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Fix-Mahonian Calculus, I: two transformations

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aedificavisti Lacim Uqam tuam.

To Pierre Leroux, Montreal, Sept. 2006,
on the occasion of the LerouxFest.

Abstract. We construct two bijections of the symmetric group $\mathfrak{S}_n$ onto itself that enable us to show that three new three-variable statistics are equidistributed with classical statistics involving the number of fixed points. The first one is equidistributed with the triplet $(\mathrm{fix}, \mathrm{des}, \mathrm{maj})$, the last two with $(\mathrm{fix}, \mathrm{exc}, \mathrm{maj})$, where "fix," "des," "exc" and "maj" denote the number of fixed points, the number of descents, the number of excedances and the major index, respectively.

1. Introduction

In this paper *Fix-Mahonian Calculus* is understood to mean the study of multivariable statistics on the symmetric group $\mathfrak{S}_n$, which involve the number of fixed points "fix" as a marginal component. As for the two transformations mentioned in the title, they make it possible to show that the new statistics defined below are equidistributed with the classical ones.

The *descent set*, $\mathrm{DES}\, w$, and *rise set*, $\mathrm{RISE}\, w$, of a word $w = x_1 x_2 \cdots x_n$, whose letters are nonnegative integers, are respectively defined as being the *subsets*:

(1.1) $$\mathrm{DES}\, w := \{i : 1 \le i \le n-1, x_i > x_{i+1}\};$$

(1.2) $$\mathrm{RISE}\, w := \{i : 1 \le i \le n, x_i \le x_{i+1}\}.$$

By convention, $x_0 = x_{n+1} = +\infty$. If $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ is a permutation of $12\cdots n$, we can then consider $\mathrm{RISE}\,\sigma$, but also a new statistic $\mathrm{RIZE}\,\sigma$, which is simply the rise set of the word w derived from σ by replacing each *fixed point* $\sigma(i) = i$ by 0. For instance, with $\sigma = 3\,2\,5\,4\,1$ having the two fixed points 2, 4, we get $w = 3\,0\,5\,0\,1$, so that $\mathrm{RISE}\,\sigma = \{2, 5\}$ and $\mathrm{RIZE}\,\sigma = \mathrm{RISE}\, w = \{2, 4, 5\}$. It is quite unexpected to notice that the two set-theoretic statistics "RISE" and "RIZE" are equidistributed on each symmetric group $\mathfrak{S}_n$. Such an equidistribution property, which is a consequence of our Theorem 1.4, has become a powerful tool, as it has enabled

Xin and the second author [HaXi07] to give an immediate proof of a conjecture of Stanley [St06] on alternating permutations. More importantly, the equidistribution properties are proved by means of new transformations of the symmetric group, the bijections Φ and F_3 introduced in the sequel, whose properties will be fully exploited in our next paper [FoHa07]. Those transformations will be described not directly on $\mathfrak{S}_n$, but on classes of *shuffles*, as now introduced.

Let $0 \le m \le n$ and let v be a nonempty word of length m, whose letters are *positive* integers (with possible repetitions). Designate by $\mathrm{Sh}(0^{n-m}v)$ the set of all *shuffles* of the words 0^{n-m} and v, that is, the set of all rearrangements of the juxtaposition product $0^{n-m}v$, whose longest *subword* of positive letters is v. Let $w = x_1x_2\cdots x_n$ be a word from $\mathrm{Sh}(0^{n-m}v)$. It is convenient to write: $\mathrm{Pos}\, w := v$, $\mathrm{Zero}\, w := \{i : 1 \le i \le n,\, x_i = 0\}$, $\mathrm{zero}\, w := \#\,\mathrm{Zero}\, w$ $(= n-m)$, so that w is completely characterized by the pair $(\mathrm{Zero}\, w, \mathrm{Pos}\, w)$.

The *major index* of w is defined by

$$\mathrm{maj}\, w := \sum_{i\ge 1} i \quad (i \in \mathrm{DES}\, w), \tag{1.3}$$

and a new integral-valued statistic "mafz" by

$$\mathrm{mafz}\, w := \sum_{i\in \mathrm{Zero}\, w} i - \sum_{i=1}^{\mathrm{zero}\, w} i + \mathrm{maj}\,\mathrm{Pos}\, w. \tag{1.4}$$

Note that the first three definitions are also valid for each arbitrary word with nonnegative letters. The link of "mafz" with the statistic "maf" introduced in [CHZ97] for permutations will be further mentioned.

We shall also be interested in shuffle classes $\mathrm{Sh}(0^{n-m}v)$ when the word v is a *derangement* of the set $[\,m\,] := \{1, 2, \ldots, m\}$, that is, when the word $v = y_1y_2\cdots y_m$ is a permutation of $12\cdots m$ and $y_i \ne i$ for all i. For short, v is a *derangement of order* m. Let $w = x_1x_2\cdots x_n$ a be word from the shuffle class $\mathrm{Sh}(0^{n-m}v)$. Then $v = y_1y_2\cdots y_m = x_{j_1}x_{j_2}\cdots x_{j_m}$ for a certain sequence $1 \le j_1 < j_2 < \cdots < j_m \le n$. Let "red" be the increasing bijection of $\{j_1, j_2, \ldots, j_m\}$ onto $[\,m\,]$. Say that each positive letter x_k of w is *excedent* (resp. *subexcedent*) if and only if $x_k > \mathrm{red}\, k$ (resp. $x_k < \mathrm{red}\, k$). Another kind of rise set, denoted by $\mathrm{RISE}^\bullet\, w$, can then be introduced as follows.

Say that $i \in \mathrm{RISE}^\bullet\, w$ if and only if $1 \le i \le n$ and if one of the following conditions holds (assuming that $x_{n+1} = +\infty$):

(1) $0 < x_i < x_{i+1}$;
(2) $x_i = x_{i+1} = 0$;
(3) $x_i = 0$ and x_{i+1} is excedent;
(4) x_i is subexcedent and $x_{i+1} = 0$.

Note that if $x_i = 0$ and x_{i+1} is subexcedent, then $i \in \mathrm{RISE}\, w \setminus \mathrm{RISE}^\bullet\, w$, while if x_i is subexcedent and $x_{i+1} = 0$, then $i \in \mathrm{RISE}^\bullet\, w \setminus \mathrm{RISE}\, w$.

Example. Let $v = 5\,1\,2\,3\,6\,4$ be a derangement of order 6. Its excedent letters are 5, 6. Let $w = 5\,0\,1\,2\,0\,0\,3\,6\,4 \in \mathrm{Sh}(0^3 v)$. Then, $\mathrm{RISE}\, w = \{2, 3, 5, 6, 7, 9\}$ and $\mathrm{RISE}^\bullet\, w = \{3, 4, 5, 7, 9\}$. Also $\mathrm{mafz}\, w = (2 + 5 + 6) - (1 + 2 + 3) + \mathrm{maj}(512364) = 7 + 6 = 13$.

