \( \lambda \)-weak topology \( \sigma_\lambda(E, E') \) on \( E \) and the \( \lambda \)-weak* topology \( \sigma_{\lambda^*}(E', E) \) on \( E' \). This is the basis to develop the theory of polar sets for subsets of a \( \lambda \)-module . With the developed tools we prove the bipolar theorem for \( \lambda \)-modules in section 9.

Once the bipolar theorem established, we generalize the Krein-Šmulian theorem for \( \lambda \)-modules in section 10. In order to continue with general locally convex \( \lambda \)-modules, we require completeness only with respect to Cauchy nets relative to a \( \lambda \)-convex neighborhood. This seems to be a slightly more general version compared to previous formulations.

Weakening the topology even more to a weak-weak* topology on \( E \), resp. to a weak*-weak* topology on \( E' \), we get a module analogy to the Alaoglu-Bourbaki theorem in section 11. Theorems of this kind can also be found in [17]. Finally, we give in section 12 a module version of Goldstine’s theorem and a characterization of reflexive Banach \( \lambda \)-modules. In [11] we continued the present work by considering an additional order or lattice structure on \( \lambda \)-modules.
2. Rings of random variables

In this section, we follow the ideas of [12], but instead of regarding the subspace $L^0(\mathcal{G})$ with the topology of almost sure dominance as a ring of a module $E$, we use the space of bounded $\mathcal{G}$-measurable random variables with the $L^\infty$-norm as ring. We adopt this space to be able to find a general version of Alaoglu’s theorem. Some, but not all proofs from [12] can be transformed to our situation. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a sub-$\sigma$-algebra $\mathcal{G} \subset \mathcal{F}$. By $(\mathcal{G} \cap A)^+$ we denote the set of all $A' \in \mathcal{G}$ with $A' \subset A$ and $\mathbb{P}(A') > 0$; similar to $\mathcal{F}^+$.

By $L^0$ (resp. $L^0(\mathcal{G})$) we denote the set of all $\mathcal{F}$-measurable (resp. $\mathcal{G}$-measurable) real random variables, while $L^0_+$ is the set of all $\mathcal{F}$-measurable variables with values in $[-\infty, \infty]$. The set $L^0_+$ is the one of non-negative variables in $L^0$ and $L^0_+ := L^0 \cap L^0_+$; similar for $L^0(\mathcal{G})$, $L^0_+(\mathcal{G})$, $L^0_+ \mathcal{G})$. All equalities and inequalities are intended to hold $\mathbb{P}$ almost surely. For random variables $X, Y$, we mean by $X \sim Y$, that $\mathbb{P}(X \sim Y) = 1$. For any space $L$ of random variables we define $L_+ := \{X \in L \mid \exists \varepsilon > 0 \text{ with } X \geq \varepsilon\}$. We always treat 0 in multiplications as dominant element: i.e. $\frac{0}{0} = 0 \cdot \infty = 0 \cdot (-\infty) = 0$.

In the rest of the paper, we consider the dual pair $(\kappa := L^1(\mathcal{G}), \lambda := L^\infty(\mathcal{G}))$. Most of the time a generic element of $\lambda$ will be denoted by $\zeta$. We write $\mathbb{1}$ for the element in $\lambda$ being constant 1. Further, let $\lambda_+ := \{\zeta \in \lambda \mid \zeta \geq 0\}$.

The space $\kappa$, resp. $\lambda$, is equipped with the topology induced by the norm

$$
\|\theta\|_\kappa := \|\theta\|_1 \quad \theta \in \kappa \quad \text{resp.} \\
\|\zeta\|_\lambda := \|\zeta\|_\infty, \quad \zeta \in \lambda.
$$

(2.1)

Thus $\kappa$ and $\lambda$ have countable neighborhood bases of 0.

For $p \in [1, \infty]$ and $\mathcal{G} \subset \mathcal{F}_1 \subset \mathcal{F}$, we introduce the conditional norms $\|\cdot\|_{p, \mathcal{F}_1}$ on $L^0$ by

$$
\|X\|_{p, \mathcal{F}_1} := \begin{cases} 
\lim_{n \to \infty} \mathbb{E}[|X|^p \wedge n \mathcal{F}_1]^{1/p} & \text{for } p < \infty, \\
\text{ess.inf} \left\{\zeta \in L^0_+(\mathcal{F}_1) \mid \zeta \geq |X|\right\} & \text{for } p = \infty
\end{cases}
$$

(2.2)

where $X \in L^0$.

3. $\lambda$-Modules

Our starting point for the study of modules is the following definition:

Definition 3.1.

(i) A $\lambda$-module $E$ is a set with an additive operation $+$ and a multiplication $\cdot$ by the elements of the ring $\lambda$:

A) $E \times E \to E$, $(X_1, X_2) \mapsto X_1 + X_2$ and

B) $\lambda \times E \to E$, $(\zeta, X) \mapsto \zeta \cdot X$.

(ii) We call a $\lambda$-module $E$ topological if $E$ is endowed with a topology $\mathcal{T}$ so that the module operations

(a) and (b) are continuous with respect to the corresponding product topologies.

In the following, we will always consider the $\lambda$-module $E$ as a topological one.

Example 3.1. For $0 < q \leq \infty$, the spaces $L^q$ over the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the $L^q$-norm $\|\cdot\|_q$ are topological $\lambda$-modules.

Let $E$ be a $\lambda$-module. For $X \in E$ and a subset $C \subset E$, we define

$$
X \cap C := \text{ess.sup}\{A \in \mathcal{G}, \mathbb{1}_A \cdot X \in C\}.
$$

(3.1)
If $C$ is a $\lambda$-submodule of $E$, the collection $\{A \in G| \mathbb{I}_A \cdot X \subset C\}$ is directed upwards for all $X \in E$, since $\mathbb{I}_{A_i} \cdot X \subset C$ for $i = 1, 2$ implies $\mathbb{I}_{A_1 \cup A_2} \cdot X = \mathbb{I}_{A_1} \cdot (\mathbb{I}_{A_1} \cdot X) + \mathbb{I}_{A_2} \cdot (\mathbb{I}_{A_2} \cdot X) \subset C$. Therefore, there exists an increasing sequence $(A_n) \subset G$ with

$$\mathbb{I}_{A_n} \cdot X \subset C \quad \text{and} \quad X \cap C = \bigcup_{n \in \mathbb{N}} A_n.$$  

(3.2)

**Definition 3.2.** We say that a set $C \subset E$ has the closure property if

$$\mathbb{I}_{X \cap C} \cdot X \subset C \quad \text{for all } X \in E.$$  

(3.3)

By $\hat{C}$, we denote the smallest subset of $E$ that has closure property and contains $C$. Note that $\hat{C}$ is given by

$$\hat{C} = \{\mathbb{I}_{X \cap C} \cdot X, \ X \in E\}.$$  

(3.4)

For a set $C \subset E$, we denote by

$$\text{span}_\lambda(C) := \{\sum_{i=1}^{n} \zeta_i \cdot X_i | X_i \in C, \zeta_i \in \lambda, 0 \leq i \leq n \in \mathbb{N}\}$$  

(3.5)

the $\lambda$-submodule of $E$ generated by $C$.

**Proposition 3.1.** Let $C \subset E$ be a $\lambda$-submodule of $E$, $X \in E$ and $\overline{X} := \mathbb{I}_{(X \cap C)^c} \cdot X$. Then

(i) $X \cap C = \overline{X} \cap C$,

(ii) If $Y_1 + \zeta_1 \cdot X = Y_2 + \zeta_2 \cdot X$ for some $Y_1, Y_2 \in C$ and $\zeta_1, \zeta_2 \in \lambda$, then we have on $(X \cap C)^c$,

$$Y_1 = Y_2 \quad \text{and} \quad \zeta_1 = \zeta_2.$$

(iii) Let $Z$ be a $\lambda$-linear function from $C$ to $\lambda$ and $\xi \in \mathbb{I}_{(X \cap C)^c} \cdot \lambda$. Then

$$\langle Y + \zeta \cdot \overline{X}, Z \rangle := \langle Y, Z \rangle + \zeta \cdot \xi \quad \text{for all } Y \in C \quad \text{and} \quad \zeta \in \lambda,$$  

(3.6)

defines the unique $\lambda$-linear extension of $Z$ to $\text{span}_\lambda(C, X)$ which satisfies $\langle \overline{X}, Z \rangle = \xi$.

(iv) Suppose now that $C$ has in addition the closure property. Then

$$\text{span}_\lambda(C, X) = \text{span}_\lambda(C, X).$$

Proof. (i) Since $C$ is a $\lambda$-submodule and $X \cap C \subset G$, it follows that $\mathbb{I}_A \cdot X \subset C$ implies $\mathbb{I}_{(X \cap C)^c} \cdot \mathbb{I}_A \cdot X \subset C$. Therefore, $X \cap C \subset X \cap C$. If $P(X \cap C \setminus X \cap C) > 0$, then for some $A \in G^+$, $A \subset \overline{X} \cap C \setminus X \cap C$, $\mathbb{I}_A \cdot \overline{X} \subset C$ but $\mathbb{I}_A \cdot \overline{X} = \mathbb{I}_A \cdot (\mathbb{I}_{(X \cap C)^c} \cdot X) = \mathbb{I}_A \cdot X$. Hence $\mathbb{I}_A \cdot X \subset C$ which contradicts the definition of $X \cap C$.

(ii) Let $Y_1 + \zeta_1 \cdot X = Y_2 + \zeta_2 \cdot X$ for some $Y_1, Y_2 \in C$ and $\zeta_1, \zeta_2 \in \lambda$. Then on $(X \cap C)^c$, we get

$$Y_1 - Y_2 = (\zeta_2 - \zeta_1) \cdot X.$$

Suppose that for some $\varepsilon > 0$ we have $D_\varepsilon := \{\zeta_2 - \zeta_1 \geq \varepsilon\} \cap (X \cap C)^c \in G^+$. Then $\mathbb{I}_{D_\varepsilon} \cdot (\zeta_2 - \zeta_1)^{-1} \subset \lambda$ and $\mathbb{I}_{D_\varepsilon} \cdot X = \mathbb{I}_{D_\varepsilon} \cdot \mathbb{I}_{(X \cap C)^c} \cdot X = \mathbb{I}_{D_\varepsilon} \cdot (\zeta_2 - \zeta_1)^{-1} \cdot (Y_2 - Y_1) \subset C$ since $C$ is a $\lambda$-submodule. But this contradicts the definition of $X \cap C$. Therefore, $\zeta_2 = \zeta_1$ and $Y_1 = Y_2$ on $(X \cap C)^c$.

(iii) If $Y_1 + \zeta_1 \cdot \overline{X} = Y_2 + \zeta_2 \cdot \overline{X}$, it follows by (ii) that $\zeta_1 = \zeta_2$ on $(X \cap C)^c$ and since $\xi \in \mathbb{I}_{(X \cap C)^c} \cdot \lambda$, we have $\zeta_1 \cdot \xi = \zeta_2 \cdot \xi$ and also

$$\langle Y_1, Z \rangle = \langle Y_1 + \zeta_1 \cdot X, Z \rangle - \zeta_1 \cdot \xi = \langle Y_2 + \zeta_2 \cdot X, Z \rangle - \zeta_2 \cdot \xi = \langle Y_2, Z \rangle.$$  

This shows that $\overline{Z}$ is well defined. The uniqueness of $\overline{Z}$ follows from the right-hand side of (3.6) with $\xi$ fixed.

(iv) The fact that we have $\mathbb{I}_{(X \cap C)^c} \cdot X = \overline{X}$ implies that $\text{span}_\lambda(C, X) \subset \text{span}_\lambda(C, X)$.

If $C$ has the closure property then $\mathbb{I}_{(X \cap C)^c} \cdot X \subset C$ and since $X = \mathbb{I}_{(X \cap C)^c} \cdot X + \overline{X}$, we get $\text{span}_\lambda(C, X) \subset \text{span}_\lambda(C, \overline{X})$. \hfill $\square$
4. Hahn-Banach theorem for $\lambda$-modules

As mentioned before, extension results of the classical Hahn-Banach theorem have existed for a long time in several directions. In the case $L^0$-modules, extension theorems are given in [12] and [18]. We state and prove the corresponding theorem for $\lambda$-modules in the form we need it in the following sections. Let’s start with the definition of sublinear functions.

**Definition 4.1.** Let $E$ be a $\lambda$-module. A function $q : E \to \lambda$ is $\lambda$-sublinear if:

(i) $q(\zeta \cdot X) = \zeta \cdot q(X)$ for all $X \in E$ and $\zeta \in \lambda_+$.

