2007/08/30

Fix-Mahonian Calculus, II: further statistics

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Abstract. Using classical transformations on the symmetric group and two transformations constructed in Fix-Mahonian Calculus I, we show that several multivariable statistics are equidistributed either with the triplet (fix,des,maj), or the pair (fix,maj), where "fix," "des" and "maj" denote the number of fixed points, the number of descents and the major index, respectively.

1. Introduction

First, recall the traditional notations for the q-ascending factorials

$$(a;q)_n := \begin{cases} 1, & \text{if } n=0; \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & \text{if } n\ge 1; \end{cases}$$
$$(a;q)_\infty := \prod_{n\ge 1}(1-aq^{n-1});$$

and the q-exponential (see [GaRa90, chap. 1])

$$e_q(u)=\sum_{n\ge 0}\frac{u^n}{(q;q)_n}=\frac{1}{(u;q)_\infty}.$$

Furthermore, let $(A_n(Y,t,q))$ and $(A_n(Y,q))$ $(n\ge 0)$ be the sequences of polynomials respectively defined by the factorial generating functions

$$\sum_{n\ge 0}A_n(Y,t,q)\frac{u^n}{(t;q)_{n+1}} := \sum_{s\ge 0}t^s\Bigl(1-u\sum_{i=0}^{s}q^i\Bigr)^{-1}\frac{(u;q)_{s+1}}{(uY;q)_{s+1}}; \tag{1.1}$$

$$\sum_{n\ge 0}A_n(Y,q)\frac{u^n}{(q;q)_n} := \Bigl(1-\frac{u}{1-q}\Bigr)^{-1}\frac{(u;q)_\infty}{(uY;q)_\infty}. \tag{1.2}$$

Of course, (1.2) can be derived from (1.1) by letting the variable t tend to 1, so that $A_n(Y,q)=A_n(Y,1,q)$. The classical combinatorial interpretation for those classes of polynomials has been found by Gessel and Reutenauer [GeRe93] (see Theorem 1.1 below). For each permutation $\sigma=\sigma(1)\sigma(2)\cdots\sigma(n)$ from the symmetric group $\mathfrak{S}_n$ let $\mathbf{i}\,\sigma:=\sigma^{-1}$ denote the inverse of σ; then let its *set of fixed points*, $\mathrm{FIX}\,\sigma$, *descent set*, $\mathrm{DES}\,\sigma$, *idescent set*, $\mathrm{IDES}\,\sigma$, be defined as the subsets:

$$\begin{aligned}\mathrm{FIX}\,\sigma &:= \{i : 1 \le i \le n,\ \sigma(i) = i\};\\ \mathrm{DES}\,\sigma &:= \{i : 1 \le i \le n-1,\ \sigma(i) > \sigma(i+1)\};\\ \mathrm{IDES}\,\sigma &:= \mathrm{DES}\,\sigma^{-1}.\end{aligned}$$

Note that $\mathrm{IDES}\,\sigma$ is also the set of all i such that $i+1$ is on the left of i in the linear representation $\sigma(1)\sigma(2)\cdots\sigma(n)$ of σ. Also let $\mathrm{fix}\,\sigma := \#\,\mathrm{FIX}\,\sigma$ (the *number of fixed points*), $\mathrm{des}\,\sigma := \#\,\mathrm{DES}\,\sigma$ (the *number of descents*), $\mathrm{maj}\,\sigma := \sum_i i\ (i \in \mathrm{DES}\,\sigma)$ (the *major index*), $\mathrm{imaj}\,\sigma := \sum_i i\ (i \in \mathrm{IDES}\,\sigma)$ (the *inverse major index*).

Theorem 1.1 (Gessel, Reutenauer)**.** *For each $n \ge 0$ the generating polynomial for $\mathfrak{S}_n$ by $(\mathrm{fix}, \mathrm{des}, \mathrm{maj})$ (resp. by $(\mathrm{fix}, \mathrm{maj})$) is equal to $A_n(Y,t,q)$ (resp. to $A_n(Y,q)$). Accordingly,*

$$A_n(Y,t,q) = \sum_{\sigma\in\mathfrak{S}_n} Y^{\mathrm{fix}\,\sigma} t^{\mathrm{des}\,\sigma} q^{\mathrm{maj}\,\sigma}; \tag{1.3}$$

$$A_n(Y,q) = \sum_{\sigma\in\mathfrak{S}_n} Y^{\mathrm{fix}\,\sigma} q^{\mathrm{maj}\,\sigma}. \tag{1.4}$$

The purpose of this paper is to show that there are several other three-variable (resp. two-variable) statistics on $\mathfrak{S}_n$, whose distribution is given by the generating polynomial $A_n(Y,t,q)$ (resp. $A_n(Y,q)$). For proving that those statistics are equidistributed with $(\mathrm{fix}, \mathrm{des}, \mathrm{maj})$ (resp. $(\mathrm{fix}, \mathrm{maj})$) we make use the properties of the classical bijections $\mathrm{F}_2^{\mathrm{loc}}$, F_2', CHZ, $\mathrm{DW}^{\mathrm{glo}}$, $\mathrm{DW}^{\mathrm{loc}}$, plus two transformations F_3, Φ, constructed in our previous paper [FoHa07], finally a new transformation F_3' described in Section 3.

All those bijections appear as arrows in the diagram of Fig. 1. The nodes of the diagram are pairs or triplets of statistics, whose definitions have been, or will be, given in the paper. The integral-valued statistics are written in lower case, such as "fix" or "maj", while the set-valued ones appear in capital letters, such as "FIX" or "DES". We also introduce two mappings "Der" and "Desar" of $\mathfrak{S}_n$ into $\mathfrak{S}_m$ with $m \le n$.

