Time-Consistent Supervisory Accounting

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Introductory Remarks

This work intends to connect the theory of risk measures with the ongoing solvency discussion: Solvency II and Swiss Solvency Test (SST) by:

1. constructing a **top-down** model of risk for insurance companies starting from
   - a market-consistent best-estimate operator, in particular for non-hedgeable cash-flows,
   - a multiperiod risk assessment operator;

This construction includes successive precise definitions of:
   - free capital ($F$),
   - supervisory provision ($SP$) and optimal replicating portfolio (ORP) of given obligations,
   - Technical provision ($TP$),
   - risk margin ($RM$), and
   - solvency capital requirement ($SCR$).

2. Comparing these items with the corresponding items of the industry **bottom-up** constructions in Solvency II and SST.
Financial Market

We work with a discrete time space \( \{0, \ldots, T\} \) and a finite filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \leq T}, \mathbb{P})\).

The financial market is given by processes:

\[
\left\{ S^i, i = 0, \ldots, d \right\}
\]

with \( S^0 = r \) as numéraire.

\[
r(s) := \frac{1}{r_s} \mathbb{1}_{s \geq s} \cdot r
\]

are the saving processes starting at \( s \) in the asset \( r \).

\( \mathcal{M} \neq \emptyset \) is the set of risk-neutral pricing probabilities.

The set of zero-cost asset values at \( t \) is

\[
N_t := \left\{ D_t = \sum_{i=0}^{d} S^i_t \cdot \xi^i_{t-1} \mid \sum_{i=0}^{d} S^i_{t-1} \cdot \xi^i_{t-1} = 0, \ \xi^i_{t-1} \text{ are } \mathcal{F}_{t-1}\text{-measurable} \right\}.
\]

(2)
We understand by a time-consistent risk assessment $\Psi = (\Psi_t)_{t \leq T}$ a family of functionals $\Psi_t$ of adapted processes $X$ to $\mathcal{F}_t$-measurable random variables with the following properties:

1. **Final assessment:**
2. **Localization:**
3. **Monotonicity:**
4. **Dependence on future values:**
5. **Cash invariance:**
6. **Lower semi-continuity:**
7. **Time-consistency:**
We understand by a **time-consistent risk assessment** \( \Psi = (\Psi_t)_{t \leq T} \) a family of functionals \( \Psi_t \) of adapted processes \( X \) to \( \mathcal{F}_t \)-measurable random variables with the following properties:

1. **Final assessment**: \( \Psi_T = 0 \),
2. **Localization**: \( \Psi_t(X \cdot 1_A) = \Psi_t(X) \cdot 1_A \),
3. **Monotonicity**: If \( X \leq Y \) then \( \Psi_t(X) \leq \Psi_t(Y) \),
4. **Dependence on future values**: If \( X1_{t>t} = Y1_{t>t} \), then \( \Psi_t(X) = \Psi_t(Y) \),
5. **Cash invariance**: \( \Psi_t(X + Y_t \cdot r(t)) = \Psi_t(X) + Y_t \), \( (3) \)
6. **Lower semi-continuity**: If \( X_n \rightharpoonup X \) for \( n \to \infty \), then \( \Psi_t(X_n) \rightharpoonup \Psi_t(X) \), \( t \in T \).
7. **Time-consistency**: For all \( 0 \leq s \leq t < T \)

\[
\Psi_s(X) = \Psi_s(X1_{t \leq t} + \Psi_t(X) \cdot r(t)1_{t>t}).
\] \( (4) \)
Remark 1

*Note that the definition does not contain any convexity or coherence condition.*
Supervisory Arbitrage

Definition 2

A family $\Phi = (\Phi_t)_{t \leq T}$ of functionals $\Phi_t$ on processes $X$ allows supervisory arbitrage if for some $X$ and $t < T$

$$
\sup_{D_u \in N_u, \ t < u \leq T} \Phi_t \left( X + \sum_{u > t} D_u \cdot r(u) \right) = +\infty.
$$

(5)

Proposition 1

Let $\Phi$ as before with

$$
\Phi_t (X) \leq \mathbb{E}_Q \left[ X_T \frac{r_t}{r_T} \bigg| \mathcal{F}_t \right]
$$

(6)

for some $Q \in \mathcal{M}$. Then $\Phi$ doesn’t allow supervisory arbitrage.
Remark 2

*For the risk assessment $\psi$, we shall assume*

$$\psi_t(X) \leq \mathbb{E}_Q \left[X_T \frac{r_t}{r_T} \mid \mathcal{F}_t \right] \tag{7}$$

*for some $Q \in \mathcal{M}$, which is weaker than coherence and at the same time excludes supervisory arbitrage.*

We fix $Q \in \mathcal{M}$ with (7) for the rest of the talk.
Definition 3

A family $\Phi = (\Phi_t)_{t \leq T}$ of functionals on processes $X$ is called market consistent if for all $X$, $t < u \leq T$ and $D_u \in N_u$

$$\Phi_t(X) = \Phi_t(X + D_u \cdot r(u)).$$  \hspace{1cm} (8)
Business Plan Accounting

The business plan \((C, Z) = (C_t, Z_t)_{t\in\mathbb{T}}\) of an insurance company is a sequence of current assets values \(C_t\) and obligations cash-flows \(Z_t\). \(Z_t\) is the obligation that company has to pay at time \(t\) to its policy-holders according to the contracts signed at \(t = 0\). \(Z\) is exogenously given.