(ii) $q(X_1 + X_2) \leq q(X_1) + q(X_2)$ for all $X_1, X_2 \in E$.

If we have equality in (ii), then the function $q$ is $\lambda$-linear.

**Theorem 4.1.** Consider a $\lambda$-sublinear function $q : E \to \lambda$, a $\lambda$-submodule $C$ of $E$ and a $\lambda$-linear function $Z : C \to \lambda$ so that

$$\langle X, Z \rangle \leq q(X) \quad \text{for all } X \in C.$$  

Then $Z$ extends to a $\lambda$-linear function $Z : E \to \lambda$ so that $\langle X, Z \rangle \leq q(X)$ for all $X \in E$.

Before proceeding to the proof of the theorem 4.1, we need the following result.

**Lemma 4.2.** Let $C, Z, q$ as in theorem 4.1. Then $Z$ extends uniquely to a $\lambda$-linear function $\hat{Z} : \hat{C} \to \lambda$ so that $\langle X, \hat{Z} \rangle \leq q(X)$ for all $X \in \hat{C}$.

**Proof.** For $X \in E$, let

$$\langle \mathbb{I}_{(X\cap C)} \cdot X, \hat{Z} \rangle := \lim_{n \to \infty} \langle \mathbb{I}_{M_n} \cdot X, Z \rangle,$$

where $X \cap C = \cup_{n \in \mathbb{N}} M_n$ as in (3.2). Since for all $n \leq m$

$$\langle \mathbb{I}_{M_n} \cdot X, Z \rangle = \langle \mathbb{I}_{M_m} \cdot X, Z \rangle \quad \text{on } M_n,$$

(4.1) defines the $\lambda$-linear extension $\hat{Z} : \hat{C} \to \lambda$ of $Z$ to $\hat{C}$. Further, $\langle \mathbb{I}_{M_n} \cdot X, Z \rangle \leq q(\mathbb{I}_{M_n} \cdot X) = \mathbb{I}_{M_n} \cdot q(X)$ implies that $\langle \mathbb{I}_{(X\cap C)} \cdot X, Z \rangle \leq \mathbb{I}_{(X\cap C)} \cdot q(X) = q(\mathbb{I}_{(X\cap C)} \cdot X)$ for all $X \in \hat{C}$. \hfill $\square$

**Proof.** of theorem 4.1:

Step 1 (one-step extension): In view of lemma 4.2, we can assume that $C$ has the closure property. Suppose now that $X \in E \setminus C$. Then $\overline{X} = \mathbb{I}_{(X\cap C)} + X \not\in C$ and $X \not= 0$. We will show that $Z$ extends to a $\lambda$-linear function $Z : \text{span}_\lambda(C, X) \to \lambda$ so that

$$\langle Y, Z \rangle \leq q(Y) \quad \forall Y \in \text{span}_\lambda(C, X).$$

Let $\xi = \mathbb{I}_{(X\cap C)^c} \text{ ess.} \inf_{Y \in C} \{q(\overline{X} + Y) - \langle Y, Z \rangle \}$ and define

$$\langle Y + \zeta \cdot \overline{X}, Z \rangle := \langle Y, Z \rangle + \zeta \cdot \xi \quad \forall Y \in C \text{ and } \zeta \in \lambda.$$

By proposition 3.1, $Z$ is the unique extension of $Z$ to $\text{span}_\lambda(C, X) = \text{span}_\lambda(C, X)$ with $\langle \overline{X}, Z \rangle = \xi$. Moreover on $D_\varepsilon = \{ |\zeta| \geq \varepsilon \} \cap (X \cap C)^c \in \mathcal{G}$, we have $\mathbb{I}_{D_\varepsilon} \zeta^{-1} \in \lambda$ and for $Y \in C$

$$\mathbb{I}_{D_\varepsilon} \cdot \langle Y + \zeta \cdot \overline{X}, Z \rangle = \mathbb{I}_{D_\varepsilon} \cdot \langle Y, Z \rangle + \mathbb{I}_{D_\varepsilon} \cdot \zeta \cdot \xi = \mathbb{I}_{D_\varepsilon} \cdot \zeta \cdot (\langle \mathbb{I}_{D_\varepsilon} \cdot \zeta^{-1} \cdot Y, Z \rangle + \mathbb{I}_{D_\varepsilon} \cdot \xi) \leq \mathbb{I}_{D_\varepsilon} \cdot \zeta \cdot (\langle \mathbb{I}_{D_\varepsilon} \cdot \zeta^{-1} \cdot Y, Z \rangle + \mathbb{I}_{D_\varepsilon} \cdot q(\overline{X} + \mathbb{I}_{D_\varepsilon} \cdot \zeta^{-1} \cdot Y) - \langle \mathbb{I}_{D_\varepsilon} \cdot \zeta^{-1} \cdot Y, Z \rangle) = \mathbb{I}_{D_\varepsilon} \cdot q(\zeta \cdot \overline{X} + Y).$$
Therefore on \((X \cap C)^c\), we get \(<Y + \zeta \cdot \overline{X}, \overline{Z}> \leq q(Y + \zeta \cdot \overline{X})\) and since \(\mathbf{1}_{(X \cap C)_c} \overline{X} = 0\), we have on \(Z \cap C, \langle Y, \overline{Z} \rangle = \langle Y, \overline{Z} \rangle \leq q(Y)\). This means
\[
\langle Y + \zeta \cdot \overline{X}, \overline{Z} \rangle \leq q(Y + \zeta \cdot \overline{X}) \quad \forall Y \in C \text{ and } \zeta \in \lambda.
\]
We have extended \(Z\) from \(C\) to \(\overline{Z}\) on \(\operatorname{span}_\lambda(C, \overline{X}) = \operatorname{span}_\lambda(C, X)\) satisfying \(Z \leq q\) on \(\operatorname{span}_\lambda(C, X)\).

Step 2: The set
\[
\mathcal{I} := \left\{ (D, \overline{Z}) : \text{D with } C \subseteq D \subseteq E \text{ is a submodule of } E \text{ with closure property and } \overline{Z} \text{ is } \lambda - \text{linear on } D, \overline{Z} \upharpoonright C = Z \text{ and } \overline{Z} \leq q \text{ on } D \right\}
\]
is partially ordered by
\[
(D, \overline{Z}) \leq (D', \overline{Z}') \quad \text{if and only if} \quad D \subseteq D' \quad \overline{Z}' \upharpoonright D = \overline{Z}.
\]
If for some \(I \subseteq \mathcal{I}\) the family \(\{(D_i, \overline{Z}_i)_{i \in I}, \leq\}\) is totally ordered, then \(D := \bigcup_{i \in I} D_i \subseteq E\) is a \(\lambda\)-module containing \(C\). The hypotheses of lemma 4.2 are verified. Indeed, for \(\overline{Z} : D \to \lambda\) given by \(\overline{Z} \upharpoonright D_i := \overline{Z}_i\), we have \(\overline{Z} \upharpoonright C = Z\) is \(\lambda\)-linear and \(\overline{Z} \leq q\) on \(D\). So we can assume that \(D\) has the closure property and thus
\[
(D_i, \overline{Z}_i) \leq (D, \overline{Z}) \quad \forall i \in I.
\]
Hence \((D, \overline{Z})\) is an upper bound for the family \((D_i, \overline{Z}_i)_{i \in I}\). Now, we apply Zorn’s lemma to have the existence of a maximal element \((D_{\max}, \overline{Z}_{\max}) \in \mathcal{I}\), i.e.
\[
(D_{\max}, \overline{Z}_{\max}) \leq (D, \overline{Z}) \in \mathcal{I} \quad \text{implies} \quad (D_{\max}, \overline{Z}_{\max}) = (D, \overline{Z}).
\]
In order to show that \(D_{\max} = E\), suppose that \(D_{\max} \neq E\). By the first step of the proof \(\overline{Z}_{\max}\) extends to
\[
\overline{Z}'_{\max} : \operatorname{span}_\lambda(D_{\max}, X) \to \lambda.
\]
for any \(X \in E \setminus D_{\max}\). This contradicts the maximality of \((D_{\max}, \overline{Z}_{\max})\). Hence \(D_{\max} = E\). \(\square\)

5. Gauge functions

We start with some well-known definitions.

**Definition 5.1.** Let \(D\) be a subset of the \(\lambda\)-module \(E\). We call \(D\)

(i) \(\lambda\)-**absorbing** if for all \(X \in E\) there is \(\zeta \in \lambda_+\) so that \(X \in \zeta \cdot D\),

(ii) \(\lambda\)-**balanced** if \(\zeta \cdot X \in D\) for all \(X \in E\) and \(\zeta \in \lambda\) with \(||\zeta||_\lambda \leq 1\),

(iii) \(\lambda\)-**convex** if \(\zeta \cdot X_1 + (1 - \zeta) \cdot X_2 \in D\) for all \(X_1, X_2 \in D\) and \(\zeta \in \lambda\) with \(0 \leq \zeta \leq 1\).

Let us first recall that the properties of a set \(D\) to be \(\lambda\)-convex and \(\lambda\)-balanced can be combined:

**Lemma 5.1.** A subset \(D\) of the \(\lambda\)-module \(E\) is \(\lambda\)-convex and \(\lambda\)-balanced if and only if it is \(\lambda\)-**absolute-convex** in the following sense:

\[
\text{For all } X_1, X_2 \in D \text{ and } \zeta_1, \zeta_2 \in \lambda \text{ with } |\zeta_1| + |\zeta_2| \leq 1: \quad \zeta_1 \cdot X_1 + \zeta_2 \cdot X_2 \in D. \quad (5.1)
\]

**Proof.** We just need to show the necessity of condition (5.1); its sufficiency is obvious. So for \(i = 1, 2\) let \(X_i \in D\) and \(\zeta_i \in \lambda\) with \(|\zeta_1| + |\zeta_2| \leq 1\). We define \(\zeta' := |\zeta_1| / (|\zeta_1| + |\zeta_2|) \leq 1\) and \(X'_i := \mathbf{1}_{\{\zeta \geq 0\}} \cdot X_i + \mathbf{1}_{\{\zeta < 0\}} \cdot (-X_i)\) so that
\[
\zeta_1 \cdot X_1 + \zeta_2 \cdot X_2 = |\zeta_1| \cdot X'_1 + |\zeta_2| \cdot X'_2 = (|\zeta_1| + |\zeta_2|) \left( \zeta' \cdot X'_1 + (1 - \zeta') \cdot X'_2 \right).
\]
Definition 5.2. For a subset \( D \) of \( E \), the gauge function \( q_D := q_D|\cdot| : E \to \lambda_+ \) is defined by
\[
q_D|X| := \operatorname{ess}\inf\{\lambda|X| : \lambda \in \lambda_+ \text{ and } X \in \lambda_+\}.
\] (5.2)

Proposition 5.3. Let \( D \) be a \( \lambda \)-absorbent subset of \( E \). Then the gauge function \( q_D \) satisfies for all \( X \in E \), \( A \in \mathcal{G} \), and \( \lambda \in \lambda_+ \):

(i) \( q_D|X| \leq 1 \) if \( X \in D \).

(ii) \( \mathbb{I}_A \cdot q_D|\mathbb{I}_A \cdot X| \leq \mathbb{I}_A \cdot q_D|X| \).

(iii) \( \lambda \cdot q_D|\lambda|X| = q_D|\lambda \cdot X| \).

in particular \( \lambda \cdot q_D|X| = q_D|\lambda \cdot X| \) if \( \lambda \in \lambda_+ \).

Proof. (i) We use the definition of \( q_D \) in particular for \( \lambda = 1 \in \lambda_+ \).