Each arrow goes from one node to another node with the following meaning that we shall explain by means of an example: the vertical arrow $(\mathrm{fix}, \mathrm{maz}, \mathrm{Der}) \xrightarrow{\mathrm{F}_3} (\mathrm{fix}, \mathrm{maf}, \mathrm{Der})$ (here rewritten horizontally for typographical reasons!) indicates that the bijection F_3 maps $\mathfrak{S}_n$ onto itself and has the property: $(\mathrm{fix}, \mathrm{maz}, \mathrm{Der})\,\sigma = (\mathrm{fix}, \mathrm{maf}, \mathrm{Der})\,\mathrm{F}_3(\sigma)$ for all σ. The remaining statistics are now introduced together with the main two decompositions of permutations: the *fixed* and *pixed decompositions*.

$$
\begin{tikzcd}
 & (\mathrm{FIX}, \mathrm{maf}) \arrow[r, "\mathrm{DW}^{\mathrm{loc}}"] & (\mathrm{PIX}, \mathrm{mag}) \arrow[dd, "\mathrm{F}_2^{\mathrm{loc}}"] & (\mathrm{pix}, \mathrm{mag}, \mathrm{Desar}) \\
(\mathrm{fix}, \mathrm{maf}, \mathrm{Der}) \arrow[dr, "\mathrm{CHZ}"] & & & \\
 & \begin{matrix}(\mathrm{fix}, \mathrm{maj}, \mathrm{Der}) \\ (\mathrm{fix}, \mathrm{DES})\end{matrix} & \begin{matrix}(\mathrm{PIX}, \mathrm{inv}) \\ (\mathrm{pix}, \mathrm{inv})\end{matrix} \arrow[d, "(\mathrm{F}_2')^{-1}"] & \\
(\mathrm{fix}, \mathrm{maz}, \mathrm{Der}) \arrow[uu, "\mathrm{F}_3"] & \begin{matrix}(\mathrm{fix}, \mathrm{DEZ}) \\ (\mathrm{FIX}, \mathrm{DEZ})\end{matrix} \arrow[u, "\Phi"] \arrow[r, "\mathrm{DW}^{\mathrm{loc}}"] & \begin{matrix}(\mathrm{pix}, \mathrm{imaj}) \\ (\mathrm{pix}, \mathrm{IDES}) \\ (\mathrm{PIX}, \mathrm{IDES})\end{matrix} \arrow[ul, "\mathrm{DW}^{\mathrm{glo}}"'] & (\mathrm{pix}, \mathrm{imaj}, \mathrm{Desar}) \arrow[uuu, "\mathrm{F}_3'"']
\end{tikzcd}
$$

Fig. 1

1.1. *The fixed decomposition.* Let $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ be a permutation and let $(i_1, i_2, \ldots, i_{n-m})$ (resp. $(j_1, j_2, \ldots, j_m)$) be the increasing sequence of the integers k (resp. k') such that $1 \le k \le n$ and $\sigma(k) = k$ (resp. $1 \le k' \le n$ and $\sigma(k') \neq k'$). Also let "red" denote the increasing bijection of $\{j_1, j_2, \ldots, j_m\}$ onto $[\,m\,]$. Let $\mathrm{ZDer}(\sigma) = x_1x_2\cdots x_n$ be the word derived from $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ by replacing each fixed point $\sigma(i_k)$ by 0 and each other letter $\sigma(j_{k'})$ by $\mathrm{red}\,\sigma(j_{k'})$. As "DES," "des" and "maj" can also be defined for arbitrary words with nonnegative letters, we further introduce:

$$\mathrm{DEZ}\,\sigma := \mathrm{DES}\ \mathrm{ZDer}(\sigma); \tag{1.5}$$

$$\mathrm{dez}\,\sigma := \mathrm{des}\ \mathrm{ZDer}(\sigma), \quad \mathrm{maz}\,\sigma := \mathrm{maj}\ \mathrm{ZDer}(\sigma); \tag{1.6}$$

$$\mathrm{Der}\,\sigma := \mathrm{red}\,\sigma(j_1)\ \mathrm{red}\,\sigma(j_2)\ \cdots\ \mathrm{red}\,\sigma(j_m); \tag{1.7}$$

$$\mathrm{maf}\,\sigma := \sum_{k=1}^{n-m} (i_k - k) + \mathrm{maj} \circ \mathrm{Der}\,\sigma. \tag{1.8}$$

The subword $\mathrm{Der}\,\sigma$ of $\mathrm{ZDer}(\sigma)$ can be regarded as a permutation from $\mathfrak{S}_m$ (with the above notations). It is important to note that $\mathrm{FIX}\,\mathrm{Der}\,\sigma = \emptyset$, so that $\mathrm{Der}\,\sigma$ is a *derangement* of order m. Also note that σ is fully characterized by the pair $(\mathrm{FIX}\,\sigma, \mathrm{Der}\,\sigma)$, which is called the *fixed decomposition* of σ. Finally, we can rewrite $\operatorname{maf}\sigma$ as

$$\operatorname{maf}\sigma := \sum_{i\in \mathrm{FIX}\,\sigma} i - \sum_{i=1}^{\operatorname{fix}\sigma} i + \operatorname{maj}\circ \operatorname{Der}\sigma. \tag{1.9}$$

Example. With $\sigma = \begin{pmatrix} 1\,\mathbf{2}\,3\,4\,\mathbf{5}\,\mathbf{6}\,7\,8\,9 \\ 8\,\mathbf{2}\,1\,3\,\mathbf{5}\,\mathbf{6}\,4\,9\,7\end{pmatrix}$ we have red $= \begin{pmatrix} 1\,3\,4\,7\,8\,9 \\ 1\,2\,3\,4\,5\,6\end{pmatrix}$, $\mathrm{ZDer}(\sigma) = 5\,0\,1\,2\,0\,0\,3\,6\,4$, $\mathrm{DEZ}\,\sigma = \{1,4,8\}$, $\operatorname{dez}\sigma = 3$, $\operatorname{maz}\sigma = 13$, $\operatorname{Der}\sigma = 5\,1\,2\,3\,6\,4$, $\mathrm{FIX}\,\sigma = \{\mathbf{2},\mathbf{5},\mathbf{6}\}$ and $\operatorname{maf}\sigma = (2-1)+(5-2)+(6-3)+\operatorname{maj}(5\,1\,2\,3\,6\,4) = 7+6 = 13$.