Theorem 1.1. *For each derangement v of order m and each integer $n \geq m$ the transformation $\mathbf{\Phi}$ constructed in Section 2 is a bijection of $\mathrm{Sh}(0^{n-m}v)$ onto itself having the property that*

$$\mathrm{RISE}\, w = \mathrm{RISE}^\bullet\, \mathbf{\Phi}(w) \tag{1.5}$$

holds for every $w \in \mathrm{Sh}(0^{n-m}v)$.

Theorem 1.2. *For each arbitrary word v of length m with positive letters and each integer $n \geq m$ the transformation $\mathbf{F}_3$ constructed in Section 4 is a bijection of $\mathrm{Sh}(0^{n-m}v)$ onto itself having the property that*

$$\mathrm{maj}\, w = \mathrm{mafz}\, \mathbf{F}_3(w); \tag{1.6}$$

$$L\, w = L\, \mathbf{F}_3(w) \quad (\text{“}L\text{” for “last” or rightmost letter}); \tag{1.7}$$

hold for every $w \in \mathrm{Sh}(0^{n-m}v)$.

We emphasize the fact that Theorem 1.1 is restricted to the case where v is a derangement, while Theorem 1.2 holds for an arbitrary word v with possible repetitions. In Fig. 1 we can see that “RISE” and “$\mathrm{RISE}^\bullet$” (resp. “maj” and “mafz”) are equidistributed on the shuffle class $\mathrm{Sh}(0^2312)$ (resp. $\mathrm{Sh}(0^2121)$).

$\mathrm{RISE}\, w$	w	$\mathbf{\Phi}(w)$	$\mathrm{RISE}^\bullet \mathbf{\Phi}(w)$
1, 2, 4, 5	0 0 3 1 2	0 0 3 1 2	1, 2, 4, 5
1, 3, 4, 5	0 3 0 1 2	0 3 1 2 0	1, 3, 4, 5
1, 4, 5	0 3 1 0 2	0 3 0 1 2	1, 4, 5
1, 3, 5	0 3 1 2 0	0 3 1 0 2	1, 3, 5
2, 3, 4, 5	3 0 0 1 2	3 1 2 0 0	2, 3, 4, 5
2, 4, 5	3 0 1 0 2 3 1 2 0 0	3 0 0 1 2 3 1 0 2 0	2, 4, 5
2, 3, 5	3 0 1 2 0	3 0 1 0 2	2, 3, 5
3, 4, 5	3 1 0 0 2	3 0 1 2 0	3, 4, 5
3, 5	3 1 0 2 0	3 1 0 0 2	3, 5

$\mathrm{Sh}(0^2312)$

$\mathrm{maj}\, w$	w	$\mathbf{F}_3(w)$	$\mathrm{mafz}\, \mathbf{F}_3(w)$
2	1 2 0 0 1	0 0 1 2 1	2
3	0 1 2 0 1	0 1 0 2 1	3
4	0 0 1 2 1 1 0 2 0 1	1 0 0 2 1 0 1 2 0 1	4
5	1 0 0 2 1 1 2 1 0 0	1 0 2 0 1 0 1 2 1 0	5
6	0 1 0 2 1 1 2 0 1 0	1 2 0 0 1 1 0 2 1 0	6
7	0 1 2 1 0	1 2 0 1 0	7
8	1 0 2 1 0	1 2 1 0 0	8

$\mathrm{Sh}(0^2121)$

Fig. 1

Those two transformations are fully exploited once we know how to map those shuffle classes onto the symmetric groups. The permutations from the symmetric group $\mathfrak{S}_n$ will be regarded as linear words $\sigma =$

$\sigma(1)\sigma(2)\cdots\sigma(n)$. If σ is such a permutation, let $\mathrm{FIX}\,\sigma$ denote the set of its fixed points, i.e., $\mathrm{FIX}\,\sigma := \{i : 1 \le i \le n,\, \sigma(i) = i\}$ and let $\mathrm{fix}\,\sigma := \#\,\mathrm{FIX}\,\sigma$. Let $(j_1, j_2, \dots, j_m)$ be the increasing sequence of the integers k such that $1 \le k \le n$ and $\sigma(k) \ne k$ and "red" be the increasing bijection of $\{j_1, j_2, \dots, j_m\}$ onto $[m]$. The word $w = x_1x_2\cdots x_n$ derived from $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ by replacing each fixed point by 0 and each other letter $\sigma(j_k)$ by $\mathrm{red}\,\sigma(j_k)$ will be denoted by $\mathrm{ZDer}(\sigma)$. Also let

$$\mathrm{Der}\,\sigma := \mathrm{red}\,\sigma(j_1)\ \mathrm{red}\,\sigma(j_2)\ \cdots\ \mathrm{red}\,\sigma(j_m), \tag{1.8}$$

so that $\mathrm{Der}\,\sigma$ is the word derived from $\mathrm{ZDer}(\sigma)$ by deleting all the zeros. Accordingly, $\mathrm{Der}\,\sigma = \mathrm{Pos}\ \mathrm{ZDer}(\sigma)$.

It is important to notice that $\mathrm{Der}\,\sigma$ is a *derangement* of order m. Also $\sigma(j_k)$ is excedent in σ (*i.e.* $\sigma(j_k) > j_k$) if and only $\mathrm{red}\,\sigma(j_k)$ is excedent in $\mathrm{Der}\,\sigma$ (*i.e.* $\mathrm{red}\,\sigma(j_k) > \mathrm{red}\,j_k$)

Recall that the statistics "DES," "RISE" and "maj" are also valid for permutations $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ and that the statistics "des" (*number of descents*) and "exc" (*number of excedances*) are defined by

$$\mathrm{des}\,\sigma := \#\,\mathrm{DES}\,\sigma; \tag{1.9}$$

$$\mathrm{exc}\,\sigma := \#\{i : 1 \le i \le n-1,\, \sigma(i) > i\}. \tag{1.10}$$

We further define:

$$\mathrm{DEZ}\,\sigma := \mathrm{DES}\ \mathrm{ZDer}(\sigma); \tag{1.11}$$

$$\mathrm{RIZE}\,\sigma := \mathrm{RISE}\ \mathrm{ZDer}(\sigma); \tag{1.12}$$

$$\mathrm{dez}\,\sigma := \#\,\mathrm{DEZ}\,\sigma = \mathrm{des}\ \mathrm{ZDer}(\sigma); \tag{1.13}$$

$$\mathrm{maz}\,\sigma := \mathrm{maj}\ \mathrm{ZDer}(\sigma); \tag{1.14}$$

$$\mathrm{maf}\,\sigma := \mathrm{mafz}\ \mathrm{ZDer}(\sigma). \tag{1.15}$$

As the zeros of $\mathrm{ZDer}(\sigma)$ correspond to the fixed points of σ, we also have

$$\mathrm{maf}\,\sigma := \sum_{i\in \mathrm{FIX}\,\sigma} i - \sum_{i=1}^{\mathrm{fix}\,\sigma} i + \mathrm{maj}\,\mathrm{Der}\,\sigma. \tag{1.16}$$