(ii) For \( X \in E \) and \( A \in \mathcal{G} \) we have
\[
\mathbb{I}_A \cdot q_D|X| = \mathbb{I}_A \cdot \operatorname{ess}\inf_{X \in D} \lambda = \mathbb{I}_A \cdot \mathbb{I}_A \cdot q_D|X| \]
\[
\geq \mathbb{I}_A \cdot \operatorname{ess}\inf_{X \in D} \mathbb{I}_A \cdot \lambda = \mathbb{I}_A \cdot \mathbb{I}_A \cdot \operatorname{ess}\inf_{X \in D} \mathbb{I}_A \cdot \lambda = \mathbb{I}_A \cdot \mathbb{I}_A \cdot q_D|\lambda \cdot X|.
\]

(iii) Let \( X \in E \), \( \lambda \in \lambda_+ \) and for \( \varepsilon \geq 0 \) define \( A_\varepsilon := \{\lambda \geq \varepsilon\} \in \mathcal{G} \). It suffices to show that for all \( \varepsilon > 0 \) we find \( \mathbb{I}_{A_\varepsilon} \cdot q_D|X| = \mathbb{I}_{A_\varepsilon} \cdot q_D|\lambda \cdot X| \). Using the fact that \( \mathbb{I}_{A_\varepsilon} \cdot \lambda \in \lambda_+ \), we have
\[
\mathbb{I}_{A_\varepsilon} \cdot \lambda \cdot q_D|X| = \mathbb{I}_{A_\varepsilon} \cdot \lambda \cdot \operatorname{ess}\inf_{X \in D} \lambda = \mathbb{I}_{A_\varepsilon} \cdot \lambda \cdot \operatorname{ess}\inf_{X \in D} \lambda = \mathbb{I}_{A_\varepsilon} \cdot \lambda \cdot q_D|\lambda \cdot X|.
\]

\[\square\]

Proposition 5.3. Let \( D \) be a \( \lambda \)-absorbent and \( \lambda \)-convex subset of \( E \). We first remark that \( 0 \in D \) and therefore
\[
\zeta \cdot D \subset D
\] (5.3)
for all \( \zeta \in \lambda \) with \( 0 \leq \zeta \leq 1 \).

For such a subset \( D \) the gauge function \( q_D \) satisfies for all \( X \), \( X_1 \), \( X_2 \in E \) and \( \zeta \in \lambda_+ \):

(i) \( q_D|X| = \operatorname{ess}\inf\{\zeta' \in \lambda_+| X \in \zeta' \cdot D\} \).

(ii) \( \zeta \cdot q_D|X| = q_D|\zeta \cdot X| \).

(iii) \( q_D|X_1 + X_2| \leq q_D|X_1| + q_D|X_2| \).

Parts (ii) and (iii) imply that \( q_D \) is a \( \lambda \)-sublinear function.

(iv) there exists a sequence \( (\zeta_n)_n \) in \( \lambda \) so that
\[
\zeta_n \wedge q_D|X| \text{ a.s.}
\] (5.4)

(v) If in addition \( D \) is \( \lambda \)-balanced then \( q_D \) satisfies for all \( \zeta \in \lambda \) and \( X \in E \):
\[
q_D|\zeta \cdot X| = |\zeta| \cdot q_D|X|.
\] (5.5)
(vi) For all $A \in \mathcal{G}^+$ and all $X \in E$ with $\mathbb{1}_B \cdot X \notin \mathbb{1}_B \cdot D$ for all $B \subset A, B \in \mathcal{G}^+$ we have
\[
\mathbb{1}_A \cdot q_D |X| \geq \mathbb{1}_A.
\] (5.6)

Proof. By the $\lambda$-absorbing of $D$, we find for any $X \in E, \zeta_1, \zeta_2 \in \lambda_+$ and $Y_1, Y_2 \in D$ with $X = \zeta_1 \cdot Y_1$ and $-X = \zeta_2 \cdot Y_2$. With the convention $0/0 = 0$ we have that $\xi := \zeta_1/(\zeta_1 + \zeta_2) \in \lambda_+$ with $0 \leq \xi \leq 1$. Therefore, $0 = X - X = \xi \cdot Y_1 + (1 - \xi) \cdot Y_2 \in D$. The relation (5.3) follows now immediately.

(i) " $\leq $" follows from the definition of $q_D$. To prove the reverse inequality, let $\zeta \in \lambda_+$ with $X = \zeta \cdot Y$ for some $Y \in D$. For $0 < \varepsilon \in \mathbb{R}$ we set $A := \{\zeta > \varepsilon\} \in \mathcal{G}$ and $\zeta_\varepsilon := \mathbb{1}_{A_\varepsilon} \cdot \zeta + \varepsilon \cdot \mathbb{1}_{A_\varepsilon^c} \in \lambda_\varepsilon$ to have with (5.3)
\[
X = (\mathbb{1}_{A_\varepsilon} \cdot \zeta + \mathbb{1}_{A_\varepsilon^c} \cdot \varepsilon) \cdot Y \in \zeta_\varepsilon \cdot (\mathbb{1}_{A_\varepsilon} + \mathbb{1}_{A_\varepsilon^c} \cdot \varepsilon) \cdot D \subset \zeta_\varepsilon \cdot D.
\]
Since $\inf_{\varepsilon>0} \zeta_\varepsilon = \zeta$, we get $q_D |X| \geq \text{ess.inf.}\{\zeta' \in \lambda_\varepsilon | X \in \zeta' \cdot D\}$.

(ii) We first show that for all $X \in E$ and $A \in \mathcal{G}$
\[
\mathbb{1}_A \cdot q_D |\mathbb{1}_A \cdot X| = \mathbb{1}_A \cdot q_D |X|.
\] (5.7)

From proposition 5.2 (ii) we know that $\mathbb{1}_A \cdot q_D |\mathbb{1}_A \cdot X| \leq \mathbb{1}_A \cdot q_D |X|$. For the inverse inequality let $\mathbb{1}_A \cdot X = \mathbb{1}_A \cdot \zeta \cdot Y$ for some $\zeta \in \lambda_+$ and $Y \in D$. If we can show that $\mathbb{1}_A \cdot \zeta \geq \mathbb{1}_A \cdot q_D |X|$, it follows by taking the ess.inf over such $\zeta$s that $\mathbb{1}_A \cdot q_D |\mathbb{1}_A \cdot X| \geq \mathbb{1}_A \cdot q_D |X|$ and (5.7) is proved. Now we consider the representation $X = \zeta' \cdot Y'$ with $\zeta' \in \lambda_+$ and $Y' \in D$. Obviously, $\mathbb{1}_{A_\varepsilon} \cdot X = \mathbb{1}_{A_\varepsilon} \cdot \zeta' \cdot Y'$ which implies
\[
X = (\mathbb{1}_{A_\varepsilon} + \mathbb{1}_{A_\varepsilon^c}) \cdot X = \mathbb{1}_{A_\varepsilon} \cdot \zeta \cdot Y + \mathbb{1}_{A_\varepsilon^c} \cdot \zeta' \cdot Y' = (\mathbb{1}_{A_\varepsilon} \cdot \zeta + \mathbb{1}_{A_\varepsilon^c} \cdot \zeta') \cdot (\mathbb{1}_{A_\varepsilon} \cdot Y + \mathbb{1}_{A_\varepsilon^c} \cdot Y') \in (\mathbb{1}_{A_\varepsilon} \cdot \zeta + \mathbb{1}_{A_\varepsilon^c} \cdot \zeta') \cdot D,
\]
since $(\mathbb{1}_{A_\varepsilon} \cdot Y + \mathbb{1}_{A_\varepsilon^c} \cdot Y') \in D$ by the $\lambda$-convexity of $D$. Therefore, $\mathbb{1}_A \cdot q_D |X| \leq \mathbb{1}_A \cdot (\mathbb{1}_{A_\varepsilon} \cdot \zeta + \mathbb{1}_{A_\varepsilon^c} \cdot \zeta') = \mathbb{1}_A \cdot \zeta$, the inequality we needed to complete the proof of (5.7).

Now proposition 5.2 (iii) together with (5.7) yields
\[
q_D |\zeta \cdot X| = \zeta \cdot q_D |\mathbb{1}_{\{\zeta>0\}} \cdot X| = \zeta \cdot \mathbb{1}_{\{\zeta>0\}} \cdot q_D |X| = \zeta \cdot q_D |X|.
\]

(iii) Let $X_1, X_2 \in E$ have the representations $X_i = \zeta_i \cdot Y_i$ with $\zeta_i \in \lambda_\varepsilon$ and $Y_i \in D$, $i = 1, 2$. We set $\xi := \zeta_1/(\zeta_1 + \zeta_2) \leq 1$ so that $\xi \in \lambda_\varepsilon$. The $\lambda$-convexity of $D$ implies $(X_1 + X_2)/(\zeta_1 + \zeta_2) = \xi \cdot Y_1 + (1 - \xi) \cdot Y_2 \in D$. By proposition 5.2 (i) it follows further that $1 \geq q_D |(X_1 + X_2)/(\zeta_1 + \zeta_2)|$. Since $(\zeta_1 + \zeta_2)^{-1} \in \lambda_\varepsilon$, proposition 5.2 (iii) implies
\[
1 \geq q_D |(X_1 + X_2)/(\zeta_1 + \zeta_2)| = (\zeta_1 + \zeta_2)^{-1} \cdot q_D |X_1 + X_2|
\]
This shows $\zeta_1 + \zeta_2 \geq q_D |X_1 + X_2|$. Taking the ess.inf’s over $\zeta_i \in \lambda_\varepsilon$, $i = 1, 2$, in part (i) we find $q_D |X_1| + q_D |X_2| \geq q_D |X_1 + X_2|$. (iv) If $X$ has two representations $X = \zeta_1 \cdot Y_1$ with $\zeta_1 \in \lambda_+$ and $Y_1 \in D$, $i = 1, 2$ and $A := \{\zeta_1 < \zeta_2\}$, then $X = (\mathbb{1}_A \cdot \zeta_1 + \mathbb{1}_{A_\varepsilon} \cdot \zeta_2)(\mathbb{1}_A \cdot Y_1 + \mathbb{1}_{A_\varepsilon} \cdot Y_2) = \zeta' \cdot Y'$ with $\zeta' \in \lambda_+$ and $Y' \in D$ shows that the system $\{\zeta \in \lambda_+ | X = \zeta \cdot Y \in D\}$ is downward directed. (It is also upward directed, which however is not of great interest).

(v) For a $\lambda$-balanced $D$ we have $D = -D$ with implies $q_D |-X| = q_D |X| = q_D |X|$. Therefore, the element $\zeta \cdot X$ in $E, \zeta \in \lambda$ and $X \in E$ can be rewritten with $A := \{\zeta \geq 0\}$ as $\mathbb{1}_A |\zeta| \cdot X + \mathbb{1}_{A^c} |\zeta| \cdot (-X)$. Now (5.7) and (ii) imply
\[
q_D |\zeta \cdot X| = \mathbb{1}_A \cdot q_D |\mathbb{1}_A |\zeta| \cdot X| + \mathbb{1}_{A^c} \cdot q_D |\mathbb{1}_{A^c} |\zeta| \cdot (-X)|
= \mathbb{1}_A \cdot |\zeta| \cdot q_D |X| + \mathbb{1}_{A^c} \cdot |\zeta| \cdot q_D |-X| = |\zeta| \cdot q_D |X|.
\]

(vi) Suppose $A \cap \{q_D |X| < 1\} \in \mathcal{G}^+$. This means that there exists $\zeta \in \lambda_+$ with $B := A \cap \{\zeta < 1\} \in \mathcal{G}^+$ and $\mathbb{1}_A \cdot X \in \mathbb{1}_A \cdot \zeta \cdot D$. But by (5.3) it follows $\mathbb{1}_B \cdot X \in \mathbb{1}_B \cdot \zeta \cdot D \subset \mathbb{1}_B \cdot D$, contradicting the assumption. □
The preceding proof of part (i) shows the following stronger version of absorption:

**Corollary 5.4.** If $D$ is $\lambda$-absorbent and -convex, then it is also $\lambda_\#$-absorbent:

$$\text{For all } X \in E \text{ there exists } \zeta \in \lambda_\# \text{ with } X \in \zeta \cdot D. \quad (5.8)$$

### 6. Seminorms and locally convex $\lambda$-modules

If the subset $D$ is a $\lambda$-convex, -absorbent, and -balanced subset of $E$, then its gauge function $q_D$ is a $\lambda$-seminorm in the following sense:

**Definitions 6.1.**

(i) A function $q := q|\cdot| : E \to \lambda_+$ is a $\lambda$-seminorm on $E$ if $X, X_1, X_2 \in E$ and $\zeta \in \lambda$:

(i.a) $q|\zeta \cdot X| = |\zeta| \cdot q|X|,$

(i.b) $q|X_1 + X_2| \leq q|X_1| + q|X_2|.$

Moreover

(iii.a) if $q|X| = 0$ implies $X = 0$,

then $q$ is a $\lambda$-norm on $E$. In this case $(E, q)$ is called a **normed $\lambda$-module**.

(iv) For $\eta \in \lambda_\#$, we let

$$B_{q,\eta}(X) := \{Y \in E| q|Y - X| \leq \eta\} \quad (6.1)$$

denote the $q$-ball of radius $\eta$ around $X \in E$. We write $B_{q,\eta}(0)$ and $B_{q,r}(X) := B_{q,r\|X\|}(X)$ for $r \in \mathbb{R}$.

Indeed, proposition 5.3 (iii) and (v) show that $q_D$ is a seminorm for a $\lambda$-convex, -absorbent, and -$\lambda$-balanced subset $D$.