1.2. *The pixed decomposition.* Let $w = y_1y_2\cdots y_n$ be a word having no repetitions, without necessarily being a permutation of $12\cdots n$. Say that w is a *desarrangement* if $y_1 > y_2 > \cdots > y_{2k}$ and $y_{2k} < y_{2k+1}$ for some $k \geq 1$. By convention, $y_{n+1} = \infty$. We could also say that the *leftmost trough* of w occurs at an *even* position. This notion was introduced by Désarménien [De84] and elegantly used in a subsequent paper [DW88]. A further refinement is due to Gessel [Ge91].

Let $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ be a permutation. Unless σ is increasing, there is always a nonempty right factor of σ which is a desarrangement. It then makes sense to define σ^d as the *longest* such a right factor. Hence, σ admits a unique factorization $\sigma = \sigma^p\sigma^d$, called the *pixed factorization*, where σ^p is *increasing* and σ^d is the longest right factor of σ which is a desarrangement. The set (resp. number) of the letters in σ^p is denoted by $\mathrm{PIX}\,\sigma$ (resp. $\operatorname{pix}\sigma$).

If $\sigma^d = \sigma(n-m+1)\sigma(n-m+2)\cdots\sigma(n)$ and if "red" is the increasing bijection mapping the set $\{\sigma(n-m+1), \sigma(n-m+2), \ldots, \sigma(n)\}$ onto $\{1,2,\ldots,m\}$, define

$$\operatorname{Desar}\sigma := \operatorname{red}\sigma(n-m+1)\ \operatorname{red}\sigma(n-m+2)\ \ldots\ \operatorname{red}\sigma(n); \tag{1.10}$$

$$\operatorname{mag}\sigma := \sum_{i\in \mathrm{PIX}\,\sigma} i - \sum_{i=1}^{\operatorname{pix}\sigma} i + \operatorname{imaj}\circ \operatorname{Desar}\sigma. \tag{1.11}$$

Note that $\operatorname{Desar}\sigma$ is a desarrangement and belongs to $\mathfrak{S}_m$, for short, a desarrangement of order m. Also note that σ is fully characterized by the pair $(\mathrm{PIX}\,\sigma, \operatorname{Desar}\sigma)$, which will called the *pixed decomposition* of σ. Form the inverse $(\operatorname{Desar}\sigma)^{-1} = y_1y_2\cdots y_m$ of $\operatorname{Desar}\sigma$ and define $\mathrm{ZDesar}(\sigma)$ to be the unique shuffle $x_1x_2\cdots x_n$ of 0^{n-m} and $y_1y_2\cdots y_m$, where $x_i = 0$ if and only if $i \in \mathrm{PIX}\,\sigma$.

Example. With $\sigma = 3\,5\,7\,4\,2\,8\,1\,9\,6$, then $\sigma^p = 3\,5\,7$, $\sigma^d = 4\,2\,8\,1\,9\,6$, $\mathrm{PIX}\,\sigma = \{3,5,7\}$, $\operatorname{pix}\sigma = 3$. Also, $\operatorname{Desar}\sigma = 3\,2\,5\,1\,6\,4$, $\operatorname{imaj}\circ\operatorname{Desar}\sigma = 1+2+4 = 7$, $(\operatorname{Desar}\sigma)^{-1} = 4\,2\,1\,6\,3\,5$, $\mathrm{ZDesar}\,\sigma = 4\,2\,0\,1\,0\,6\,0\,3\,5$ and $\operatorname{mag}\sigma = (3+5+7)-(1+2+3)+7 = 16$.

Referring to the diagram in Fig. 1 the purpose of this paper is to prove the next two theorems.

Theorem 1.2. *In each of the following four groups the pairs of statistics are equidistributed on* $\mathfrak{S}_n$*:*

(1) $(\mathrm{fix}, \mathrm{maj})$, $(\mathrm{fix}, \mathrm{maf})$, $(\mathrm{fix}, \mathrm{maz})$, $(\mathrm{pix}, \mathrm{mag})$, $(\mathrm{pix}, \mathrm{inv})$, $(\mathrm{pix}, \mathrm{imaj})$*;*
(2) $(\mathrm{FIX}, \mathrm{maf})$, $(\mathrm{PIX}, \mathrm{mag})$, $(\mathrm{PIX}, \mathrm{inv})$*;*
(3) $(\mathrm{fix}, \mathrm{DEZ})$, $(\mathrm{fix}, \mathrm{DES})$, $(\mathrm{pix}, \mathrm{IDES})$*;*
(4) $(\mathrm{FIX}, \mathrm{DEZ})$, $(\mathrm{PIX}, \mathrm{IDES})$.

Theorem 1.3. *In each of the following two groups the triplets of statistics are equidistributed on* $\mathfrak{S}_n$*:*

(1) $(\mathrm{fix}, \mathrm{maf}, \mathrm{Der})$ *and* $(\mathrm{fix}, \mathrm{maz}, \mathrm{Der})$*;*
(2) $(\mathrm{pix}, \mathrm{mag}, \mathrm{Desar})$ *and* $(\mathrm{pix}, \mathrm{imaj}, \mathrm{Desar})$.