Example. Let $\sigma = 8\,\mathbf{2}\,1\,3\,\mathbf{5}\,\mathbf{6}\,4\,9\,7$; then $\mathrm{DES}\,\sigma = \{1,2,6,8\}$, $\mathrm{des}\,\sigma = 4$, $\mathrm{maj}\,\sigma = 17$, $\mathrm{exc}\,\sigma = 2$. Furthermore, $\mathrm{ZDer}(\sigma) = w = 5\,0\,1\,2\,0\,0\,3\,6\,4$ and $\mathrm{Pos}\,w = \mathrm{Der}\,\sigma = 5\,1\,2\,3\,6\,4$ is a derangement of order 6. We have $\mathrm{FIX}\,\sigma = \{2,5,6\}$, $\mathrm{fix}\,\sigma = 3$, $\mathrm{DEZ}\,\sigma = \{1,4,8\}$, $\mathrm{RIZE}\,w = \{2,3,5,6,7,9\}$, $\mathrm{dez} = 3$, $\mathrm{maz}\,\sigma = 13$ and $\mathrm{maf}\,\sigma = (2{+}5{+}6){-}(1{+}2{+}3){+}\mathrm{maj}(512364) = 7{+}6 = 13$.

For each $n \ge 0$ let D_n be the set of all derangements of order n and $\mathfrak{S}_n^{\mathrm{Der}}$ be the union: $\mathfrak{S}_n^{\mathrm{Der}} := \bigcup\limits_{m,v} \mathrm{Sh}(0^{n-m}v) \quad (0 \le m \le n,\, v \in D_m)$.

Proposition 1.3. *The map* ZDer *is a bijection of* $\mathfrak{S}_n$ *onto* $\mathfrak{S}_n^{\mathrm{Der}}$ *having the following properties:*

$$\mathrm{RIZE}\,\sigma = \mathrm{RISE}\;\mathrm{ZDer}(\sigma) \quad \textit{and} \quad \mathrm{RISE}\,\sigma = \mathrm{RISE}^{\bullet}\;\mathrm{ZDer}(\sigma). \tag{1.17}$$

Proof. It is evident to verify that ZDer is bijective and to define its inverse ZDer^{-1}. On the other hand, we have $\mathrm{RIZE} = \mathrm{RISE}\,\mathrm{ZDer}$ by definition. Finally, let $w = x_1x_2\cdots x_n = \mathrm{ZDer}(\sigma)$ and $\sigma(i) < \sigma(i+1)$ for $1 \le i \le n-1$. Four cases are to be considered:

(1) both i and $i+1$ are not fixed by σ and $0 < x_i < x_{i+1}$;

(2) both i and $i+1$ are fixed points and $x_i = x_{i+1} = 0$;

(3) $\sigma(i) = i$ and $\sigma(i+1)$ is excedent; then $x_i = 0$ and x_{i+1} is also excedent;

(4) $\sigma(i) < i < i+1 = \sigma(i+1)$; then x_i is subexcedent and $x_{i+1} = 0$.

We recover the four cases considered in the definition of $\mathrm{RISE}^{\bullet}$. The case $i = n$ is banal to study. ∎

We next form the two chains:

$$\Phi : \sigma \overset{\mathrm{ZDer}}{\mapsto} w \overset{\boldsymbol{\Phi}}{\mapsto} w' \overset{\mathrm{ZDer}^{-1}}{\mapsto} \sigma'; \tag{1.18}$$

$$\mathrm{F}_3 : \sigma \overset{\mathrm{ZDer}}{\mapsto} w \overset{\mathbf{F}_3}{\mapsto} w'' \overset{\mathrm{ZDer}^{-1}}{\mapsto} \sigma''. \tag{1.19}$$

The next theorem is then a consequence of Theorems 1.1 and 1.2 and Propositions 1.3.

Theorem 1.4. *The mappings* Φ, F_3 *defined by* (1.18) *and* (1.19) *are bijections of* $\mathfrak{S}_n$ *onto itself and have the following properties*

$$(\mathrm{fix}, \mathrm{RIZE}, \mathrm{Der})\,\sigma = (\mathrm{fix}, \mathrm{RISE}, \mathrm{Der})\,\Phi(\sigma); \tag{1.20}$$

$$(\mathrm{fix}, \mathrm{maz}, \mathrm{Der})\,\sigma = (\mathrm{fix}, \mathrm{maf}, \mathrm{Der})\,\mathrm{F}_3(\sigma); \tag{1.21}$$

$$(\mathrm{fix}, \mathrm{maj}, \mathrm{Der})\,\sigma = (\mathrm{fix}, \mathrm{maf}, \mathrm{Der})\,\mathrm{F}_3 \circ \Phi^{-1}(\sigma); \tag{1.22}$$

for every σ *from* $\mathfrak{S}_n$.

It is evident that if $\mathrm{Der}\,\sigma = \mathrm{Der}\,\tau$ holds for a pair of permutations σ, τ of order n, then $\mathrm{exc}\,\sigma = \mathrm{exc}\,\tau$. Since $\mathrm{DES}\,\sigma = [\,n\,] \setminus \mathrm{RISE}\,\sigma$ and $\mathrm{DEZ}\,\sigma = [\,n\,] \setminus \mathrm{RIZE}\,\sigma$ it follows from (1.20) that

$$(\mathrm{fix}, \mathrm{DEZ}, \mathrm{exc})\,\sigma = (\mathrm{fix}, \mathrm{DES}, \mathrm{exc})\,\Phi(\sigma); \tag{1.23}$$

$$(\mathrm{fix}, \mathrm{dez}, \mathrm{maz}, \mathrm{exc})\,\sigma = (\mathrm{fix}, \mathrm{des}, \mathrm{maj}, \mathrm{exc})\,\Phi(\sigma). \tag{1.24}$$

On the other hand, (1.21) implies that

$$(\mathrm{fix}, \mathrm{maz}, \mathrm{exc})\,\sigma = (\mathrm{fix}, \mathrm{maf}, \mathrm{exc})\,\mathrm{F}_3(\sigma). \tag{1.25}$$

As a consequence we obtain the following Corollary.

Corollary 1.5. *The two triplets* (fix, dez, maz) *and* (fix, des, maj) *are equidistributed over* $\mathfrak{S}_n$. *Moreover, the three triplets* (fix, exc, maz), (fix, exc, maj) *and* (fix, exc, maf) *are also equidistributed over* $\mathfrak{S}_n$.