Conversely, if $q$ is a $\lambda$-seminorm then for each $\eta \in \lambda_\#$ the $q$-ball $B_{q,\eta}$ is a $\lambda$-convex, -absorbent, and -$\lambda$-balanced subset of $E$. Since the class of $\lambda$-convex, -absorbent, and -$\lambda$-balanced subset of $E$ is closed under finite intersection, we can generalize the simple set $B_{q,\eta}$ to the following system:

Let $Q$ be a family of $\lambda$-seminorms on $E$. Then

$$U_Q := \left\{ B_Q f, \eta \left| f \text{ is a finite subset of } Q \text{ and } \eta \in \lambda_\# \right. \right\} \quad \text{where} \quad (6.2)$$

$$B_{Q f,\eta} := \left\{ X \in E| \sup_{q \in Q f} q|X| \leq \eta \right\} \quad (6.3)$$

defines a system of $\lambda$-convex, -absorbent, and -$\lambda$-balanced subset of $E$, closed under finite intersections. If we generate a topology $T_Q$ on the $\lambda$-module $E$ by using the system $U_Q$ as a neighborhood base of $0 \in E$, then the properties (i) and (ii) of definition 6.1 show that $E$ is a topological $\lambda$-module. Therefore $E$ with the topology $T_Q$ is a locally convex $\lambda$-module in the sense of the following definition.

**Definition 6.2.** The $\lambda$-module $E$ is **locally convex** if it has a neighborhood base $U$ of $0 \in E$ with the following properties:

(i) $0 \in U$ for all $U \in U$,

(ii) $U$ is downward filtrated: for all $U_1, U_2 \in U$ there exists $U \in U$ with $U \subset U_1 \cap U_2$,

(iii) for all $U \in U$ there exists $U' \in U$ with $U' + U' \subset U$,

(iv) for all $U \in U$ and $\zeta \in \lambda_\#$ there exists $U' \in U$ with $\zeta \cdot U' \subset U$,

(v) - (vii) all $U \in U$ are $\lambda$-absorbent, $\lambda$-balanced, and $\lambda$-convex,
and its topology \( \mathcal{T}_U \) is defined by the fact that a subset \( O \subset E \) is open if and only if for all \( X \in O \) there exists \( U \in \mathcal{U} \) with \( X + U \subset O \).

It is well known that the properties (i) to (vi) imply that \( (E, \mathcal{T}_U) \) is a topological \( \lambda \)-module in the sense of definition 3.1 (see e.g. [27] chapter 8, the proof given there can be transferred literally to our case if we only replace \( \varepsilon > 0 \) by \( \eta \in \lambda_\varepsilon \) using corollary 5.4).

**Proposition 6.1.** Let \( E \) be a locally convex \( \lambda \)-module with a neighborhood base \( \mathcal{U} \) of 0 \( \in E \) as in definition 6.2. Then the topology \( \mathcal{T}_U \) is Hausdorff if and only if

\[
\bigcap_{U \in \mathcal{U}} U = \{0\}.
\]  

(6.4)

**Proof.** The necessity of condition (6.4) is obvious. Conversely, if \( X \neq X' \) in \( E \), then there exists \( U \in \mathcal{U} \) with \( X - X' \not\subset U \). By property (ii) of definition 6.2 we get \( U' \in \mathcal{U} \) with \( U' + U'' \subset U \). Now, \( (X + U') \cap (X' + U'') = \emptyset \) since \( (X' + U') - (X + U') \subset (X' - X) + U \) and \( 0 \notin (X' - X) + U \). \( \square \)

**Assumption 6.1.** If we consider in the following sections a locally convex \( \lambda \)-module \( E \) we always assume condition (6.4) to be satisfied.

Having shown that a family \( \Omega \) of \( \lambda \)-semimodules creates via (6.2) a neighborhood base \( \mathcal{U}_\Omega \) of a locally convex topology on \( E \), we now want to show that any locally convex \( \lambda \)-module is indeed given by a family of \( \lambda \)-semimodules:

**Theorem 6.2.** A topological \( \lambda \)-module \( (E, T) \) is locally convex if and only if \( T \) is induced by a family \( \Omega \) of \( \lambda \)-semimodules.

**Proof.** If \( \mathcal{U} \) is a neighborhood base as in definition 6.2 then proposition 5.3 shows that \( \Omega := \{q_U|U \in \mathcal{U}\} \) is a family of semimodules. Moreover, the fact that \( q_U \geq q_{U_1 \cap U_2} \geq \sup(q_{U_1}, q_{U_2}) \) for \( U \subset U_1 \cap U_2 \) shows that the topology generated by \( \Omega \) is invariant if we replace \( \Omega \) by \( \Omega' := \{\sup_{i \leq n} q_{U_i}|U_i \in \mathcal{U}, i \leq n \in \mathbb{N}\} \). Finally, since by proposition 5.3 (i) and (vi)

\[
\frac{1}{2} \cdot U \subset \{X|q_U|X| \leq \frac{1}{2}\} \subset \{X|q_U|X| < 1\} \subset U \subset \{X|q_U|X| \leq 1\}
\]

we conclude that \( \mathcal{U} \) and \( \mathcal{U}_\Omega' \) generate the same topology. \( \square \)

For a \( \lambda \)-semimodule \( q \in \Omega \) it is evident that \( \{X|q|X| \leq \eta\} \subset \{X|\mathbf{I}_A \cdot q|X| \leq \eta\} \) for all \( A \in \mathcal{G}^+ \). Of course, \( \mathbf{I}_A \cdot q \) is equally a \( \lambda \)-semimodule. In view of theorem 6.2, there is no restriction in assuming that the family \( \Omega \) of \( \lambda \)-semimodules is closed under \( q \mapsto \mathbf{I}_A \cdot q \) for all \( A \in \mathcal{G}^+ \). Even more: since we use in (6.3) the \( \sup_{\Omega_f} \) over finite subsets \( \Omega_f \) of \( \Omega \), we can and do make the following assumption:

**Assumption 6.2.** If \( q_i \in \Omega \) and \( A_i \in \mathcal{G}^+ \) with \( A_i \cap A_j = \emptyset \) for all \( i \neq j \leq n \), then \( \sum_{i=1}^n \mathbf{I}_{A_i} \cdot q_i \in \Omega \).

The locally convex topology on the \( \lambda \)-module \( E \) is now defined by the neighborhood base

\[
\mathcal{U}_\Omega = \{B_{q,\eta} := \{X \in E|q|X| \leq \eta\}|q \in \Omega, \eta \in \lambda_\eta\}.
\]  

(6.5)

We still note the following result for \( \lambda \)-absorbent and -convex subset of \( E \):

**Proposition 6.3.** Let \( E \) be a locally convex \( \lambda \)-module and \( D \) a \( \lambda \)-absorbent and \( \lambda \)-convex subset of \( E \). For \( A \in \mathcal{G}^+ \) and all \( X \in E \)

\[
\text{if } \mathbf{I}_A \cdot X \in \mathbf{I}_A \cdot D \text{ then there exists } \zeta \in \lambda_\zeta, \zeta \leq 1 \text{ with } \mathbf{I}_A \cdot q_D|X| \leq \mathbf{I}_A \cdot (1 - \zeta)
\]

(6.6)

where \( D \) denotes the interior of \( D \).
Proof. Let the topology of $E$ be given by $\mathcal{U}_\Omega$ for a family $\Omega$ of $\lambda$-seminorms, satisfying assumption 6.2. Now, $\mathbb{I}_A \cdot X \in \mathbb{I}_A \cdot D$ means that there exists $q \in \Omega$ and $\eta \in \lambda_1$ with $\mathbb{I}_A \cdot (X + B_{q,\eta}) \in \mathbb{I}_A \cdot D$. Corollary 5.4 implies that $X \in \xi \cdot B_{q,\eta}$ for some $\xi \in \lambda_2$. Therefore $\mathbb{I}_A \cdot (1 + \xi^{-1}) \cdot X \in \mathbb{I}_A \cdot D$. With $\zeta := 1/(1 + \xi) \in \lambda_2$ we get

$$\mathbb{I}_A \cdot q_D \cdot |X| \leq \mathbb{I}_A \cdot \frac{1}{1 + \xi^{-1}} = \mathbb{I}_A \cdot (1 - \zeta).$$

\[\Box\]

7. Separation theorems for convex subsets

In the following sections we study locally convex $\lambda$-modules $E$ whose topology is induced by a set $\Omega$ of $\lambda$-seminorms, satisfying the assumption 6.2 (assumption 6.1 is tacitly supposed). We refer to it as $(E, \Omega)$. Our first result is the following separation theorem:

**Theorem 7.1.** Let $E$ be a locally convex $\lambda$-module and $A \in \mathcal{G}^+$. For a $\lambda$-convex non-empty subsets $C$ and $D$ of $E$ we suppose that $\mathbb{I}_A \cdot D$ is relatively open in $\mathbb{I}_A \cdot E$ and that

$$\mathbb{I}_B \cdot C \cap \mathbb{I}_B \cdot D = \emptyset \quad \text{for all } B \in (\mathcal{G} \cap A)^+. \quad (7.1)$$

Then there exists a continuous $\lambda$-linear function $Z : E \to \lambda$ with $\langle \hat{X}, Z \rangle = \mathbb{I}_A$ for some $\hat{X} \in E$ and for all $X \in C$ and $Y \in D$ there exists $\eta \in \lambda_2$ so that

$$\mathbb{I}_A \cdot (Y, Z) \leq \mathbb{I}_A \cdot ((X, Z) - \eta). \quad (7.2)$$

As a consequence, we get

$$\mathbb{I}_A \cdot \text{ess sup}_{Y \in D} \langle Y, Z \rangle \leq \mathbb{I}_A \cdot \text{ess inf}_{X \in C} \langle X, Z \rangle. \quad (7.3)$$

**Proof.** 1.Step: Let $C = \{\hat{X}\}$ for some $\hat{X} \in E$. By a translation of both $C$ and $D$ we can suppose that $0 \in D$ which implies that $\mathbb{I}_A \cdot D$ is $\lambda$-convex and $\mathbb{I}_A$-absorbent for $\mathbb{I}_A \cdot E$. By hypothesis $\mathbb{I}_B \cdot \hat{X} \notin \mathbb{I}_B \cdot D$ for all $B \in (\mathcal{G} \cap A)^+$. Now $\mathbb{I}_A \cdot \zeta_1 \cdot \hat{X} = \mathbb{I}_A \cdot \zeta_2 \cdot \hat{X}$ implies $\mathbb{I}_A \cdot \zeta_1 = \mathbb{I}_A \cdot \zeta_2$ since otherwise for $i = 1$ or 2 and some $\varepsilon > 0$ we get $B := \{\varepsilon \leq \zeta_i - \zeta_{3-i}\} \cap A \in \mathcal{G}^+$ so that $\mathbb{I}_B \cdot \hat{X} = \mathbb{I}_B \cdot (\zeta_i - \zeta_{3-i})^{-1}(\zeta_i - \zeta_{3-i}) \cdot \hat{X} = 0 \in \mathbb{I}_B \cdot D$, a contradiction. It follows that $\langle \zeta \cdot \hat{X}, Z \rangle := \mathbb{I}_A \cdot \zeta$ defines well a $\lambda$-linear functional on $\text{span}_\lambda \{\hat{X}\}$ with $\langle \hat{X}, Z \rangle = \mathbb{I}_A$.