Furthermore, the following diagram, involving the bijections $\mathrm{DW}^{\mathrm{loc}}$, F_3 *and* F_3' *is commutative.*

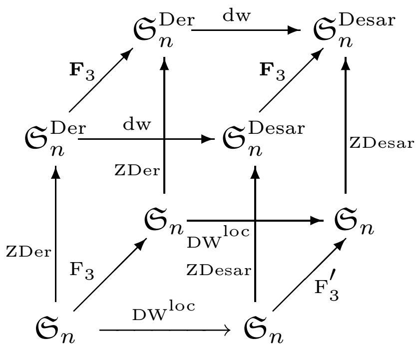

Fig. 2

2. The bijections

2.1. *The transformations* Φ *and "*CHZ*".* In our preceding paper [FoHa07] we have given the constructions of two bijections Φ, F_3 of $\mathfrak{S}_n$ onto itself. The latter one will be re-studied and used in Section 3. As was shown in our previous paper [FoHa07], the first one has the following property:

$$(\mathrm{fix}, \mathrm{DEZ}, \mathrm{Der})\, \sigma = (\mathrm{fix}, \mathrm{DES}, \mathrm{Der})\, \Phi(\sigma) \quad (\sigma \in \mathfrak{S}_n). \tag{2.1}$$

This shows that over $\mathfrak{S}_n$ the pairs $(\mathrm{fix}, \mathrm{maz})$ and $(\mathrm{fix}, \mathrm{maj})$ are equidistributed over $\mathfrak{S}_n$, their generating polynomial being given by the polynomial $A_n(Y, q)$ introduced in (1.2). Also the triplets $(\mathrm{fix}, \mathrm{dez}, \mathrm{maz})$ and $(\mathrm{fix}, \mathrm{des}, \mathrm{maj})$ are equidistributed, with generating polynomial $A_n(Y, t, q)$ introduced in (1.1).

In [CHZ97] the authors have constructed a bijection, here called "CHZ", satisfying

$$(\mathrm{fix}, \mathrm{maf}, \mathrm{Der})\,\sigma = (\mathrm{fix}, \mathrm{maj}, \mathrm{Der})\,\mathrm{CHZ}(\sigma) \quad (\sigma \in \mathfrak{S}_n). \tag{2.2}$$

2.2. *The Désarménien-Wachs bijection.* For each $n \geq 0$ let D_n denote the set of permutations σ from $\mathfrak{S}_n$ such that $\mathrm{FIX}\,\sigma = \emptyset$. The elements of D_n are referred to as the *derangements* of order n. Let K_n be the set of permutations σ from $\mathfrak{S}_n$ such that $\mathrm{PIX}\,\sigma = \emptyset$. The class K_n was introduced by Désarménien [De84], who called its elements *desarrangements* of order n. He also set up a one-to-one correspondence between D_n and K_n. Later, by means of a symmetric function argument Désarménien and Wachs [DW88] proved that for every subset $J \subset [n-1]$ the equality

$$\#\{\sigma \in D_n : \mathrm{DES}\,\sigma = J\} = \#\{\sigma \in K_n : \mathrm{IDES}\,\sigma = J\} \tag{2.3}$$

holds. In a subsequent paper [DW93] they constructed a bijection $\mathrm{DW} : D_n \to K_n$ having the expected property, that is,

$$\mathrm{IDES} \circ \mathrm{DW}(\sigma) = \mathrm{DES}\,\sigma. \tag{2.4}$$

Although their bijection is based on an inclusion-exclusion argument, leaving the door open to the discovery of an explicit correspondence, we use it as such in the sequel. For the very definition of "DW" we refer the reader to their original paper [DW93].

We now make a full use of the fixed and pixed decompositions introduced in §§ 1.1 and 1.2. Let $\tau \in \mathfrak{S}_n$ and consider the chain

$$\tau \mapsto (\mathrm{FIX}\,\tau, \mathrm{Der}\,\tau) \mapsto (\mathrm{FIX}\,\tau, \mathrm{DW} \circ \mathrm{Der}\,\tau) \mapsto \sigma, \tag{2.5}$$

where σ is the permutation defined by

$$(\mathrm{PIX}\,\sigma, \mathrm{Desar}\,\sigma) := (\mathrm{FIX}\,\tau, \mathrm{DW} \circ \mathrm{Der}\,\tau). \tag{2.6}$$

Then, the mapping $\mathrm{DW}^{\mathrm{loc}}$ defined by

$$\mathrm{DW}^{\mathrm{loc}}(\tau) := \sigma, \tag{2.7}$$

is a bijection of $\mathfrak{S}_n$ onto itself satisfying $\mathrm{FIX}\,\tau = \mathrm{PIX}\,\sigma$ and $\mathrm{DES} \circ \mathrm{Der}\,\tau = \mathrm{IDES} \circ \mathrm{Desar}\,\sigma$. In particular, $\mathrm{fix}\,\tau = \mathrm{pix}\,\sigma$. Taking the definitions of "maf" and "mag" given in (1.9) and (1.11) into account we have:

$$\begin{aligned}\mathrm{maf}\,\tau &= \sum_{i \in \mathrm{FIX}\tau} i - \sum_{i=1}^{\mathrm{fix}\,\tau} i + \mathrm{maj} \circ \mathrm{Der}\,\tau \\ &\qquad = \sum_{i \in \mathrm{PIX}\sigma} i - \sum_{i=1}^{\mathrm{pix}\,\sigma} i + \mathrm{imaj} \circ \mathrm{Desar}\,\sigma = \mathrm{mag}\,\sigma.\end{aligned}$$

We have then proved the following proposition.