The distributions of (fix, des, maj) and (fix, exc, maj) have been calculated by Gessel-Reutenauer ([GeRe93], Theorem 8.4) and by Shareshian and Wachs [ShWa06], respectively, using the algebra of the q-series (see, e.g., Gasper and Rahman ([GaRa90], chap. 1). Let

$$A_n(s,t,q,Y) := \sum_{\sigma\in\mathfrak{S}_n} s^{\operatorname{exc}\sigma} t^{\operatorname{des}\sigma} q^{\operatorname{maj}\sigma} Y^{\operatorname{fix}\sigma} \quad (n\ge 0).$$

Then, they respectively derived the identities:

$$\text{(1.26)}\quad \sum_{n\ge0} A_n(1,t,q,Y)\frac{u^n}{(t;q)_{n+1}} = \sum_{r\ge0} t^r\Bigl(1-u\sum_{i=0}^{r} q^i\Bigr)^{-1}\frac{(u;q)_{r+1}}{(uY;q)_{r+1}};$$

$$\text{(1.27)}\quad \sum_{n\ge0} A_n(s,1,q,Y)\frac{u^n}{(q;q)_n} = \frac{(1-sq)e_q(Yu)}{e_q(squ)-sqe_q(u)}.$$

In our third paper [FoHa07a] we have shown that the factorial generating function for the four-variable polynomials $A_n(s,t,q,Y)$ could be evaluated under the form

$$\text{(1.28)}\quad \sum_{n\ge0} A_n(s,t,q,Y)\frac{u^n}{(t;q)_{n+1}} = \sum_{r\ge0} t^r \frac{(1-sq)\,(u;q)_r\,(usq;q)_r}{((u;q)_r - sq(usq;q)_r)(uY;q)_{r+1}},$$

the two identities (1.26) and (1.27) becoming simple specializations. We then know the distributions over $\mathfrak{S}_n$ of the triplets (fix, dez, maz), (fix, exc, maz) and (fix, exc, maf) and also the distribution of the quadruplet (fix, dez, maz, exc). Note that the statistic “maf” was introduced by Clarke *et al.* [CHZ97]. Although it was not explicitly stated, their bijection “CHZ” of $\mathfrak{S}_n$ onto itself satisfies identity (1.22) when $\mathrm{F}_3\circ\Phi^{-1}$ is replaced by “CHZ.”

As is shown in Section 2, the transformation $\mathbf{\Phi}$ is described as a composition product of bijections ϕ_l. The image $\phi_l(w)$ of each word w from a shuffle class $\mathrm{Sh}(0^{n-m}v)$ is obtained by moving its l-th zero, to the right or to the left, depending on its preceding and following letters. The description of the inverse bijection $\mathbf{\Psi}$ of $\mathbf{\Phi}$ follows an analogous pattern. The verification of identity (1.5) requires some attention and is made in Section 3. The construction of the transformation $\mathbf{F}_3$ is given in Section 4. Recall that $\mathbf{F}_3$ maps each shuffle class $\mathrm{Sh}(0^{n-m}v)$ onto itself, the word v being an *arbitrary* word with nonnegative letters. Very much like the *second fundamental transformation* (see, e.g., [Lo83], p. 201, Algorithm 10.6.1) the construction of $\mathbf{F}_3$ is defined by induction on the length of the words and preserves the *rightmost* letter.

2. The bijection $\mathbf{\Phi}$

Let v be a *derangement* of order m and $w = x_1 x_2 \cdots x_n$ be a word from the shuffle class $\mathrm{Sh}(0^{n-m}v)$ $(0 \le n \le m)$, so that $v = x_{j_1} x_{j_2} \cdots x_{j_m}$ for $1 \le j_1 < j_2 < \cdots < j_m \le n$. Let "red" ("reduction") be the increasing bijection of $\{j_1, j_2, \ldots, j_m\}$ onto the interval $[\,m\,]$. Remember that a positive letter x_k of w is said to be excedent (resp. subexcedent) if and only if $x_k > \operatorname{red} k$ (resp. $x_k < \operatorname{red} k$). Accordingly, a letter is non-subexcedent if it is either equal to 0 or excedent.

We define n bijections ϕ_l $(1 \le l \le n)$ of $\mathrm{Sh}(0^{n-m}v)$ onto itself in the following manner: for each l such that $n-m+1 \le l \le n$ let $\phi_l(w) := w$. When $1 \le l \le n-m$, let x_j denote the l-th letter of w, equal to 0, when w is read *from left to right*. Three cases are next considered (by convention, $x_0 = x_{n+1} = +\infty$):

(1) x_{j-1}, x_{j+1} both non-subexcedent;

(2) x_{j-1} non-subexcedent, x_{j+1} subexcedent; or x_{j-1}, x_{j+1} both subexcedent with $x_{j-1} > x_{j+1}$;

(3) x_{j-1} subexcedent, x_{j+1} non-subexcedent; or x_{j-1}, x_{j+1} both subexcedent with $x_{j-1} < x_{j+1}$.

When case (1) holds, let $\phi_l(w) := w$.

When case (2) holds, determine the *greatest* integer $k \ge j+1$ such that

$$x_{j+1} < x_{j+2} < \cdots < x_k < \operatorname{red}(k),$$

so that

$$w = x_1 \cdots x_{j-1}\, 0\, x_{j+1} \cdots x_k\, x_{k+1} \cdots x_n$$

and define:

$$\phi_l(w) := x_1 \cdots x_{j-1}\, x_{j+1} \cdots x_k\, 0\, x_{k+1} \cdots x_n.$$

When case (3) holds, determine the *smallest* integer $i \le j-1$ such that

$$\operatorname{red}(i) > x_i > x_{i+1} > \cdots > x_{j-1},$$

so that

$$w = x_1 \cdots x_{i-1}\, x_i \cdots x_{j-1}\, 0\, x_{j+1} \cdots x_n$$

and define:

$$\phi_l(w) := x_1 \cdots x_{i-1}\, 0\, x_i \cdots x_{j-1}\, x_{j+1} \cdots x_n.$$

It is important to note that ϕ_l has no action on the 0's other than the l-th one. Then the mapping $\mathbf{\Phi}$ in Theorem 1.1 is defined to be the composition product

$$\mathbf{\Phi} := \phi_1 \phi_2 \cdots \phi_{n-1} \phi_n.$$

Example. The following word w has four zeros, so that $\mathbf{\Phi}(w)$ can be reached in four steps:

$$\begin{array}{rlll}
\mathrm{Id} & = 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11 & \\
w & = 5\ \mathbf{0}\ 1\ 2\ \mathbf{0}\ \mathbf{0}\ 3\ 6\ \mathbf{0}\ 7\ \ 4 & j=9, \text{apply } \phi_4, \text{case (1)}; \\
 & \ 5\ \mathbf{0}\ 1\ 2\ \mathbf{0}\ \mathbf{0}\ 3\ 6\ \mathbf{0}\ 7\ \ 4 & j=6, \text{apply } \phi_3, \text{case (2)},\ k=7; \\
 & \ 5\ \mathbf{0}\ 1\ 2\ \mathbf{0}\ 3\ \mathbf{0}\ 6\ \mathbf{0}\ 7\ \ 4 & j=5, \text{apply } \phi_2, \text{case (3)},\ i=4; \\
 & \ 5\ \mathbf{0}\ 1\ \mathbf{0}\ 2\ 3\ \mathbf{0}\ 6\ \mathbf{0}\ 7\ \ 4 & j=2, \text{apply } \phi_1, \text{case (2)},\ k=3; \\
\mathbf{\Phi}(w) & = 5\ 1\ \mathbf{0}\ \mathbf{0}\ 2\ 3\ \mathbf{0}\ 6\ \mathbf{0}\ 7\ \ 4. &
\end{array}$$

We have: $\mathrm{RISE}\, w = \mathrm{RISE}^{\bullet}\, \mathbf{\Phi}(w) = \{2,3,5,6,7,9,11\}$, as desired.