Next we show that $Z \leq q_D$ on $\text{span}_\lambda \{\hat{X}\}$. Indeed by proposition 5.2 (iii) and proposition 5.3 (vi) we get

$$\langle \zeta \cdot \hat{X}, Z \rangle \geq \mathbb{I}_A \cdot q_D \cdot |\zeta \cdot \hat{X}| = \mathbb{I}_{A \cap (\zeta > 0)} \cdot q_D \cdot |\zeta \cdot \hat{X}| + \mathbb{I}_{A \cap (\zeta \leq 0)} \cdot q_D \cdot |\zeta \cdot \hat{X}| \geq \mathbb{I}_{A \cap (\zeta > 0)} \cdot \zeta \cdot q_D \cdot |\zeta \cdot \hat{X}| \geq \mathbb{I}_{A \cap (\zeta > 0)} \cdot \zeta \geq \mathbb{I}_A \cdot \zeta = \langle \zeta \cdot \hat{X}, Z \rangle.$$

According to theorem 4.1 we can extend $Z$ on $E$ satisfying $Z \leq q_D$. In particular, for all $Y \in D = \emptyset$ we find by (6.6) some $\eta \in \lambda_2$ with

$$\mathbb{I}_A \cdot (Y, Z) \leq \mathbb{I}_A \cdot q_D \cdot |Y| \leq \mathbb{I}_A \cdot (1 - \eta) = \mathbb{I}_A \cdot (\langle \hat{X}, Z \rangle - \eta).$$

2. Step: Now let $C$ be a non-empty $\lambda$-convex subset of $E$. Since $D - C$ is not empty, we pick $-\hat{X} \in D - C$ and define $D' := D - C - (-\hat{X}) = \bigcup_{X \in C} D - (X - \hat{X})$. Now $D'$ is a non-empty $\lambda$-convex open subset of $E$ with $0 \in D'$. Since hypothesis (7.1) can be rewritten as $0 \notin \mathbb{I}_B \cdot (D - C)$ for all $B \in (\mathcal{G} \cap A)^+$, we see that $\mathbb{I}_B \cdot \hat{X} \notin \mathbb{I}_B \cdot D'$ for all $B \in \mathcal{G}^+, B \subset A$. From step 1 it follows that there
exists a $\lambda$-linear functional $Z : E \to \lambda$ with $\langle \hat{X}, Z \rangle = I_A$ and $Z \leq q_{D'}$ and so that for all $Y' \in D'$ there exists $\eta \in \lambda_2$ with $I_A \cdot \langle Y', Z \rangle \leq I_A \cdot q_{D'} |Y'| \leq I_A \cdot (\langle \hat{X}, Z \rangle - \eta)$ which implies that for all $X \in C$ and $Y \in D$ there exists $\eta \in \lambda_2$ with $I_A \cdot \langle Y, Z \rangle \leq I_A \cdot (\langle X, Z \rangle - \eta)$.

Moreover, since $Z \leq q_{D'}$ for the $\lambda$-convex open subset $D'$ with $0 \in D'$ implies the continuity of $Z$ by the next proposition. □

**Proposition 7.2.** Let $D$ be a $\lambda$-convex open subset of a locally convex $\lambda$-module $E$ with $0 \in D$ and $Z : E \to \lambda$ a $\lambda$-linear functional with $Z \leq q_D$ on $E$. Then $Z$ is continuous.

**Proof.** Let $0 < \varepsilon \in \mathbb{R}$ and set $D' := \varepsilon \cdot D \cap (-\varepsilon) \cdot D$ which is an open neighborhood of 0. For $Y' \in D'$ we find by (6.6) some $\eta_1, \eta_2 \in \lambda_2$ with $\langle Y', Z \rangle \leq q_{D'}|Y'| \leq \varepsilon \cdot (1 - \eta_1)$ and $\langle -Y', Z \rangle \leq q_{D'}|-Y'| \leq \varepsilon \cdot (1 - \eta_2)$. Therefore $\|\langle Y', Z \rangle\|_\lambda < \varepsilon$ for all $Y' \in D'$ which shows the continuity of $Z$. □

**Proposition 7.3.** On the locally convex $\lambda$-module $(E, \Omega)$, a $\lambda$-linear function $Z : E \to \lambda$ is continuous if and only if there exists $\eta \in \lambda_2$ so that for all $X \in E$

$$\|\langle X, Z \rangle\|_\lambda \leq \eta \cdot q_{|X|}.$$  \hspace{1cm} (7.4)

**Proof.** If $Z : E \to \lambda$ is continuous then for all $\varepsilon > 0$ there exists $\eta \in \Omega$ and $\zeta \in \lambda_2$ so that for all $X \in E$ $q_{|X|} \leq \zeta$ implies $\|\langle X, Z \rangle\|_\lambda \leq \varepsilon$. But then

$$\|\langle X, Z \rangle\|_\lambda = q_{|X|} / \zeta \cdot \left\| \left\langle \frac{X}{q_{|X|}} \cdot \zeta, Z \right\rangle \right\|_\lambda \leq \varepsilon / \zeta \cdot q_{|X|}$$

for all $X$ which shows (7.4) with $\eta = \varepsilon / \zeta \in \lambda_2$.

Conversely, if (7.4) holds with $q \in \Omega$ and $\eta \in \lambda_2$, we set $D = \{X \mid q_{|X|} < 1 / \eta\}$ which is an open $\lambda$-convex set containing 0. Since its gauge function $q_D$ satisfies $q_D = \eta \cdot q$, we get $\|\langle X, Z \rangle\| \leq q_D |X|$ and the assertion follows from proposition 7.2. □

For a compact convex set $C$ and a closed convex subset $D$, we can — under some conditions — reinforce the first separation theorem 7.1 to get a strong separation, meaning that the “gap” $\eta$ in (7.2) between $C$ and $D$ becomes uniform. We start with following partial result:

**Theorem 7.4.** Let $D$ be a closed $\lambda$-convex non-empty subset of a locally convex $\lambda$-module $(E, \Omega)$. Suppose that for some $X \in E \setminus D$ and $A \in G^+$ there exists a neighborhood $U$ of $X$ so that

$$\mathbb{I}_B \cdot U \cap \mathbb{I}_B \cdot D = \emptyset$$  \hspace{1cm} (7.5)

for all $B \in (G \cap A)^+$. Then there exists a $\lambda$-linear continuous function $Z : E \to \lambda$ with $\langle \hat{X}, Z \rangle = I_A$ for some $\hat{X} \in E$ and $\eta \in \lambda_2$ so that

$$\mathbb{I}_A \cdot (\langle Y, Z \rangle + \eta) \leq \mathbb{I}_A \cdot \langle X, Z \rangle$$  \hspace{1cm} (7.6)

for all $Y \in D$.

**Proof.** Let $A \in G^+$. Without loss of generality we assume $X = 0$ so that $0 \notin \mathbb{I}_B \cdot D$ for all $B \in (G \cap A)^+$. By assumption 6.2, we may assume that there exists a neighborhood $U_{q,\eta_0}$ of 0 with $\mathbb{I}_B \cdot U_{q,\eta_0} \cap \mathbb{I}_B \cdot D = \emptyset$ for all $B \in (G \cap A)^+$. This implies that $\mathbb{I}_B \cdot U_{q,\eta_0/2} \cap \mathbb{I}_B \cdot (D \cup U_{q,\eta_0/2}) = \emptyset$ for all such $B$. Since $D \cup U_{q,\eta_0/2}$ is an open $\lambda$-convex subset theorem 7.1 yields a continuous $\lambda$-linear functional $Z$ with
\[ \langle \tilde{X}, Z \rangle = I_A \text{ for some } \tilde{X} \in E \text{ and } I_A \cdot \langle Y, Z \rangle \leq I_A \cdot \langle Y', Z \rangle \text{ for all } Y \in D + U_{q_0/2} \text{ and } Y' \in U_{q_0/2}. \]

Since \( \eta \cdot \tilde{X} \in U_{q_0/2} \) for some \( \eta \in \lambda \), we get

\[ I_A \cdot \langle Y + \eta \cdot \tilde{X}, Z \rangle = I_A \cdot \langle (Y, Z) + \eta \rangle \leq I_A \cdot \langle X, Z \rangle \]

for all \( Y \in D \).

Now we come to the strong separation of compact versus closed convex sets in a \( \lambda \)-module.

**Theorem 7.5.** Let \( C \) and \( D \) be \( \lambda \)-convex non-empty subsets of a locally convex \( \lambda \)-module \( (E, \Omega) \) with \( C \) compact and \( D \) closed. Suppose that for \( A \in G^+ \) we have

\[ I_B \cdot C \cap I_B \cdot D = \emptyset \]  

(7.7)

for all \( B \in (G \cap A)^+ \).

Then for all \( \varepsilon > 0 \) there exists a set \( A_\varepsilon \subset A \) with \( \mathbb{P}(A \setminus A_\varepsilon) < \varepsilon \), a \( \lambda \)-linear continuous function \( Z : E \to \lambda \), and \( \eta \in \lambda \) so that

\[ I_{A_\varepsilon} \cdot \langle (Y, Z) + \eta \rangle \leq I_{A_\varepsilon} \cdot \langle X, Z \rangle \]  

(7.8)

for all \( X \in C \) and \( Y \in D \).

**Proof.** We first note that the set \( D' = \{ Y - X \mid X \in C, Y \in D \} \) is obviously \( \lambda \)-convex, but it is also closed. The later property is proved as in the scalar case (see [2], 5.3). Property (7.7) is now equivalent to

\[ 0 \notin I_B \cdot D'. \]  

(7.9)

for all \( B \in (G \cap A)^+ \). Following ideas in [12], we define

\[ \zeta := \text{ess.sup}_{q \in \Omega} \text{ess.inf} \left\{ \eta \in \lambda \mid \eta \leq I_A \text{ and } I_A \cdot (B_{q, \eta} \cap D') \neq \emptyset \right\}. \]

We claim that \( \zeta > 0 \) on \( A \). Suppose \( B := A \cap \{ \zeta = 0 \} \in G^+ \). This means that for all \( q \in \Omega \) and all \( \eta \in \lambda \) there exists \( X_{q, \eta} \) with \( I_B \cdot X_{q, \eta} \in I_A \cdot (B_{q, \eta} \cap D') \). We fix \( X_0 \in D' \) so that the \( \lambda \)-convexity of \( D' \) implies that \( X'_{q, \eta} := I_B \cdot X_{q, \eta} + I_{A_B} \cdot X_0 \in I_A \cdot D' \). Now the net \( (X'_{q, \eta})_{q \in \Omega, \eta \in \lambda} \) converges to \( I_{A_B} \cdot X_0 \in I_A \cdot D' \) since \( D' \) is closed. It follows that \( 0 = I_B \cdot I_{A_B} \cdot X_0 \subset I_B \cdot D' \) in contradiction to (7.9).

Now, let \( \varepsilon > 0 \). We find successively \( \gamma > 0 \) with \( \mathbb{P}(A \cap \{ \zeta < \gamma \}) < \varepsilon / 2 \), next a \( q \in \Omega \) so that

\[ \mathbb{P} \left( A \cap \left\{ 2/3 \cdot \zeta > \text{ess.inf} \{ \eta \in \lambda \mid \eta \leq I_A \text{ and } I_A \cdot (B_{q, \eta} \cap D') \neq \emptyset \} \right\} \right) < \varepsilon / 2, \]

and finally the set \( A_\varepsilon := A \cap \{ \zeta \geq \gamma \} \cap \left\{ 2/3 \cdot \zeta \leq \text{ess.inf} \{ \eta \in \lambda \mid \eta \leq I_A \text{ and } I_A \cdot (B_{q, \eta} \cap D') \neq \emptyset \} \right\} \) so that

\[ \mathbb{P}(A \setminus A_\varepsilon) < \varepsilon. \]

Setting \( \zeta^* := (I_{A_\varepsilon} \cdot \zeta + \gamma \cdot I_{A_\varepsilon}) / 2 \) we see that \( \zeta^* \in \lambda \). Moreover, we claim that

\[ I_B \cdot (B_{q, \zeta^*} \cap D') = \emptyset \]  

(7.10)

for all \( B \in (G \cap A)^+ \), since otherwise we would find \( B \in (G \cap A)^+ \) and \( X \in E \) with \( I_B \cdot X \in I_B \cdot (B_{q, \zeta^*} \cap D') \). This would mean that

\[ I_B \cdot \text{ess.inf} \{ \eta \in \lambda \mid \eta \leq I_A \text{ and } I_A \cdot (B_{q, \eta} \cap D') \neq \emptyset \} \leq I_B \cdot \zeta^* \leq I_B \cdot \zeta / 2 \]

\[ \leq I_B \cdot \left( 2/3 \zeta - \gamma / 6 \right) \leq I_B \cdot \left( \text{ess.inf} \{ \eta \in \lambda \mid \eta \leq I_A \text{ and } I_A \cdot (B_{q, \eta} \cap D') \neq \emptyset \} - \gamma / 6 \right) \]

which is a contradiction. Since with (7.10) we proved the condition of theorem 7.4 with respect to \( A_\varepsilon \), theorem 7.5 is equally proved. \( \square \)
Remark 7.1. In the case where \( C = \{ X \} \) for some \( X \in E \) and \( X \notin D \), we note that the condition (7.7) is satisfied with \( A := (X \cap D)^\circ \), since \( A \in G^+ \) and \( \mathbb{I}_B \cdot X \notin \mathbb{I}_B \cdot D \) for all \( B \in (\mathcal{G} \cap A)^+ \).