Proposition 2.1. *Let* $\sigma := \mathrm{DW}^{\mathrm{loc}}(\tau)$. *Then*

$$\mathrm{FIX}\,\tau = \mathrm{PIX}\,\sigma, \quad \mathrm{DES}\circ \mathrm{Der}\,\tau = \mathrm{IDES}\circ \mathrm{Desar}\,\sigma, \quad \mathrm{maf}\,\tau = \mathrm{mag}\,\sigma. \tag{2.8}$$

Corollary 2.2. *The pairs* $(\mathrm{FIX}, \mathrm{maf})$ *and* $(\mathrm{PIX}, \mathrm{mag})$ *are equidistributed over* $\mathfrak{S}_n$.

Example. Assume that the bijection "DW" maps the derangement 512364 onto the desarrangement 623145. On the other hand, the fixed decomposition of $\tau = 182453697$ is equal to $(\{1,4,5\}, 512364)$ and $(\{1,4,5\}, 623145)$ is the pixed decomposition of the permutation $\sigma = 145936278$. Hence $\mathrm{DW}^{\mathrm{loc}}(182453697) = 145936278$.

We verify that $\mathrm{DES}\circ\mathrm{Der}\,\tau = \mathrm{DES}(512364) = \{1,5\} = \mathrm{IDES}(623145) = \mathrm{IDES}\circ\mathrm{Desar}\,\sigma$. Also $\mathrm{maf}\,\tau = (1+4+5)-(1+2+3)+(1+5) = 10 = \mathrm{mag}\,\sigma$.

Proposition 2.3. *Let* $\sigma := \mathrm{DW}^{\mathrm{loc}}(\tau)$. *Then*

$$(\mathrm{FIX}, \mathrm{DEZ})\,\tau = (\mathrm{PIX}, \mathrm{IDES})\,\sigma; \tag{2.9}$$

$$(\mathrm{fix}, \mathrm{maz})\,\tau = (\mathrm{pix}, \mathrm{imaj})\,\sigma. \tag{2.10}$$

Proof. It suffices to prove (2.9) and in fact only $\mathrm{DEZ}\,\tau = \mathrm{IDES}\,\sigma$. Let $\sigma^p\sigma^d$ be the pixed factorization of σ. Then σ^p is the *increasing* sequence of the elements of $\mathrm{PIX}\,\sigma = \mathrm{FIX}\,\tau$. We have $i \in \mathrm{DEZ}\,\tau$ if and only if $\tau(i) \neq i$ and one of the following conditions holds:

(1) $\tau(i) > \tau(i+1)$ and $\tau(i+1) \neq i+1$;

(2) $\tau(i+1) = i+1$.

In case (1) the letters $\mathrm{red}\,\tau(i)$ and $\mathrm{red}\,\tau(i+1)$ are adjacent letters in $\mathrm{Der}\,\tau$ and $\mathrm{red}\,\tau(i) > \mathrm{red}\,\tau(i+1)$. As $\mathrm{DES}\circ\mathrm{Der}\,\tau = \mathrm{IDES}\circ\mathrm{Desar}\,\sigma$, the letter $\mathrm{red}(i+1)$ is to the left of the letter $\mathrm{red}(i)$ in $\mathrm{Desar}\,\sigma$ and then $(i+1)$ is to the left of i is σ, so that $i \in \mathrm{IDES}\,\sigma$.

In case (2) we have $(i+1) \in \mathrm{FIX}\,\tau = \mathrm{PIX}\,\sigma$ and $\mathrm{red}\,i$ is a letter of $\mathrm{Desar}\,\sigma$. Again $i \in \mathrm{IDES}\,\sigma$. $\square$

Corollary 2.4. *The pairs* $(\mathrm{FIX}, \mathrm{DEZ})$ *and* $(\mathrm{PIX}, \mathrm{IDES})$ *are equidistributed over* $\mathfrak{S}_n$.

Using the same example as above we have: $\tau = 182453697$, $\mathrm{FIX}\,\tau = \{1,4,5\}$, $\mathrm{ZDer}\,\tau = 082003697$, so that $\mathrm{DEZ}\,\tau = \{2,3,8\}$. Moreover, $\sigma = \mathrm{DW}^{\mathrm{loc}}(182453697) = 145 \mid 936278$. Hence $2, 8 \in \mathrm{IDES}\,\sigma$ (case (1)) and $3 \in \mathrm{IDES}\,\sigma$ (case (2)).

2.3. *The second fundamental transformation.* As described in [Lo83, p. 201, Algorithm 10.6.1] by means of an algorithm, the second fundamental transformation, further denoted by F_2, can be defined on permutations as well as on words. Here we need only consider the case of permutations. As usual, the *number of inversions* of a permutation $\sigma=\sigma(1)\sigma(2)\cdots\sigma(n)$ is defined by $\mathrm{inv}\,\sigma := \#\{(i,j)\mid 1\le i<j\le n, \sigma(i)>\sigma(j)\}$. Its construction was given in [Fo68]. Further properties have been proved in [FS78], [BjW88]. Here we need the following result.

Theorem 2.5 [FS78]. *The transformation* F_2 *defined on the symmetric group* $\mathfrak{S}_n$ *is bijective and the following identities hold for every permutation* $\sigma\in\mathfrak{S}_n$*:* $\mathrm{inv}\,\mathrm{F}_2(\sigma)=\mathrm{maj}\,\sigma$; $\mathrm{IDES}\,\mathrm{F}_2(\sigma)=\mathrm{IDES}\,\sigma$.

Using the composition product $\mathrm{F}_2' := \mathbf{i}\circ\mathrm{F}_2\circ\mathbf{i}$ we therefore have:

$$\mathrm{inv}\,\mathrm{F}_2'(\sigma)=\mathrm{imaj}\,\sigma;\quad \mathrm{DES}\,\mathrm{F}_2'(\sigma)=\mathrm{DES}\,\sigma \tag{2.11}$$

for every $\sigma\in\mathfrak{S}_n$.