To verify that $\mathbf{\Phi}$ is bijective, we introduce a class of bijections ψ_l, whose definitions are parallel to the definitions of the ϕ_l's. Let $w = x_1x_2\cdots x_n \in \mathrm{Sh}(0^{n-m}v)$ $(0 \le m \le n)$. For each l such that $n-m+1 \le l \le n$ let $\psi_l(w) := w$. When $1 \le l \le n-m$, let x_j denote the l-th letter of w, equal to 0, when w is read *from left to right.* Consider the following three cases (remember that $x_0 = x_{n+1} = +\infty$ by convention):

(1′) = (1) x_{j-1}, x_{j+1} both non-subexcedent;

(2′) x_{j-1} subexcedent, x_{j+1} non-subexcedent; or x_{j-1}, x_{j+1} both subexcedent with $x_{j-1} > x_{j+1}$;

(3′) x_{j-1} non-subexcedent, x_{j+1} subexcedent; or x_{j-1}, x_{j+1} both subexcedent with $x_{j-1} < x_{j+1}$.

When case (1′) holds, let $\psi_l(w) := w$.

When case (2′) holds, determine the *smallest* integer $i \le j-1$ such that

$$x_i < x_{i+1} < \cdots < x_{j-1} < \mathrm{red}(j-1),$$

so that

$$w = x_1 \cdots x_{i-1}\ x_i \cdots x_{j-1}\ 0\ x_{j+1} \cdots x_n$$

and define:

$$\psi_l(w) := x_1 \cdots x_{i-1}\ 0\ x_i \cdots x_{j-1}\ x_{j+1} \cdots x_n.$$

When case (3′) holds, determine the *greatest* integer $k \ge j+1$ such that

$$\mathrm{red}(j+1) > x_{j+1} > x_{j+2} > \cdots > x_k,$$

so that

$$w = x_1 \cdots x_{j-1}\ 0\ x_{j+1} \cdots x_k\ x_{k+1} \cdots x_n$$

and define:

$$\psi_l(w) := x_1 \cdots x_{j-1}\ x_{j+1} \cdots x_k\ 0\ x_{k+1} \cdots x_n.$$

We now observe that when case (2) (resp. (3)) holds for w, then case (2′) (resp. (3′)) holds for $\phi_l(w)$. Also, when case (2′) (resp. (3′)) holds for w, then case (2) (resp. (3)) holds for $\psi_l(w)$. Therefore

$$\phi_l\psi_l = \psi_l\phi_l = \text{Identity map}$$

and the product $\mathbf{\Psi} := \psi_n\psi_{n-1}\cdots\psi_2\psi_1$ is the inverse bijection of Φ.

3. Verification of $\mathrm{RISE}\, w = \mathrm{RISE}^\bullet\, \mathbf{\Phi}(w)$

Let us introduce an alternate definition for $\mathbf{\Phi}$. Let w belong to $\mathrm{Sh}(0^{n-m}v)$ and w' be a nonempty left factor of w, of length n'. Let w' have p' letters equal to 0. If $p' \geq 1$, write

$$w' = x_1 \cdots x_{j-1}\, 0^h x_{j+h}x_{j+h+1}\cdots x_{n'},$$

where $1 \leq h \leq p'$, $x_{j-1} \neq 0$ and where the right factor $x_{j+h}x_{j+h+1}\cdots x_{n'}$ contains no 0. By convention $x_{j+h} := +\infty$ if $j+h = n'+1$. If x_{j+h} is subexcedent let k be the *greatest* integer $k \geq j+h$ such that $x_{j+h} < x_{j+h+1} < \cdots < x_k < \mathrm{red}(k)$. If x_{j-1} is subexcedent let i be the *smallest* integer $i \leq j-1$ such that $\mathrm{red}(i) > x_i > x_{i+1} > \cdots > x_{j-1}$. Examine four cases:

(1) if x_{j-1} and x_{j+h} are both excedent, let

$$u := x_1\cdots x_{j-1}, \quad u' := 0^h x_{j+h}x_{j+h+1}\cdots x_{n'}$$

and define

$$\theta(u') := u'.$$

(2) if x_{j-1} is excedent and x_{j+h} subexcedent, or if x_{j-1}, x_{j+h} are both subexcedent with $x_{j-1} > x_{j+h}$, let

$$u := x_1\cdots x_{j-1}, \quad u' := 0^h\, x_{j+h}\cdots x_k\, x_{k+1}\cdots x_{n'}$$

and define

$$\theta(u') := x_{j+h}\cdots x_k\, 0^h\, x_{k+1}\cdots x_{n'}.$$

(3) if x_{j-1}, x_{j+h} are both subexcedent with $x_{j-1} < x_{j+h}$, let

$$u := x_1\cdots x_{i-1}, \quad u' := x_i\cdots x_{j-1}0^h x_{j+h}\cdots x_k\, x_{k+1}\cdots x_{n'}$$

and define

$$\theta(u') := 0\, x_i\cdots x_{j-1}x_{j+h}\cdots x_k\, 0^{h-1}\, x_{k+1}\cdots x_{n'}.$$

(4) if x_{j-1} is subexcedent and x_{j+h} excedent, let

$$u := x_1\cdots x_{i-1}, \quad u' := x_i\cdots x_{j-1}0^h x_{j+h}\cdots x_{n'}$$

and define

$$\theta(u') := 0\, x_i\cdots x_{j-1}\, 0^{h-1}\, x_{j+h}\cdots x_{n'}.$$

By construction $w' = uu'$. Call it the *canonical factorization* of w'. In the four cases we evidently have:

$$\mathrm{RISE}\, u' = \mathrm{RISE}^{\bullet}\, \theta(u'). \tag{3.1}$$

Define $\Theta(w')$ to be the three-term sequence:

$$\Theta(w') := (u, u', \theta(u')). \tag{3.2}$$

Let q be the length of u. Then $q = j-1$ in cases (1) and (2) and $q = i-1$ in cases (3) and (4).