For locally convex \( \lambda \)-modules we get the following consequence of theorem 7.1:

**Theorem 7.6.** Let \((E, \Omega)\) be a locally convex \( \lambda \)-module, \( D \) a closed \( \lambda \)-convex subset, and \( C \) an open \( \lambda \)-convex subset containing 0. Suppose that for some \( X_0 \in E \), \( A \in G^+ \), and \( \eta_1, \eta_2 \in \lambda_2 \) we have

\[
\mathbb{I}_B \cdot ((\eta_1 + \eta_2) \cdot C + X_0) \cap \mathbb{I}_B \cdot D = \emptyset
\]

for all \( B \in (\mathcal{G} \cap A)^+ \).

Then there exists a \( \lambda \)-linear continuous function \( Z : E \to \lambda \) and \( \eta \in \lambda_2 \) so that

\[
\mathbb{I}_A \cdot \langle (Y, Z) + \eta \rangle \leq \mathbb{I}_A \cdot \langle X, Z \rangle
\]

for all \( Y \in D \) and all \( X \in \eta_1 \cdot C + X_0 \).

**Proof.** With \( C = \{ X_0 \} \) and the open \( \lambda \)-convex set \( \tilde{D} := D - (\eta_1 + \eta_2) \cdot C \), we can apply theorem 7.1 since \( \mathbb{I}_B \cdot \{ X_0 \} \cap \mathbb{I}_B \cdot \tilde{D} = \emptyset \) for all \( B \in (\mathcal{G} \cap A)^+ \). We get a continuous \( \lambda \)-linear function \( Z : E \to \lambda \) with \( \langle \tilde{X}, Z \rangle = \mathbb{I}_A \) for some \( \tilde{X} \in E \) so that

\[
\mathbb{I}_A \cdot \langle Y - \tilde{X}, Z \rangle \leq \mathbb{I}_A \cdot \langle X_0, Z \rangle
\]

for all \( Y \in D \) and \( \tilde{X} \in (\eta_1 + \eta_2) \cdot C \). Since there exists \( \eta \in \lambda_2 \) with \( \eta \cdot \tilde{X} \in \eta_2 \cdot C \) it follows that

\[
\mathbb{I}_A \cdot \langle Y - (-\eta \cdot \tilde{X}), Z \rangle \leq \mathbb{I}_A \cdot \langle X_0 + \tilde{X}, Z \rangle
\]

for all \( Y \in D \) and \( \tilde{X} \in \eta_1 \cdot C \), hence

\[
\mathbb{I}_A \cdot \langle (Y, Z) + \eta \rangle \leq \mathbb{I}_A \cdot \langle X, Z \rangle
\]

for all \( Y \in D \) and \( X \in \eta_1 \cdot C + X_0 \). \( \square \)

8. Linear duality of locally convex \( \lambda \)-modules

Let \((E, \Omega)\) be a locally convex \( \lambda \)-module. Before we investigate below the dual \( \lambda \)-module of \( E \) we introduce the following definitions:

**Definitions 8.1.** Let \( B \) be a closed \( \lambda \)-convex subset of \( E \) containing 0.

(i) A net \((X_i)_{i \in I}\) in \( E \) is a **\( B \)-Cauchy net** if for all \( \eta \in \lambda_2 \) there exists \( \iota_\eta \in I \) so that for all \( \iota_1, \iota_2 \geq \iota_\eta \) we have \( X_{\iota_1} - X_{\iota_2} \in \eta \cdot B \).

(ii) The \( \lambda \)-module \( E \) is **\( B \)-complete** if every \( B \)-Cauchy net has a limit in \( E \).

(iii) A net \((X_i)_{i \in I}\) in \( E \) is a **Cauchy net** if it is a \( B_{q,1} \)-Cauchy net for all \( q \in \Omega \).

(iv) We simply say that \( E \) is **complete**, if every Cauchy net has a limit in \( E \).

(v) In the case \( \Omega = \{ ||\cdot|| \} \) with a \( \lambda \)-norm \( ||\cdot|| \), the complete \( \lambda \)-module \((E, ||\cdot||)\) is called a **Banach \( \lambda \)-module**.

By \( E' \) we denote the \( \lambda \)-dual space of \( E \) which is the \( \lambda \)-module of all continuous \( \lambda \)-linear functions \( Z : E \to \lambda \).

**Definition 8.2.**
Proposition 8.1. The dual $\xi \in X$

Proof. This shows that $(\lambda, \lambda)$-weak topology uniformly in $X$. For any $D \subseteq \mathcal{U}$ base convex subsets of $X$ we choose $\iota, \iota \in \lambda, \nu = 1, 2,$

$$\langle \zeta_1 \cdot X_1 + \zeta_2 \cdot X_2, \lim_{\iota} Z_i \rangle = \lim_{\iota} \left( \langle \zeta_1 \cdot X_1, Z_i \rangle + \zeta_2 \cdot \langle X_2, Z_i \rangle \right)$$

$$= \zeta_1 \cdot \langle X_1, \lim_{\iota} Z_i \rangle + \zeta_2 \cdot \langle X_2, \lim_{\iota} Z_i \rangle.$$

This shows that $\lim_{\iota} Z_i$ is $\lambda$-linear.

To show that $\lim_{\iota} Z_i$ is continuous, let $q' \in \Omega'$ and $\eta \in \lambda_2$. We choose $\iota_{q', \eta} \in I$ so that for all $\iota_1, \iota_2 \geq \iota_{q', \eta}$

$$Z_{i_1} - Z_{i_2} \in B_{q', \eta}. \quad (8.3)$$

For any $X \in B_{q, 1}$, we choose $\iota_X \in I$, $\iota_X \geq \iota_{q, \eta}$ so that $|\langle X, Z_i \rangle - \langle X, Z_i^X \rangle| \leq \eta$ for all $\iota \geq \iota_X$. Now, for $\iota \geq \iota_{q, \eta}$ we get

$$|\lim_{\iota} \langle X, Z_i \rangle| \leq |\lim_{\iota} \langle X, Z_i \rangle - \langle X, Z_i^X \rangle| + |\langle X, Z_i^X - Z_i^q, \eta \rangle| + |\langle X, Z_i^{q, \eta} \rangle| \leq \eta + q'\left|Z_i^X - Z_i^{q, \eta}\right| + q'\left|Z_i^{q, \eta}\right| \leq 2 \cdot \eta + q'\left|Z_i^{q, \eta}\right|$$

uniformly in $X \in B_{q, 1}$. Hence, $q'\left|\lim_{\iota} Z_i \right| \leq 2 \cdot \eta + q'\left|Z_i^{q, \eta}\right|$. This shows the continuity of $\lim_{\iota} Z_i$, hence $\lim_{\iota} Z_i \in E'$. \qed

9. Weak topologies and polar sets for $\lambda$-modules

The $\lambda$-weak topology $\sigma_\lambda(E, E')$ on $E$ is the locally $\lambda$-convex topology generated by the family of $\lambda$-seminorms $(q_Z : E \to \lambda)_{Z \in E'}$ defined by $q_Z[X] := |\langle X, Z \rangle|$, $X \in E$. By theorem 7.1, the $\lambda$-weak topology $\sigma_\lambda(E, E')$ is Hausdorff. Moreover, theorem 7.4 implies the following results for closed $\lambda$-convex subsets of $E'$:

Theorem 9.1. Let $E$ be a locally convex $\lambda$-module with a topology $\mathcal{T}_U$ generated by a neighborhood base $\mathcal{U}$ of $0 \in E$ as in definition 6.2 and $D \subseteq E$ a $\lambda$-convex subset of $E$. Then $D$ is closed in the topology $\mathcal{T}_U$ if and only if it is closed in the $\lambda$-weak topology $\sigma_\lambda(E, E')$.

Proof. Since the $\lambda$-weak topology $\sigma_\lambda(E, E')$ is coarser than the topology $\mathcal{T}_U$, the sufficiency of the last condition is obvious. Conversely, if $D$ is closed in $\mathcal{T}_U$ and $X \not\in D$ then $\mathbb{I}_B \cdot X \not\in \mathbb{I}_B \cdot D$ for all
Proposition 9.2. Let $A \in \mathcal{G}^+$, $A \subset (X \cap D)^c$, $Z \in E'$, and $\eta \in \lambda_\sharp$ with $\mathbf{I}_A \cdot ((Y, Z) + \eta) \leq \mathbf{I}_A \cdot (X, Z)$ for all $Y \in D$. Therefore

$$X \notin C_X := \{Y \mid \langle Y, \mathbf{I}_A \cdot Z \rangle \leq \langle X, \mathbf{I}_A \cdot Z \rangle - \mathbf{I}_A \cdot \eta \} \supset D.$$ 

where the set $C_X$ is closed in the $\lambda$-weak topology $\sigma_\lambda(E, E')$. Therefore $D = \bigcap_{X \notin D} C_X$ shows that $D$ is also closed in the $\lambda$-weak topology $\sigma_\lambda(E, E')$. \hfill $\square$

Similarly, the $\lambda$-weak* topology $\sigma_\lambda(E', E)$ on $E'$ is the locally $\lambda$-convex topology generated by the family of $\lambda$-seminorms $(q'_X : E' \to \lambda)_{X \in E}$ defined by $q'_X|Z| := |\langle X, Z \rangle|$, $Z \in E'$. The $\lambda$-weak* topology $\sigma_\lambda(E', E)$ is also Hausdorff.

**Definition 9.1.** Let’s consider the $\lambda$-dual pair $(E, E')$.

(i) For a subset $D \subset E$ we define its $\lambda$-polar set $D^\circ$ as

$$D^\circ := \{Z \in E \mid \langle X, Z \rangle \leq 1, \text{ for all } X \in D\}.$$ \hspace{1cm} (9.1)

(ii) Similarly, for a subset $D' \subset E'$ its $\lambda$-polar set $D'^\circ$ (with respect to $(E, E')$) is

$$D'^\circ := \{X \in E \mid \langle X, Z \rangle \leq 1, \text{ for all } Z \in D'\}.$$ \hspace{1cm} (9.2)

For subsets $D \subset E$ we let $co_\lambda D$ be the $\lambda$-convex hull of $D$ in $E$ and $\overline{co_\lambda D}$ be the closure of $co_\lambda D$ in $E$. Theorem 9.1 shows that this closure is the same in the two topologies $\mathcal{T}_\lambda$ or $\sigma_\lambda(E, E')$.

Similarly for a subset $D' \subset E'$, we denote by $co_\lambda D'$ the $\lambda$-convex hull of $D'$ in $E'$ and by $\overline{co_\lambda D'}$ closure of it in $E'$, however only in the $\lambda$-weak* topology $\sigma_\lambda(E', E)$.

The following properties of $\lambda$-polar sets are well-known in the scalar case.

**Proposition 9.2.** Let $E$ be a locally convex $\lambda$-module and $D, D_i \subset E$ where $i$ runs through an index set $I \ni 1, 2$.

(i) $D^\circ$ is $\lambda$-convex and closed in $\sigma_\lambda(E', E)$. Similarly, $D'^\circ$ is $\lambda$-convex and closed in $E$ (or equivalently in $\sigma_\lambda(E, E')$).

(ii) $D^\circ = (\overline{co_\lambda(D \cup \{0\})})^\circ$.

(iii) $0 \in D^\circ$, $D \subset D'^\circ$, and $D_1 \subset D_2$ implies $D_1^\circ \subset D_2^\circ$.

(iv) For $\eta \in \lambda_\sharp$ we have $(\eta \cdot D)^\circ = \frac{1}{\eta} \cdot D^\circ$.

(v) $(\bigcup_{i \in I} D_i)^\circ = \bigcap_{i \in I} D_i^\circ$.

(vi) $(\bigcap_{i \in I} D_i)^\circ \supset \overline{co_\lambda \bigcup_{i \in I} D_i}$.

**Proof.** (i) Since $D^\circ = \bigcap_{X \in D} \{Z \mid \langle X, Z \rangle \leq 1\}$, it is the intersection of $\lambda$-convex and $\sigma_\lambda(E', E)$-closed sets. As such, it has also these properties. The same reasoning holds for $D'^\circ$.

(ii) It is obvious that $D^\circ = (D \cup \{0\})^\circ$. Now, let $\eta \in \lambda_\sharp$ with $0 \leq \eta \leq 1$. It follows that

$$\{Z \mid \langle \eta \cdot X_1 + (1 - \eta) \cdot X_2, Z \rangle \leq 1\} \supset \bigcap_{i=1}^2 \{Z \mid \langle X_i, Z \rangle \leq 1\}$$

for all $X_i \in E$. Otherwise, if $(X_i)_{i \in I}$ is a net converging to $X$ then $\{Z \mid \langle X, Z \rangle \leq 1\} \supset \bigcap_{i \in I} \{Z \mid \langle X_i, Z \rangle \leq 1\}$. These both inclusions show (ii).