As the descent set "DES" is preserved under the transformation F_2', each desarrangement is mapped onto another desarrangement. It then makes sense to consider the chain:

$$\sigma\mapsto(\mathrm{PIX}\,\sigma,\mathrm{Desar}\,\sigma)\mapsto(\mathrm{PIX}\,\sigma,\mathrm{F}_2'\circ\mathrm{Desar}\,\sigma)\mapsto\rho,$$

where $(\mathrm{PIX}\,\rho,\mathrm{Desar}\,\rho):=(\mathrm{PIX}\,\sigma,\mathrm{F}_2'\circ\mathrm{Desar}\,\sigma)$. The mapping $\mathrm{F}_2^{\mathrm{loc}}:\sigma\mapsto\rho$ is a bijection of $\mathfrak{S}_n$ onto itself. Moreover, $\mathrm{PIX}\,\sigma=\mathrm{PIX}\,\rho$, $\mathrm{imaj}\circ\mathrm{Desar}\,\sigma=\mathrm{inv}\circ\mathrm{Desar}\,\rho$ and $\mathrm{DES}\circ\mathrm{Desar}\,\sigma=\mathrm{DES}\circ\mathrm{Desar}\,\rho$. Hence,

$$\begin{aligned}
\mathrm{mag}\,\sigma &= \sum_{i\in\mathrm{PIX}\,\sigma} i - \sum_{i=1}^{\mathrm{pix}\,\sigma} i + \mathrm{imaj}\circ\mathrm{Desar}\,\sigma\\
&= \sum_{i\in\mathrm{PIX}\,\rho} i - \sum_{i=1}^{\mathrm{pix}\,\rho} i + \mathrm{inv}\circ\mathrm{Desar}\,\rho\\
&= \#\{(i,j): 1\le i\le \mathrm{pix}\,\rho<j\le n,\ \rho(i)>\rho(j)\}\\
&\qquad\quad + \#\{(i,j): \mathrm{pix}\,\rho<i<j\le n,\ \rho(i)>\rho(j)\}\\
&= \mathrm{inv}\,\rho.
\end{aligned}$$

We have then proved the following proposition.

Proposition 2.6. *Let* $\rho:=\mathrm{F}_2^{\mathrm{loc}}(\sigma)$. *Then*

$$(\mathrm{PIX},\mathrm{mag})\,\sigma=(\mathrm{PIX},\mathrm{inv})\,\rho. \tag{2.12}$$

Corollary 2.7. *The pairs* $(\mathrm{PIX},\mathrm{mag})$ *and* $(\mathrm{PIX},\mathrm{inv})$ *are equidistributed over* $\mathfrak{S}_n$.

Finally, go back to Properties (2.11) and let $\xi:=\mathrm{F}_2'(\rho)$. Also let $\xi^p\xi^d$ and $\rho^p\rho^d$ be the pixed factorizations of ξ and ρ, respectively. We do not have $\xi^p=\rho^p$ necessarily, but as ρ and ξ have the same descent set, the factors ξ^p and ρ^p have the same length, i.e., $\mathrm{pix}\,\xi=\mathrm{pix}\,\rho$. Let us state this result in the next proposition.

Proposition 2.8. *Let* $\xi := \mathrm{F}_2'(\rho)$. *Then*

$$(\mathrm{pix}, \mathrm{inv})\,\xi = (\mathrm{pix}, \mathrm{imaj})\,\rho. \tag{2.13}$$

Corollary 2.9. *The pairs* $(\mathrm{pix}, \mathrm{inv})$ *and* $(\mathrm{pix}, \mathrm{imaj})$ *are equidistributed over* $\mathfrak{S}_n$.

Finally, $\mathrm{DW}^{\mathrm{glo}}$ attached to the unique oblique arrow in Fig. 1 refers to the global bijection constructed by Désarménien and Wachs ([DW93], § 5). It has the property: $(\mathrm{fix}, \mathrm{DES})\,\mathrm{DW}^{\mathrm{glo}}(\sigma) = (\mathrm{pix}, \mathrm{IDES})\,\sigma$ for all σ in $\mathfrak{S}_n$. The big challenge is to find two explicit bijections f and g, replacing $\mathrm{DW}^{\mathrm{glo}}$ and $\mathrm{DW}^{\mathrm{loc}}$, such that $(\mathrm{fix}, \mathrm{DES})\,g(\sigma) = (\mathrm{pix}, \mathrm{IDES})\,\sigma$ and $(\mathrm{PIX}, \mathrm{IDES})\,f(\sigma) = (\mathrm{FIX}, \mathrm{DEZ})\,\sigma$, which would make the bottom triangle commutative, that is, $\Phi = g \circ f$.

3. The bijections F_3 and F_3'

Let $0 \le m \le n$ and let v be a nonempty word of length m, whose letters are *positive* integers (with possible repetitions). Designate by $\mathrm{Sh}(0^{n-m}v)$ the set of all *shuffles* of the words 0^{n-m} and v, that is, the set of all rearrangements of the juxtaposition product $0^{n-m}v$, whose longest *subword* of positive letters is v. Let $w = x_1x_2\cdots x_n$ be a word from $\mathrm{Sh}(0^{n-m}v)$. It is convenient to write: $\mathrm{Pos}\,w := v$, $\mathrm{Zero}\,w := \{i : 1 \le i \le n,\ x_i = 0\}$, $\mathrm{zero}\,w := \#\,\mathrm{Zero}\,w$ $(= n-m)$, so that w is completely characterized by the pair $(\mathrm{Zero}\,w, \mathrm{Pos}\,w)$. Besides the statistic "maj" we will need the statistic "mafz" that associates the number

$$\mathrm{mafz}\,w := \sum_{i \in \mathrm{Zero}\,w} i - \sum_{i=1}^{\mathrm{zero}\,w} i + \mathrm{maj}\,\mathrm{Pos}\,w. \tag{3.1}$$

with each word from $\mathrm{Sh}(0^{n-m}v)$. In ([FoHa07], §4) we gave the construction of a bijection $\mathbf{F}_3$ of $\mathrm{Sh}(0^{n-m}v)$ onto itself having the following property:

$$\mathrm{maj}\,w = \mathrm{mafz}\,\mathbf{F}_3(w) \quad (w \in \mathrm{Sh}(0^{n-m}v)). \tag{3.2}$$