Lemma 3.1. *We have*

$$\mathrm{RISE}\, x_q u' = \mathrm{RISE}^{\bullet}\, x_q \theta(u'), \tag{3.3}$$

$$\mathrm{RISE}\, x_q u' = \mathrm{RISE}^{\bullet}\, 0\theta(u'), \quad \text{if } x_q \text{ is subexcedent.} \tag{3.4}$$

Proof. Let α (resp. β) be the leftmost letter of u' (resp. of $\theta(u')$). Because of (3.1) we only have to prove that $\mathrm{RISE}\, x_q\alpha = \mathrm{RISE}^{\bullet}\, x_q\beta$ for (3.3) and $\mathrm{RISE}\, x_q\alpha = \mathrm{RISE}^{\bullet}\, 0\beta$ when x_q is subexcedent for (3.4). Let us prove identity (3.3). There is nothing to do in case (1). In case (2) we have to verify $\mathrm{RISE}\, x_{j-1}0 = \mathrm{RISE}^{\bullet}\, x_{j-1}x_{j+h}$. When x_{j-1} is excedent and x_{j+h} subexcedent, then $1 \notin \mathrm{RISE}\, x_{j-1}0$ and $1 \notin \mathrm{RISE}^{\bullet}\, x_{j-1}x_{j+h}$. When x_{j-1}, x_{j+h} are both subexcedent with $x_{j-1} > x_{j+h}$, then $1 \notin \mathrm{RISE}\, x_{j-1}0$ and $1 \notin \mathrm{RISE}^{\bullet}\, x_{j-1}x_{j+h}$.

In cases (3) and (4) we have to verify $\mathrm{RISE}\, x_{i-1}x_i = \mathrm{RISE}^{\bullet}\, x_{i-1}0$. If $x_{i-1} = 0$ (resp. excedent), then $1 \in \mathrm{RISE}\, x_{i-1}x_i$ (resp. $1 \notin \mathrm{RISE}\, x_{i-1}x_i$) and $1 \in \mathrm{RISE}^{\bullet}\, x_{i-1}0$ (resp. $1 \notin \mathrm{RISE}^{\bullet}\, x_{i-1}0$). When x_{i-1} is subexcedent, then $x_{i-1} < x_i$ by definition of i. Hence $1 \in \mathrm{RISE}\, x_{i-1}x_i$ and $1 \in \mathrm{RISE}^{\bullet}\, x_{i-1}0$.

We next prove identity (3.4). In case (1) x_q is always excedent, so that identity (3.4) need not be considered. In case (2) we have to verify $\mathrm{RISE}\, x_{j-1}0 = \mathrm{RISE}^{\bullet}\, 0x_{j+h}$. But if x_{j-1} is subexcedent, then x_{j+h} is also subexcedent, so that the above two sets are empty. In cases (3) and (4) we have to verify $\mathrm{RISE}\, x_{i-1}x_i = \mathrm{RISE}^{\bullet}\, 00 = \{1\}$. But if x_{i-1} is subexcedent, then $x_{i-1} < x_i$ by definition of i, so that $1 \in \mathrm{RISE}\, x_{i-1}x_i$. □

Now, if w has p letters equal to 0 with $p \ge 1$, it may be expressed as the juxtaposition product

$$w = w_1\, 0^{h_1}\, w_2\, 0^{h_2} \,\cdots\, w_r\, 0^{h_r}\, w_{r+1},$$

where $h_1 \ge 1$, $h_2 \ge 1, \ldots, h_r \ge 1$ and where the factors $w_1, w_2, \ldots, w_r$, w_{r+1} contain no 0 and $w_2, \ldots, w_r$ are nonempty. We may define: $\Theta(w) := (u_r, u'_r, \theta(u'_r))$, where $w = u_r u'_r$ is the canonical factorization of w. As u_r

is a left factor of w, we next define $\Theta(u_r) := (u_{r-1}, u'_{r-1}, \theta(u'_{r-1}))$, where $u_r = u_{r-1}u'_{r-1}$ is the canonical factorization of u_r, and successively $\Theta(u_{r-1}) := (u_{r-2}, u'_{r-2}, \theta(u'_{r-2}))$ with $u_{r-1} = u_{r-2}u'_{r-2}$, ... , $\Theta(u_2) := (u_1, u'_1, \theta(u'_1))$ with $u_2 = u_1u'_1$, so that $w = u_1u'_1u'_2\cdots u'_r$ and $\mathbf{\Phi}(w) = u_1\,\theta(u'_1)\theta(u'_2)\cdots\theta(u'_r)$.

It can be verified that $u_r\,\theta(u'_r) = \phi_{p-h_r+1}\cdots\phi_{p-1}\phi_p(w)$, $u_{r-1}\theta(u'_{r-1})\theta(u'_r) = \phi_{p-h_r-h_{r-1}+1}\cdots\phi_{p-1}\phi_p(w)$, etc.

With identities (3.1), (3.3) and (3.4) the proof of (1.5) is now completed.

Again, consider the word w of the preceding example

$$w = 5\,0\,1\,2\,0\,0\,3\,6\,0\,7\,4,$$

so that $r = 3$, $h_1 = 1$, $h_2 = 2$, $h_3 = 1$, $w_1 = 5$, $w_2 = 1\,2$ and $w_3 = 3\,6$, $w_4 = 7\,4$. We have

$$\begin{aligned}
\Theta(w) &= (u_3, u'_3, \theta(u'_3)) = (5\,0\,1\,2\,0\,0\,3\,6;\ 0\,7\,4;\ 0\,7\,4); && \text{case (1)}\\
\Theta(u_3) &= (u_2, u'_2, \theta(u'_2)) = (5\,0\,1;\ 2\,0\,0\,3\,6;\ 0\,2\,3\,0\,6); && \text{case (3)}\\
\Theta(u_2) &= (u_1, u'_1, \theta(u'_1)) = (5;\ 0\,1;\ 1\,0); && \text{case (2)}\\
\mathbf{\Phi}(w) &= u_1\,\theta(u'_1)\theta(u'_2)\theta(u'_3) = 5 \mid 1\,0 \mid 0\,2\,3\,0\,6 \mid 0\,7\,4.
\end{aligned}$$

4. The transformation $\mathbf{F}_3$

The bijection $\mathbf{F}_3$ we are now defining maps each shuffle class $\mathrm{Sh}(0^{n-m}v)$ with v an arbitrary word of length m $(0 \le m \le n)$ onto itself. When $n = 1$ the unique element of the shuffle class is sent onto itself. Also let $\mathbf{F}_3(w) = w$ when $\mathrm{des}(w) = 0$. Let $n \ge 2$ and assume that $\mathbf{F}_3(w')$ has been defined for all words w' with nonnegative letters, of length $n' \le n - 1$. Further assume that (1.6) and (1.7) hold for all those words. Let w be a word of length n such that $\mathrm{des}(w) \ge 1$. We may write

$$w = w'a0^rb,$$

where $a \ge 1$, $b \ge 0$ and $r \ge 0$. Three cases are considered:

(1) $a \le b$; (2) $a > b$, $r \ge 1$; (3) $a > b$, $r = 0$.