We always assume the self-financing (w. r. t. \(Z\)) condition:
For each \(t > 0\)
\[
D_t := C_t + Z_t - C_{t-1} \frac{r_t}{r_{t-1}} \in N_t,
\]
(9)

\(D_t\) is the company’s trading-risk exposure

However, \(Z_t\) is generally not in the market.
The process of estimated obligations

**Definition 4**

The process of estimated obligations $\overline{Z}$ is given by

$$\overline{Z}_t := \mathbb{E}_Q \left[ \sum_{u > t} Z_u \frac{r_t}{r_u} \mid \mathcal{F}_t \right].$$

(10)

In Solvency II and SST, $\overline{Z}_t$ is the Best Estimate of the obligation.

Since we use $\mathcal{Q} \in \mathcal{M}$, we get a market consistent functional.
Acceptable business plan

A business plan \((C, Z)\) satisfies the solvency condition at time \(t\) if and only if

\[
\psi_t(C - Z) \geq 0. \tag{11}
\]

Then, \((C, Z)\) is called \(t\)-acceptable.

**Definition 5 (Free capital)**

The free capital process \(F(C, Z)\) of a business plan is

\[
F_t(C, Z) := \psi_t(C - Z). \tag{12}
\]

By cash invariance, we have

\[
\psi_t \left( C - \psi_t(C - Z) \cdot r(t) - Z \right) = \psi_t(C - Z) - \psi_t(C - Z) = 0.
\]

Thus, \((C - F_t(C, Z) \cdot r(t), Z)\) is a \(t\)-acceptable business plan.
Supervisory Provision

Definition 6

The process of supervisory provisions $SP(Z) = (SP_t(Z))_{t\leq T}$ for a cash-flow process $Z$ of obligations is given by

$$SP_t(Z) := \inf \{ C_t \mid (C, Z) \text{ is a } t\text{-acceptable plan} \}.$$ \hfill (13)

$SP_t$ is market-consistent (as a functional of $Z$).
Proposition 2

We find

\[ SP_t(Z) - \bar{Z}_t = -\psi_t^* (-\bar{Z}) \geq 0 \quad (14) \]

where

\[ \psi_t^*(X) := \sup_{\substack{D_u \in N_u \\ t < u \leq T}} \psi_t \left( X + \sum_{u=t+1}^{T} D_u \cdot r(u) \right) \quad (15) \]

is the market-consistent majorant of \( \psi_t \).
Optimal Replicating Portfolio

For each $C$, we saw that $(C - F_t(C, Z) \cdot r(t), Z)$ is a $t$-acceptable business plan, hence

$$C_t - F_t(C, Z) \geq SP_t(Z)$$

We study the case where in (13) the inf is actually reached by $C_t - F_t(C, Z)$: i.e.

$$C^*_t - F_t(C^*, Z) = SP_t(Z).$$

**Definition 7**

An process $C^*$ of current asset values is called a $t$-optimal replicating portfolio for the obligation process $Z$ if and only if $C^*$ satisfies the self-financing condition (9) and

$$C^*_t = SP_t(Z) + F_t(C^*, Z).$$

(16)
Market prudence of a risk assessment $\Psi$

There are examples where ORP does not exist.

An additional assumption on $\Psi$ called “market prudence” detects trading-risk exposures different from zero:

**Definition 8**

The risk assessment $\Psi$ is called **market prudent** if for all $0 \leq t < T$ and all $D_{t+1} \in N_{t+1} \setminus \{0\}$, we have a. s.

$$\Psi_t(D_{t+1} \cdot r(t + 1)) < 0.$$  \hspace{1cm} (17)

For a market prudent $\Psi$, $r$ is the only asset evaluated according to market prices by $\Psi$. 

Existence of an optimal replicating portfolio

Market prudence implies the existence of the extremum in the definition of the supervisory provision:

**Proposition 3**

Let $\Psi$ be a homogeneous, continuous and market prudent assessment. Then for every obligation process $Z$, $t \leq T$ and each current asset value $C_t$, there exists a self-financing $t$-optimal replicating portfolio $C^*$ with $C^*_t = C_t$. Moreover, $C^*$ is $u$-optimal replicating for all $u \geq t$. 
Acceptability by rebalancing of the asset portfolio

Definition 7 and Proposition 3 show for a market-prudent $\Psi$ that the “solvability condition”

$$C_0 \geq SP_0(Z)$$

allows for a (optimal replicating) portfolio $C^*$ with $C^*_0 = C_0$ and

$$C^*_0 = SP_0(Z) + F_0(C^*, Z),$$

hence $F_0(C^*, Z) \geq 0$ and the plan $(C^*, Z)$ is acceptable.