The bijection $\mathbf{F}_3$ is now applied to each shuffle class $\mathrm{Sh}(0^{n-m}v)$, when v is a derangement, or the *inverse* of a desarrangement. Let

$$\mathfrak{S}_n^{\mathrm{Der}} := \bigcup_{m,v} \mathrm{Sh}(0^{n-m}v) \quad (0 \le m \le n,\ v \in D_m);$$
$$\mathfrak{S}_n^{\mathrm{Desar}} := \bigcup_{m,v} \mathrm{Sh}(0^{n-m}v) \quad (0 \le m \le n,\ v^{-1} \in K_m).$$

As already seen in § 1.1, the mapping ZDer is a bijection of $\mathfrak{S}_n$ onto $\mathfrak{S}_n^{\mathrm{Der}}$ satisfying

$$
\text{(3.3)}\qquad \begin{aligned} \mathrm{FIX}\,\sigma &= \mathrm{Zero}\ \mathrm{ZDer}(\sigma); &\quad \mathrm{Der}\,\sigma &= \mathrm{Pos}\ \mathrm{ZDer}(\sigma);\\ \mathrm{maf}\,\sigma &= \mathrm{mafz}\ \mathrm{ZDer}(\sigma); &\quad \mathrm{DEZ}\,\sigma &= \mathrm{DES}\ \mathrm{ZDer}(\sigma). \end{aligned}
$$

Example 3.2. Let $\sigma = 1\,7\,3\,5\,2\,6\,4$. Then $w := \mathrm{ZDer}(\sigma) = 0\,4\,0\,3\,1\,0\,2$. We have $\mathrm{FIX}\,\sigma = \mathrm{Zero}\,w = \{1,3,6\}$; $\mathrm{Der}\,\sigma = \mathrm{Pos}\,w = 4\,3\,1\,2$; $\mathrm{maf}\,\sigma = \mathrm{mafz}\,w = (1{+}3{+}6){-}(1{+}2{+}3){+}(1{+}2{+}3) = 10$, $\mathrm{DEZ}\,\sigma = \mathrm{DES}\,w = \{2,4,5\}$.

Now define the bijection F_3 of $\mathfrak{S}_n$ onto itself by the chain

$$
\text{(3.4)}\qquad \mathrm{F}_3 : \sigma \overset{\mathrm{ZDer}}{\longmapsto} w \overset{\mathbf{F}_3}{\longmapsto} w' \overset{\mathrm{ZDer}^{-1}}{\longmapsto} \sigma'.
$$

Then, by (3.2),

$$
\begin{aligned} (\mathrm{fix},\mathrm{maz},\mathrm{Der})\,\sigma &= (\mathrm{zero},\mathrm{maj},\mathrm{Pos})\,w\\ &= (\mathrm{zero},\mathrm{mafz},\mathrm{Pos})\,w'\\ &= (\mathrm{fix},\mathrm{maf},\mathrm{Der})\,\sigma', \end{aligned}
$$

$$
\text{(3.5)}\qquad (\mathrm{fix},\mathrm{maz},\mathrm{Der})\,\sigma = (\mathrm{fix},\mathrm{maf},\mathrm{Der})\,\mathrm{F}_3(\sigma).
$$

The map "Desar" has been defined in (1.10) and it was noticed that each permutation σ was fully characterized by the pair $(\mathrm{PIX}\,\sigma, \mathrm{Desar}\,\sigma)$. Another way of deriving $\mathrm{ZDesar}(\sigma)$ introduced in §1.2 is to form the inverse $\sigma^{-1} = \sigma^{-1}(1)\sigma^{-1}(2)\cdots\sigma^{-1}(n)$ of σ. As $\sigma^{-1}(i) \ge \mathrm{pix}\,\sigma + 1$ if and only if $i \in [\,n\,] \setminus \mathrm{PIX}\,\sigma$, we see that $\mathrm{ZDesar}(\sigma)$ is also the word $w = x_1x_2\cdots x_n$, where

$$
x_i := \begin{cases} 0, & \text{if } i \in \mathrm{PIX}\,\sigma;\\ \sigma^{-1}(i) - \mathrm{pix}\,\sigma, & \text{if } i \in [\,n\,] \setminus \mathrm{PIX}\,\sigma. \end{cases}
$$

The word σ^{-1} contains the *subword* $1\,2\cdots\mathrm{pix}\,\sigma$. We then have: $i \in \mathrm{IDES}\,\sigma \Leftrightarrow i \in \mathrm{DES}\,\sigma^{-1} \Leftrightarrow \sigma^{-1}(i) \ge \mathrm{pix}\,\sigma + 1$ and $\sigma^{-1}(i) > \sigma^{-1}(i+1) \Leftrightarrow x_i \ge 1$ and $x_i > x_{i+1} \Leftrightarrow i \in \mathrm{DES}\,w$, so that $\mathrm{IDES}\,\sigma = \mathrm{DES}\,w$.