In case (1) define: $\mathbf{F}_3(w) = \mathbf{F}_3(w'a0^rb) := (\mathbf{F}_3(w'a0^r))b$.

In case (2) we may write $\mathbf{F}_3(w'a0^r) = w''0$ by Property (1.7). We then define

$$\begin{aligned}
\gamma\,\mathbf{F}_3(w'a0^r) &:= 0w'';\\
\mathbf{F}_3(w) = \mathbf{F}_3(w'a0^rb) &:= (\gamma\,\mathbf{F}_3(w'a0^r))b = 0w''b.
\end{aligned}$$

In short, add one letter "0" to the left of $\mathbf{F}_3(w'a0^r)$, then delete the rightmost letter "0" and add b to the right.

In case (3) remember that $r=0$. Write

$$\mathbf{F}_3(w'a) = 0^{m_1}x_1v_1 0^{m_2}x_2v_2\cdots 0^{m_k}x_kv_k,$$

where $m_1\ge 0, m_2,\dots,m_k$ are all positive, then $x_1, x_2, \dots, x_k$ are positive letters and $v_1, v_2, \dots, v_k$ are words with positive letters, possibly empty. Then define:

$$\begin{aligned} \delta\,\mathbf{F}_3(w'a) &:= x_1 0^{m_1} v_1 x_2 0^{m_2} v_2 x_3 \cdots x_k 0^{m_k} v_k;\\ \mathbf{F}_3(w) = \mathbf{F}_3(w'ab) &:= (\delta\,\mathbf{F}_3(w'a))b. \end{aligned}$$

In short, move each positive letter occurring just after a 0-factor of $\mathbf{F}_3(w'a)$ to the beginning of that 0-factor and add b to the right.

Example.

$$\begin{aligned} w &= 0\,0\,0\,3\,1\,2\,2\,0\,0\,1\,3 && \\ \mathbf{F}_3(0\,0\,0\,3) &= 0\,0\,0\,3 && \text{no descent}\\ \mathbf{F}_3(0\,0\,0\,3\,1) &= \delta(0\,0\,0\,3)1 = 3\,0\,0\,0\,1 && \text{case (3)}\\ \mathbf{F}_3(0\,0\,0\,3\,1\,2\,2) &= 3\,0\,0\,0\,1\,2\,2 && \text{case (1)}\\ \mathbf{F}_3(0\,0\,0\,3\,1\,2\,2\,0) &= \delta(3\,0\,0\,0\,1\,2\,2)0 = 3\,1\,0\,0\,0\,2\,2\,0 && \text{case (3)}\\ \mathbf{F}_3(0\,0\,0\,3\,1\,2\,2\,0\,0) &= \gamma(3\,1\,0\,0\,0\,2\,2\,0)0 = 0\,3\,1\,0\,0\,0\,2\,2\,0 && \text{case (2)}\\ \mathbf{F}_3(0\,0\,0\,3\,1\,2\,2\,0\,0\,1) &= \gamma(0\,3\,1\,0\,0\,0\,2\,2\,0)1 = 0\,0\,3\,1\,0\,0\,0\,2\,2\,1 && \text{case (2)}\\ \mathbf{F}_3(0\,0\,0\,3\,1\,2\,2\,0\,0\,1\,3) &= 0\,0\,3\,1\,0\,0\,0\,2\,2\,1\,3. && \end{aligned}$$

We have: $\operatorname{maj} w = \operatorname{maj}(0\,0\,0\,3\,1\,2\,2\,0\,0\,1\,3) = 4+7 = 11$ and $\operatorname{mafz}\mathbf{F}_3(w) = \operatorname{mafz}(0\,0\,3\,1\,0\,0\,0\,2\,2\,1\,3) = (1+2+5+6+7)-(1+2+3+4+5)+(1+4) = 11$.

By construction the rightmost letter is preserved by $\mathbf{F}_3$. To prove (1.6) proceed by induction. Assume that $\operatorname{mafz} w'a0^r = \operatorname{mafz}\mathbf{F}_3(w'a0^r)$ holds. In case (1) "maj" and "mafz" remain invariant when b is juxtaposed at the end. In case (2) we have $\operatorname{maj} w = \operatorname{maj}(w'a0^rb) = \operatorname{maj}(w'a0^r)$, but $\operatorname{mafz}\gamma\,\mathbf{F}_3(w'a0^r) = \operatorname{mafz}\mathbf{F}_3(w'a0^r) - |w'a0^r|_{\ge 1}$ and $\operatorname{mafz}(\gamma\,\mathbf{F}_3(w'a0^r))b = \operatorname{mafz}\gamma\,\mathbf{F}_3(w'a0^r) + |w'a0^r|_{\ge 1}$, where $|w'a0^r|_{\ge 1}$ denotes the number of *positive* letters in $w'a0^r$. Hence (1.6) holds. In case (3) remember $r=0$. We have $\operatorname{maj}(w'ab) = \operatorname{maj}(w'a) + |w'a|$, where $|w'a|$ denotes the length of the word $w'a$. But $\operatorname{mafz}\delta\mathbf{F}_3(w'a) = \operatorname{mafz}\mathbf{F}_3(w'a) + \operatorname{zero}(w'a)$ and $\operatorname{mafz}(\delta\mathbf{F}_3(w'a))b = \operatorname{mafz}\delta\mathbf{F}_3(w'a) + |w'a|_{\ge 1}$. The equality holds for $b=0$ and $b\ge 1$, as easily verified. As $\operatorname{zero}(w'a)+|w'a|_{\ge 1} = |w'a|$, we have $\operatorname{mafz}\mathbf{F}_3(w) = \operatorname{mafz}(\delta\mathbf{F}_3(w'a))b = \operatorname{mafz}\mathbf{F}_3(w'a) + |w'a| = \operatorname{maj} w'ab = \operatorname{maj} w$. Thus (1.6) holds in the three cases.

To define the inverse bijection $\mathbf{F}_3^{-1}$ of $\mathbf{F}_3$ we first need the inverses $\gamma^{-1}(w)$ and $\delta^{-1}(w)$ for each word w. Let $w = 0w'$ be a word, whose first letter is 0. Define $\gamma^{-1}(w)$ to be the word derived from w by deleting the first letter 0 and adding one letter "0" to the right of w. Clearly, $\gamma^{-1}\gamma = \gamma\gamma^{-1}$ is the identity map.

Next, let w be a word, whose first letter is positive. Define $\delta^{-1}(w)$ to be the word derived from w by moving each positive letter occurring just before a 0-factor of w to the end of that 0-factor. Again $\delta^{-1}\delta = \delta\delta^{-1}$ is the identity map.