(iii) is clear.

(iv) For $\eta \in \lambda_\sharp$, we have

$$(\eta \cdot D)^\circ = \{Z \mid (\eta \cdot X, Z) \leq 1, \forall X \in D\} = \{Z \mid \langle X, Z \rangle \leq 1 / \eta, \forall X \in D\} = 1 / \eta \cdot D^\circ.$$ 

(v) $(\bigcup_{i \in I} D_i)^\circ = \bigcap_{X \in \bigcup_{i \in I} D_i} \{Z \mid (X, Z) \leq 1\} = \bigcap_{i \in I} D_i^\circ$.

(vi) $(\bigcap_{i \in I} D_i)^\circ \supset \bigcap_{i \in I} D_i^\circ$ follows from (ii) and then (i) implies (vi). \hfill $\square$
The following theorem is a \( \lambda \)-module version of the well-known bipolar theorem (see also [5]).

**Theorem 9.3.** For \( D \subset E \) we have with respect to the \( \lambda \)-dual pair \((E, E')\)
\[
D^\circ = \overline{\sigma_\lambda(D \cup \{0\})}. \tag{9.3}
\]

The closure has to be taken in the \( \sigma_\lambda(E, E') \)-topology.

**Proof.** We set \( C := \overline{\sigma_\lambda(D \cup \{0\})} \). The inclusion \( D^\circ \supset C \) follows from proposition 9.2 (i) and (iii).

For the converse inclusion, assume \( X \not\subset C \), i.e. \((X \cap C)^c \in \mathcal{G}^+\). Again, by theorem 7.5 we find a set \( A \in \mathcal{G}^+, A \subset (X \cap D)^c \), \( Z \in E' \), and \( \eta \in \lambda_2 \) with \( \langle Y, \mathbb{I}_A \cdot Z \rangle \leq \langle X, \mathbb{I}_A \cdot Z \rangle - \mathbb{I}_A \cdot \eta \) for all \( Y \in C \). In particular, \( \langle X, \mathbb{I}_A \cdot Z \rangle \geq \mathbb{I}_A \cdot \eta \) so that \( \tilde{Z} := \mathbb{I}_A \cdot ((\langle X, \mathbb{I}_A \cdot Z \rangle - \eta/2)^{-1} \cdot Z \in E' \).

It follows that \( \langle Y, \tilde{Z} \rangle \leq \frac{\langle X, \mathbb{I}_A \cdot Z \rangle - \mathbb{I}_A \cdot \eta}{\mathbb{I}_A \cdot \eta/2} \leq 1 \) for all \( Y \in C \); thus \( \tilde{Z} \in D^\circ \). But on \( A \) we have
\[
\langle X, \tilde{Z} \rangle = \frac{\langle X, \mathbb{I}_A \cdot Z \rangle - \mathbb{I}_A \cdot \eta}{\mathbb{I}_A \cdot \eta/2} > 1 \text{ which shows that } X \not\in D^\circ. \tag*{\□}
\]

An immediate consequence of proposition 9.2 is the following

**Corollary 9.4.** Let \( E \) be a locally convex \( \lambda \)-module with the set \( \Omega \) of \( \lambda \)-seminorms. Then the dual balls
\[
B_{q', \eta}(Z) := \{ Z' \mid q'\cdot|Z' - Z| \leq \eta \} \tag{9.4}
\]
are closed in the weak*-topology \( \sigma_\lambda(E', E) \) for all \( Z \in E' \) and \( \eta \in \lambda_2 \).

**Proof.** With respect to the \( \lambda \)-dual pair \((E, E')\) we have \( B_{q', \eta} = (B_{q, 1/\eta})^\circ \). So by proposition 9.2, \( B_{q', \eta} \) is weak*-closed. \( \square \)

10. Convex dual sets of complete locally convex \( \lambda \)-modules

Let \( E \) be a locally convex \( \lambda \)-module whose topology is induced by a set \( \Omega \) of \( \lambda \)-seminorms, satisfying the assumption 6.2. Again, \( \Omega' \) is the set of conjugate \( \lambda \)-seminorms on \( E' \).

The following theorem is one of the main results of this paper. Let us first remark that if \( D' \subset E' \) is \( \sigma_\lambda(E', E) \)-closed then by proposition 9.2 (i) the intersection \( D' \cap \eta \cdot C^\circ \) is also \( \sigma_\lambda(E', E) \)-closed for all \( \eta \in \lambda_2 \) and sets \( C \) which by proposition 9.2 (ii) may be assumed to be closed, \( \lambda \)-convex and to contain 0.

Also, under additional conditions, the converse assertion is correct:

We consider an open \( \lambda \)-convex set \( C \) with \( \bigcap_{\eta \in \lambda_2} \eta \cdot C = \{0\} \). If the sets \( D' \cap \eta \cdot C^\circ \) are \( \sigma_\lambda(E', E) \)-closed for all \( \eta \in \lambda_2 \), then \( D' \) is \( \sigma_\lambda(E', E) \)-closed. The sufficient conditions for this statement are:

- \( E \) is a \( C \)-complete \( \lambda \)-module.
- The set \( D' \) is \( \lambda \)-convex.

It is important to notice that in general the set \( C^\circ \) is not a \( \sigma_\lambda(E', E) \)-neighborhood of 0. We rephrase this general version of the Krein-Šmulian theorem for \( \lambda \)-modules as follows:

**Theorem 10.1.** Let \( E \) be a locally convex \( \lambda \)-module which is complete with respect to the closure for an open \( \lambda \)-convex subset \( C \) with \( \bigcap_{\eta \in \lambda_2} \eta \cdot C = \{0\} \). Further let \( D' \) be a \( \lambda \)-convex subset of the dual \( \lambda \)-module \( E' \). Then the following are equivalent:

(i) \( D' \) is \( \sigma_\lambda(E', E) \)-closed.

(ii) \( (D' \cap \eta \cdot C^\circ) \) is \( \sigma_\lambda(E', E) \)-closed for all \( \eta \in \lambda_2 \).
Proof. The necessity of the second assertion is clear by proposition 9.2 (i). For the converse direction, we use the following shorthand writing: \( C_\eta := \eta \cdot C \) and \( C'_\eta := \eta \cdot C^\circ \). We first show that if the assertion is true for sets containing 0 then it is also true for general sets \( D' \).

Step 1: Assume the theorem is true for sets containing 0. Let \( Z_0 \in D' \subset E' \) and set \( \tilde{D}' := D' - Z_0 \). Since by assumption and proposition 9.2 (vi) the set \( C'_\eta \) is \( \lambda \)-absorbernt, we get \( Z_0 \in C'_{\eta'} \) for some \( \eta' \in \lambda_t \) so that \( C'_{\eta'} + Z_0 \subset C'_{\eta' + \eta} \). Now, \( \tilde{D}' \cap C'_{\eta'} = (D' \cap (C'_{\eta} + Z_0)) - Z_0 = (D' \cap C'_{\eta' + \eta}) \cap C'_{\eta}(Z_0) - Z_0 \).

By hypothesis \( D' \cap C'_{\eta' + \eta} \) is weak*-closed as is a priori \( C'_{\eta'}(Z_0) \) and we conclude that \( \tilde{D}' \cap C'_{\eta} \) is weak*-closed for all \( \eta \in \lambda_t \). The theorem then says that \( \tilde{D}' \) is weak*-closed and therefore also \( D' = \tilde{D}' + Z_0 \) since the map \( Z \mapsto Z + Z_0 \) is a homeomorphism in any locally convex \( \lambda \)-module. It remains to show that the theorem is true for sets \( D' \) with \( 0 \in D' \).

Step 2: We define \( D'_n := D' \cap C'_{2n+1} \) which by hypothesis is weak*-closed. Let \( D_n := (D'_n) \circ \) be the polar set of \( D'_n \) which is \( \sigma(E, E') \)-closed by proposition 9.2 (i).

Using \( 0 \in D'_n \) and the weak*-closure of \( D'_n \), the bipolar theorem 9.3 says that \( (D_n) \circ = (D'_n) \circ = D'_n \).

Also, \( D_{n+1}' \supset D'_n \) implies \( D_{n+1} \subset D_n \). We claim even that

\[
D_{n+1} \subset D_n \subset D_{n+1} - C_{2-n+1} \quad \text{(10.1)}
\]

To show (10.1), we set \( \tilde{D} := D_n - C_{2-n+1} \). Assume the existence of an \( X_0 \in D_n \setminus \tilde{D} \). This means that \( A := (X_0 \cap \tilde{D})^c \in \mathcal{G}^+ \) and \( 1_B \cdot X_0 \notin \|B \cdot \tilde{D} \) or

\[
1_B \cdot (C_{2-n+1} + X_0) \cap \|B \cdot D_{n+1} = 0
\]

for all \( B \in (\mathcal{G} \cap A)^+ \). We apply theorem 7.6 with \( \eta_1 = \eta_2 = 2^{-n} \) to get the existence of \( Z \in E' \) and \( \eta_0 \in \lambda_t \) so that

\[
\|A \cdot (\langle Y, Z \rangle + \eta_0) \leq \|A \cdot \langle X, Z \rangle \|
\]

for all \( Y \in D_{n+1} \) and \( X \in C_{2-n} + X_0 \). Since \( 0 \in D \tilde{D} \) we get \( \|A \cdot \eta_0 \leq \|A \cdot \eta_0 \|_A \cdot \esssup \langle X, Z \rangle \) :\( \eta_0 \in \|A \cdot \lambda_t \).

Replacing \( Z \) by \( \tilde{Z} := A \cdot \frac{1}{\eta_1 - \frac{1}{2} \eta_0} \cdot Z \) gives

\[
\langle Y, \tilde{Z} \rangle \leq \|A \cdot \frac{\eta_1 - \eta_0}{\eta_1 - \frac{1}{2} \eta_0} \|_A \leq \|A \cdot \frac{\eta_1 - \eta_0}{\eta_1 - \frac{1}{2} \eta_0} \|_A \cdot \langle X, \tilde{Z} \rangle
\]

for all \( Y \in D_{n+1} \) and \( X \in C_{2-n} + X_0 \). The first two inequalities imply \( \tilde{Z} \in (D_n) \circ = D'_{2-n+1} \subset D' \), in particular \( \tilde{Z} \in C'_{2-n+2} \). Since on \( A \) the second inequality is strict and \( X_0 \in D_n \), we see that \( \tilde{Z} \notin (D_n) \circ = D'_n \). Hence \( \tilde{Z} \notin C'_{2-n+2} \). This allows us to find \( Y \in C_{2-n+1} \) with \( A' := \{\langle Y, Z \rangle > 2\} \in (\mathcal{G} \cap A)^+ \). Now with \( X := X_0 - Y \) we get \( \|A' \cdot \langle X, \tilde{Z} \rangle / 2 \| = \frac{1}{2} \|A' \cdot \left[ \langle X, \tilde{Z} \rangle + \langle X, Z \rangle \right] \| \geq \frac{3}{2} \|A' \|. \) It follows that \( \tilde{Z}/2 \notin (D_n) \circ = D'_n \).

On the other hand, we know that \( 0, \tilde{Z} \in D' \) hence by convexity \( \tilde{Z}/2 \in D' \). Also \( \tilde{Z} \in C'_{2-n+2} \). Both assertions mean \( \tilde{Z}/2 \in D'_n \): a contradiction!

Step 3: Let \( D := \bigcap_{n \geq 1} D_n \). For all \( n \geq 1 \), we claim

\[
D_n \subset D - C_{2-n+2} \quad \text{(10.3)}
\]

By (10.1) any \( X_n \in D_n \) can be written as \( X_n = X_{n+1} - Y_n = \ldots = X_{n+m} - \sum_{i=0}^{m-1} Y_{n+i} \) with \( Y_{n+i} \in C_{2-n+i} \), i.e. \( \sum_{i=0}^{m-1} Y_{n+i} \in C_{2-n+2} \) for all \( m \geq 0 \). The sequence \( (X_n)_{n \geq 1} \) is a Cauchy-sequence which in \( C \)-complete \( \lambda \)-module \( E \) converges to \( \overline{X} \in \bigcap_{n} D_n = D \) since every \( D_n \subset D_{n+1} \) and \( D_n \) is closed. Thus \( X_n = \overline{X} - \sum_{i \geq 0} Y_{n+i} \in D_n - C_{2-n+2} \) which shows (10.3).