Solvability implies acceptability under ORP.
Transfer of obligations of a non-solvable company

Now assume $C_0 < SP_0(Z)$.

No plan $(\tilde{C}, Z)$ with $\tilde{C}_0 = C_0$ can be acceptable and we look for a transfer of the obligations $Z$.

The tool is the value investors are willing to pay for the “insurance option” (=terminal free capital=terminal current asset value)

$$(C^*_T)^+$$

of a candidate company choosing an optimal replicating portfolio $C^*$ with $C^*_0 \geq SP_0(Z)$.

The new empty company needs at date 0 a current asset value of at least $SP_0(Z)$. It can be funded out of two sources:
• A first amount $\text{SCR}_0^*(Z)$ (fixed by the regulator) which the market accepts to pay for the option $(C_T^*)^+$, $C^*$ being the ORP of $Z$. This $\text{SCR}_0^*(Z)$ would be of own funds (or “Eigenkapital”) type.

• A second amount:

$$TP_0(Z) := SP_0(Z) - \text{SCR}_0^*(Z), \quad (20)$$

called technical provision, provided by the old company which is then released of its obligations $Z$. This $TP_0(Z)$ is added as “foreign capital” or “Fremdkapital” to the new company, which is now acceptable under the ORP $C^*$ since

$$C_0^* = SP_0(Z) + F_0(C^*, Z) \geq \text{SCR}_0^*(Z) + TP_0(Z) = SP_0(Z). \quad (21)$$

(Acceptability by transfer under ORP)

Note that the transfer is only possible as long as the old company has a current asset value of at least $TP_0(Z)$. 21 / 26
For $u > t$ we now fix $D_u \in N_u$ and define

$$\tilde{C} := \sum_{u > t} (D_u - Z_u) \cdot r(u).$$

such that the $D_u$ are the trading-risk exposures of $\tilde{C}$. Then $(\tilde{C} - \psi_t(\tilde{C} - \bar{Z}) \cdot r(t), \bar{Z})$ is $t$-acceptable. Moreover $-\psi_t(\tilde{C} - \bar{Z})$ is the minimal asset value for any $t$-acceptable business $(C, Z)$ with the same $D_u$.

Since $\tilde{C}_t = 0$, we get

$$-\psi_t(\tilde{C} - \bar{Z}) \geq SP_t(Z) \geq TP_t(Z). \quad (22)$$
Solvency Capital Requirement (SCR)

**Definition 9**

1. The difference between the necessary asset value \(-\psi_t(\tilde{C} - \bar{Z})\) and the technical provision \(TP_t(Z)\) is the solvency capital requirement (SCR):

\[
SCR_t(C, Z) := -\psi_t(\sum_{u> t} (D_u - Z_u) \cdot r(u) - \bar{Z}) - TP_t(Z). \tag{23}
\]

According to (22), the solvency capital requirement splits into two parts:

2. \(SCR_t^*(Z) := SP_t(Z) - TP_t(Z) \tag{24}\)

is the solvency capital requirement under OPR. It depends only on the obligation cash-flows \(Z\).

3. \(SCR_t'(C, Z) := -\psi_t(\sum_{u> t} (D_u - Z_u) \cdot r(u) - \bar{Z}) - SP_t(Z) \geq 0 \tag{25}\)

is the relative solvency capital requirement (rSCR).
Characterizing the optimal replicating portfolio

Proposition 4

*C* is a $t$-optimal replicating portfolio for the obligation process *Z* if and only if

$$SCR_i^t(C, Z) = 0.$$ (26)
Supervisory accounting equality

One can show that for any business plan $(C, Z)$ with trading-risk exposures $D_u, u > t$,

$$C_t - \psi_t(C - \overline{Z}) = -\psi_t(\sum_{u > t} (D_u - Z_u) \cdot r(u) - \overline{Z})$$  \hspace{1cm} (27)

so that

$$C_t - \psi_t(C - \overline{Z}) - SP_t(Z) = SCR_t^r(C, Z).$$  \hspace{1cm} (28)

This implies:

**Proposition 5**

$$C_t = F_t(C - \overline{Z}) + SCR_t(C, Z) + TP_t(Z).$$  \hspace{1cm} (29)

This is the *supervisory accounting equality*. 
Conclusions

• We distinguish four supervisory actions:
  1. Acceptability, if \( C_0 \geq TP_0(Z) + SCR_0(C, Z) \),
  2. Rebalancing of the portfolio, if \( C_0 \geq TP_0(Z) + SCR^*_0(Z) \),
  3. Transfer to a new company, if \( C_0 \geq TP_0(Z) \),
  4. Closing of the company, if \( C_0 < TP_0(Z) \).

• The top-down approach allows to impose reasonable properties from the very beginning.

• The definitions of the provisions \( SP_t(Z), TP_t(Z) \) depend only on the obligations \( Z \).

• The concept of \( SCR^*_t(Z) \) essential for \( TP_t(Z) \) needs a precise definition of optimal replicating portfolio, which itself is based on the supervisory provision. Therefore supervisory provision is a cornerstone in the construction, even if it does not appear explicitly in supervisory accounting equation.