We may write

$$w = cw'a0^r b,$$

where $a \geq 1$, $b \geq 0$, $c \geq 0$ and $r \geq 0$. Three cases are considered:

(1) $a \leq b$; (2) $a > b$, $c = 0$; (3) $a > b$, $c \geq 1$.

In case (1) define: $\mathbf{F}_3^{-1}(w) := (\mathbf{F}_3^{-1}(cw'a0^r))b$.

In case (2) define: $\mathbf{F}_3^{-1}(w) := (\gamma^{-1}(\mathbf{F}_3^{-1}(cw'a0^r)))b$.

In case (3) define: $\mathbf{F}_3^{-1}(w) := (\delta^{-1}(\mathbf{F}_3^{-1}(cw'a0^r)))b$.

We end this section by proving a property of the transformation $\mathbf{F}_3$, which will be used in our next paper [FoHa07].

Proposition 4.1. *Let w, w'' be two words with nonnegative letters, of the same length. If* $\operatorname{Zero} w = \operatorname{Zero} w''$ *and* $\operatorname{DES}\operatorname{Pos} w = \operatorname{DES}\operatorname{Pos} w''$, *then* $\operatorname{Zero}\mathbf{F}_3(w) = \operatorname{Zero}\mathbf{F}_3(w'')$.

Proof. To derive $\mathbf{F}_3(w)$ (resp. $\mathbf{F}_3(w'')$) from w (resp. w'') we have to consider one of the three cases (1), (2) or (3), described above, at each step. Because of the two conditions $\operatorname{Zero} w = \operatorname{Zero} w''$ and $\operatorname{DES}\operatorname{Pos} w = \operatorname{DES}\operatorname{Pos} w''$, case (i) $(i = 1, 2, 3)$ is used at the j-th step in the calculation of $\mathbf{F}_3(w)$, if and only if the same case is used at that j-th step for the calculation of $\mathbf{F}_3(w'')$. Consequently the letters equal to 0 are in the same places in both words $\mathbf{F}_3(w)$ and $\mathbf{F}_3(w'')$. $\square$

By the very definition of $\Phi : \sigma \overset{\mathrm{ZDer}}{\mapsto} w \overset{\mathbf{\Phi}}{\mapsto} w' \overset{\mathrm{ZDer}^{-1}}{\mapsto} \sigma'$, given in (1.18) and of $\mathrm{F}_3 : \sigma \overset{\mathrm{ZDer}}{\mapsto} w \overset{\mathbf{F}_3}{\mapsto} w'' \overset{\mathrm{ZDer}^{-1}}{\mapsto} \sigma''$, given in (1.19) we have $\Phi(\sigma) = \sigma$ and $\mathrm{F}_3(\sigma) = \sigma$ if σ is a derangement. In the next two tables we have calculated $\Phi(\sigma) = \sigma'$ and $\mathrm{F}_3(\sigma) = \sigma''$ for the fifteen non-derangement permutations σ of order 4.

fix σ	Der σ	RIZE σ	σ	w	w'	σ'	RISE σ'	Der σ'	fix σ'
4	e	$1,2,3,4$	1234	$\mathbf{0000}$	$\mathbf{0000}$	1234	$1,2,3,4$	e	4
		$1,2,4$	1243	$\mathbf{00}21$	$\mathbf{00}21$	1243	$1,2,4$		
		$1,4$	1324	$\mathbf{0}21\mathbf{0}$	$\mathbf{0}2\mathbf{0}1$	1432	$1,4$		
		$1,3,4$	1432	$\mathbf{0}2\mathbf{0}1$	$\mathbf{0}21\mathbf{0}$	1324	$1,3,4$		
2	21	$3,4$	2134	$21\mathbf{00}$	$2\mathbf{0}1\mathbf{0}$	3214	$3,4$	21	2
		$2,4$	3214	$2\mathbf{0}1\mathbf{0}$	$2\mathbf{00}1$	4231	$2,4$		
		$2,3,4$	4231	$2\mathbf{00}1$	$21\mathbf{00}$	2134	$2,3,4$		
		$1,2,4$	1342	$\mathbf{0}231$	$\mathbf{0}231$	1342	$1,2,4$		
		$1,4$	2314	$231\mathbf{0}$	$23\mathbf{0}1$	2431	$1,4$		
1	231	$1,3,4$	2431	$23\mathbf{0}1$	$231\mathbf{0}$	2314	$1,3,4$	231	1
		$2,4$	3241	$2\mathbf{0}31$	$2\mathbf{0}31$	3241	$2,4$		
		$1,3,4$	1423	$\mathbf{0}312$	$\mathbf{0}312$	1423	$1,3,4$		
1	312	$2,4$	3124	$312\mathbf{0}$	$31\mathbf{0}2$	4132	$2,4$	312	1
		$3,4$	4132	$31\mathbf{0}2$	$3\mathbf{0}12$	4213	$3,4$		
		$2,3,4$	4213	$3\mathbf{0}12$	$312\mathbf{0}$	3124	$2,3,4$		

Calculation of $\sigma' = \Phi(\sigma)$

fix σ	Der σ	maz σ	σ	w	w''	σ''	maf σ''	Der σ''	fix σ''
4	e	0	1234	$\mathbf{0000}$	$\mathbf{0000}$	1234	0	e	4
		3	1243	$\mathbf{00}21$	$2\mathbf{00}1$	4231	3		
		5	1324	$\mathbf{0}21\mathbf{0}$	$21\mathbf{00}$	2134	5		
		2	1432	$\mathbf{0}2\mathbf{0}1$	$\mathbf{0}2\mathbf{0}1$	1432	2		
2	21	3	2134	$21\mathbf{00}$	$\mathbf{0}21\mathbf{0}$	1324	3	21	2
		4	3214	$2\mathbf{0}1\mathbf{0}$	$2\mathbf{0}1\mathbf{0}$	3214	4		
		1	4231	$2\mathbf{00}1$	$\mathbf{00}21$	1243	1		
		3	1342	$\mathbf{0}231$	$2\mathbf{0}31$	3241	3		
		5	2314	$231\mathbf{0}$	$231\mathbf{0}$	2314	5		
1	231	2	2431	$23\mathbf{0}1$	$\mathbf{0}231$	1342	2	231	1
		4	3241	$2\mathbf{0}31$	$23\mathbf{0}1$	2431	4		
		2	1423	$\mathbf{0}312$	$3\mathbf{0}12$	4213	2		
1	312	4	3124	$312\mathbf{0}$	$312\mathbf{0}$	3124	4	312	1
		3	4132	$31\mathbf{0}2$	$31\mathbf{0}2$	4132	3		
		1	4213	$3\mathbf{0}12$	$\mathbf{0}312$	3124	1		

Calculation of $\sigma'' = \mathrm{F}_3(\sigma)$

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