Step 4: For any \( 0 < \varepsilon \in \mathbb{R} \) we have for the two convex sets containing 0:

\[
D + C_{2-n+2} = (1 + \varepsilon) \cdot \left[ (1 - \frac{1}{2}) \cdot D + \frac{1}{2} C_{2-n+2} \right] \subset (1 + \varepsilon) \cdot \overline{\partial}(D \cup C_{2-n+2}/\varepsilon) \]

so that \( D \subset D_n \subset \)
(1 + \varepsilon) \cdot \sigma(D \cup C_{2^{-n+2}/\varepsilon}). Proposition 9.2 implies \( D^\circ \supset (D_n)^\circ = D'_n \supset \frac{1}{1+\varepsilon}(D^\circ \cap C'_{2n-2/\varepsilon}). \) Since \( C'_{\lambda} \) is \( \lambda \)-absorbent, by taking the union over \( n \) and then over \( \varepsilon \) one gets: \( D^\circ \supset D' \supset \bigcup_{\varepsilon > 0} \frac{1}{1+\varepsilon} D^\circ. \) Obviously, the \( \Omega' \)-closure of the last set is equal to \( D^\circ. \) On the other hand, if every \( D' \cap C'_{\eta} \) is by assumption \( \sigma_{\lambda}(E', E) \)-closed, so a fortiori \( \Omega' \)-closed, then also \( D' = \bigcup_{\eta \in \Lambda}(D' \cap C'_{\eta}) \) is \( \Omega' \)-closed. This proves that we have \( D' = D^\circ \) and again by proposition 9.2 polar sets in the dual space are \( \sigma_{\lambda}(E', E) \)-closed. \( \square \)

11. Weak-weak* topologies for \( \lambda \)-modules

We recall that \( \lambda = L^\infty(G) \) is the dual space of \( \kappa := L^1(G) \) with the dual form
\[
\langle \varphi, \zeta \rangle := E[\varphi \cdot \zeta] \tag{11.1}
\]
for \( \varphi \in \kappa \) and \( \zeta \in \lambda \). The weak*-topology \( \sigma(\lambda, \kappa) \) of \( \lambda \) is the coarsest topology for which the linear functions \( \lambda \ni \zeta \mapsto \langle \varphi, \zeta \rangle \) are continuous for all \( \varphi \in \kappa \). This allows us to introduce the weak-weak*-topology \( \sigma(E, E', \kappa) \) as the coarsest topology for which the linear functions
\[
\varphi \circ Z : E \to \mathbb{R} \quad \text{with} \quad \langle X, \varphi \circ Z \rangle := \langle \varphi, \langle X, Z \rangle \rangle \tag{11.2}
\]
are continuous for all \( Z \in E' \) and \( \varphi \in \kappa \). Obviously, the weak-weak*-topology \( \sigma(E, E', \kappa) \) is coarser than the weak-topology \( \sigma_{\lambda}(E, E'). \)

Similarly, we can consider the weak*-weak*-topology \( \sigma(E', E, \kappa) \) characterized as the coarsest topology for which the linear functions
\[
\varphi \circ X : E' \to \mathbb{R} \quad \text{with} \quad \langle \varphi \circ X, Z \rangle := \langle \varphi, \langle X, Z \rangle \rangle \tag{11.3}
\]
are continuous for all \( X \in E \) and \( \varphi \in \kappa \). Again, weak*-weak*-topology \( \sigma(E', E, \kappa) \) is coarser than the weak*-topology \( \sigma_{\lambda}(E', E) \).

Now, we are in a position to show a version of the **Alaoglu-Bourbaki theorem** for locally convex \( \lambda \)-modules.

**Theorem 11.1.** Let \( U \) be a neighborhood of 0 in a locally convex \( \lambda \)-module \( E \). Then \( U^\circ \) is compact in the weak*-weak*-topology \( \sigma(E', E, \kappa) \).

**Proof.** Here we endow \( \lambda \) with the weak*-topology \( \sigma(\lambda, \kappa) \) and consider the Cartesian product \( \lambda^E \) with its product topology. Tychonov’s theorem tells us that the Cartesian product \( \prod_{X \in E} \pi_X \) of compact subsets \( \pi_X \subset \lambda \) is itself compact in \( \lambda^E \).

We define the mapping \( \Phi : E' \to \lambda^E \) given by \( \Phi(Z) := (\langle X, Z \rangle)_{X \in E} \in \lambda^E \). Since the weak*-weak*-topology \( \sigma(E', E, \kappa) \) on \( E' \) is the topology of the pointwise convergence of the set of functions \( \langle X, \cdot \rangle : E' \to (\lambda, \sigma(\lambda, \kappa)) \) with \( X \in E \), the mapping \( \Phi \) is a homeomorphism between \( E' \) and \( \Phi(E') \subset \lambda^E \). The assertion that \( U^\circ \) is compact in the topology \( \sigma(E', E, \kappa) \) is therefore a consequence of the following two claims:

(i) \( \Phi(U^\circ) \) is contained in a compact subset \( \prod_{X \in E} \pi_X \) of \( \lambda^E \) with marginal sets \( \pi_X \) compact in \( \sigma(\lambda, \kappa) \).

(ii) \( \Phi(U^\circ) \) is closed in \( \lambda^E \).
(i) The neighborhood $U$ of 0 is $\lambda$-absorbtion which implies that for all $X \in E$ there exists $\zeta^+_X, \zeta^-_X \in \lambda_+$ so that $X \in \zeta^+_X \cdot U$ and $-X \in \zeta^-_X \cdot U$. In particular, if $X \in U$ (resp. $-X \in U$), we choose $\zeta^+_X = 1$, resp. $\zeta^-_X = 1$. By definition of $U^\circ$, it follows that

$$-\zeta^+_X \leq \langle X, Z \rangle \leq \zeta^+_X$$

for all $X \in E$ and all $Z \in U^\circ$. Therefore, $\Phi(U^\circ) = \bigcap_{X \in E} \pi_X$ with $\pi_X = \{ \zeta \in \lambda | -\zeta^+_X \leq \zeta \leq \zeta^+_X \}$; But $\pi_X$ are closed convex bounded sets in $\lambda$ and therefore compact in the $\sigma(\lambda, \kappa)$ topology. This shows (i).

(ii) Let $(Z_i)_{i \in I}$ be a net in $U^\circ$ so that $\Phi(Z_i)$ converges to $\langle \xi_X, X \in E \rangle \in \lambda^E$, i.e. $\langle X, Z_i \rangle \to \xi_X$ in $\sigma(\lambda, \kappa)$ for all $X \in E$. It follows for all $\varphi \in \kappa$ that

$$\langle \varphi, \xi_X \rangle = \lim \langle \varphi, \xi_X \rangle = \lim \langle \varphi, \xi_X \rangle = \lim \langle \varphi, \xi_X \rangle = \lim \langle \varphi, \xi_X \rangle = \lim \langle \varphi, \xi_X \rangle$$

This shows that $X \mapsto \xi_X$ is $\lambda$-linear.

By definition 6.2 we find a neighborhood $U'$ of 0 with $U' \subset U$. By our choice of $\zeta^+_X$ and $\zeta^-_X$ for $X \in U$ it follows that $|\langle X, Z \rangle| \leq 1$ for all $X \in U'$ and $Z \in U^\circ$, in particular for $Z_i$ which implies in the limit that $|\xi_X| \leq 1$ for all $X \in U'$ or $\xi^{-1}(\{ |\zeta| \leq \varepsilon \}) \cap \varepsilon \cdot U', \varepsilon > 0$. Therefore $\xi$ is a continuous $\lambda$-linear function on $E$, i.e. $\xi \in \Phi(E')$ or $\xi(\cdot) = \langle \cdot, Z \rangle$ for some $Z \in E'$.

Remember again that in (11.4) we had $\zeta^+_X = 1$ for $X \in U$ so that for all $X \in U$ we get $\langle X, Z_i \rangle \leq 1$ or $\lim_i \langle X, Z_i \rangle = \langle X, Z \rangle \leq 1$. This shows even $Z \in U^\circ$ and therefore the closure of $\Phi(U^\circ)$.

\[ \square \]

### 12. Banach $\lambda$-modules and their reflexivity

As in classical functional analysis, we can study for a normed $\lambda$-module $E$ the bidual $\lambda$-module $E''$, i.e. the $\lambda$-module of all continuous $\lambda$-linear functions $V : E' \to \lambda$, denoted by $\langle Z, V \rangle'$. The natural inclusion $\iota : E \to E''$ is given by

$$\langle Z, \iota(X) \rangle := \langle X, Z \rangle$$

for $X \in E$ and $Z \in E'$. With the weak topology $\sigma(\lambda, E')$ on $E$ and the weak* topology $\sigma(\lambda, E'')$ on $\iota(E) \subset E''$ the natural imbedding $\iota$ is obviously a homeomorphism since in both cases we have the initial topology with respect to the same set of functions. This allows us to formulate the theorem of Goldstine for $\lambda$-modules:

**Theorem 12.1.** Let $E$ be a $\lambda$-module with $\lambda$-norm $\| \cdot \|$ and $\eta \in \lambda_\pi$. Then $\iota(B_\eta$) is $\sigma(\lambda, E')$-dense in $B''_\eta$; i.e.

$$\overline{\iota(B_\eta)} = B''_\eta$$

where the closure $\overline{\iota(B_\eta)}$ has to be taken in the $\sigma(\lambda, E')$-topology. This implies that $E$ is $\sigma(\lambda, E''', E')$-dense in $E''$.

**Proof.** In the dual pair $(E', E'')$ of $\lambda$-modules we see by (8.1) that $\overline{(B_\eta)} = B''_\eta$. Then the Bipolar theorem 9.3 for $(E', E'')$ implies $\overline{\iota(B_\eta)} = \sigma(\lambda, \iota(B_\eta)) = \overline{(\iota(B_\eta)} = (\overline{(\iota(B_\eta)}) = (\overline{(\iota(B_\eta)} = B''_\eta$. Here, the closures are taken with respect to $\sigma(\lambda, E''', E')$.

The last theorem gives rise to the notion of reflexible Banach $\lambda$-modules:
Definition 12.1. A Banach $\lambda$-module is called reflexive if

$$\iota(B_1) = B''_1. \quad (12.3)$$

An immediate consequence of the theorem 11.1 is the following complement to corollary 9.4.

Corollary 12.2. For a normed $\lambda$-module $E$ the dual balls

$$B''_\eta(Z) := \{ Z' \mid \| Z' - Z \| \leq \eta \} \quad (12.4)$$

are compact in the weak*-weak*-topology $\sigma(E', E, \kappa)$ for all $Z \in E'$ and $\eta \in \lambda_\eta$.

Proof. It suffices to show the assertion for $B''_\eta$. But $B''_\eta = (B_1/\eta)^{**}$ which is weak*-closed by proposition 9.2 and weak*-weak*-compact by theorem 11.1. \qed

Now, we can characterize reflexive Banach $\lambda$-modules:

Theorem 12.3. For a Banach $\lambda$-module $E$ we have the equivalence:

$E$ is reflexive if and only if $B_\eta$ is compact in the weak*-weak*-topology $\sigma_\lambda(E, E', \kappa)$.

Proof. Since $\iota : E \rightarrow E''$ is injective, it is a homeomorphic bijection and since $\iota(B_\eta)$ is compact in the weak*-weak*-topology $\sigma_\lambda(E'', E', \kappa)$, so is $B_\eta$ in the $\sigma_\lambda(E, E', \kappa)$ topology.

Conversely: Since the imbedding $\iota : (E, \sigma_\lambda(E, E')) \rightarrow (\iota(E), \sigma_\lambda(E, E'))$ is a homeomorphism, the condition implies that $\iota(B_\eta)$ is compact in $\sigma_\lambda(\iota(E), E', \kappa)$, in particular, $\iota(B_\eta)$ is $\sigma_\lambda(\iota(E), E', \kappa)$-closed. By theorem 12.1 $\iota(B_\eta)$ is $\sigma_\lambda(\iota(E), E')$-dense in $B''_\eta$. A fortiori $\iota(B_\eta)$ is also $\sigma_\lambda(\iota(E), E', \kappa)$-dense in $B''_\eta$. The two properties of $\iota(B_\eta)$ in the $\sigma_\lambda(\iota(E), E', \kappa)$-topology imply that $\iota(B_\eta) = B''_\eta,$ i.e. $\iota$ is surjective. \qed

References


---

1Laboratoire de recherche en gestion et économie and Institute de recherche mathématique avancée
Université de Strasbourg
PEGE, 61 Avenue de la Foret-Noire, F-67085 Strasbourg Cedex, France
E-mail address: eisele@unistra.fr

2Université El-Manar, faculté des sciences, Campus Universitaire, 2092 El Manar Tunis,
E-mail address: ettaeib@yahoo.fr