On the other hand, as $\mathrm{PIX}\,\sigma = \mathrm{Zero}\,w$ and $(\mathrm{Desar}\,\sigma)^{-1} = \mathrm{Pos}\,w$, we also have, by (1.11)

$$
\begin{aligned} \mathrm{mag}\,\sigma &= \sum_{i\in \mathrm{PIX}\,\sigma} i - \sum_{i=1}^{\mathrm{pix}\,\sigma} i + \mathrm{imaj}\circ\mathrm{Desar}\,\sigma\\ &= \sum_{i\in \mathrm{Zero}\,w} i - \sum_{i=1}^{\mathrm{zero}\,w} i + \mathrm{maj}\circ\mathrm{Pos}\,w = \mathrm{mafz}\,w. \end{aligned}
$$

As a summary,

$$
\text{(3.6)}\qquad \begin{aligned} \mathrm{PIX}\,\sigma &= \mathrm{Zero}\,\mathrm{ZDesar}(\sigma); &\quad \mathrm{Desar}\,\sigma &= \mathrm{Pos}\,\mathrm{ZDesar}(\sigma);\\ \mathrm{mag}\,\sigma &= \mathrm{mafz}\,\mathrm{ZDesar}(\sigma); &\quad \mathrm{IDES}\,\sigma &= \mathrm{DES}\,\mathrm{ZDesar}(\sigma). \end{aligned}
$$

Example 3.3. Let $\sigma = 1\,3\,6\,5\,4\,7\,2$. Then $\sigma^{-1} = 1\,7\,2\,5\,4\,3\,6$; $w := \mathrm{ZDesar}(\sigma) = 0\,4\,0\,2\,1\,0\,3$, $\mathrm{PIX}\,\sigma = \mathrm{Zero}\,w = \{1,3,6\}$; $\mathrm{Desar}\,\sigma = 3\,2\,4\,1$, $(\mathrm{Desar}\,\sigma)^{-1} = \mathrm{Pos}\,w = 4\,2\,1\,3$; $\mathrm{mag}\,\sigma = \mathrm{mafz}\,w = (1+3+6)-(1+2+3)+(1+2) = 7$, $\mathrm{IDES}\,\sigma = \mathrm{DES}\,w = \{2,4,5\}$.

Next define the bijection F_3' of $\mathfrak{S}_n$ onto itself by the chain

$$\mathrm{F}_3' : \sigma \overset{\mathrm{ZDesar}}{\longmapsto} w \overset{\mathbf{F}_3}{\longmapsto} w' \overset{\mathrm{ZDesar}^{-1}}{\longmapsto} \sigma'. \tag{3.7}$$

Then, by (3.2),

$$\begin{aligned}(\mathrm{pix}, \mathrm{imaj}, \mathrm{Desar})\,\sigma &= (\mathrm{zero}, \mathrm{maj}, \mathrm{Pos})\,w\\ &= (\mathrm{zero}, \mathrm{mafz}, \mathrm{Pos})\,w'\\ &= (\mathrm{pix}, \mathrm{mag}, \mathrm{Desar})\,\sigma',\end{aligned}$$

$$(\mathrm{pix}, \mathrm{imaj}, \mathrm{Desar})\,\sigma = (\mathrm{pix}, \mathrm{mag}, \mathrm{Desar})\,\mathrm{F}_3'(\sigma). \tag{3.8}$$

With (3.5) and (3.8) the first part of Theorem 1.3 is proved.

The second part of Theorem 1.3 is proved as follows. Remember that, if $\mathrm{zero}\,w = n-m$, the pair $(\mathrm{Zero}\,w, \mathrm{Pos}\,w)$ uniquely determines the shuffle of $(n-m)$ letters equal to 0 into the letters of $\mathrm{Pos}\,w$. The bijection "dw" defined by $\mathrm{dw} := \mathrm{ZDesar} \circ \mathrm{DW}^{\mathrm{loc}} \circ \mathrm{ZDer}^{-1}$ can also be derived by the chain

$$\begin{aligned}\mathrm{dw} : w &\mapsto (\mathrm{Zero}\,w, \mathrm{Pos}\,w)\\ &\mapsto (\mathrm{Zero}\,w, (\mathrm{DW} \circ \mathrm{Pos}\,w)^{-1}) = (\mathrm{Zero}\,w', \mathrm{Pos}\,w') \mapsto w',\end{aligned} \tag{3.9}$$

where DW denotes the Désarménien-Wachs bijection. It maps $\mathfrak{S}_n^{\mathrm{Der}}$ onto $\mathfrak{S}_n^{\mathrm{Desar}}$. In particular,

$$\mathrm{Pos} \circ \mathrm{dw}\,w = (\mathrm{DW} \circ \mathrm{Pos}\,w)^{-1}; \quad \mathrm{Zero} \circ \mathrm{dw}(w) = \mathrm{Zero}\,w.$$

Because of (2.4) we also have $\mathrm{DES} \circ \mathrm{Pos}\,w = \mathrm{DES} \circ \mathrm{Pos}\,w'$. As shown in our previous paper (`[FoHa07]`, Proposition 4.1), the latter property implies that

$$\mathrm{Zero} \circ \mathbf{F}_3(w) = \mathrm{Zero} \circ \mathbf{F}_3(w').$$

Furthermore,

$$\mathrm{Pos} \circ \mathbf{F}_3(w) = \mathrm{Pos}\,w, \quad \mathrm{Pos} \circ \mathbf{F}_3(w') = \mathrm{Pos}\,w',$$

since $\mathbf{F}_3$ maps each shuffle class onto itself. Hence,

$$\begin{aligned}\mathrm{Zero} \circ \mathrm{dw} \circ \mathbf{F}_3(w) &= \mathrm{Zero} \circ \mathbf{F}_3(w);\\ \mathrm{Pos} \circ \mathrm{dw} \circ \mathbf{F}_3(w) &= (\mathrm{DW} \circ \mathrm{Pos}\,\mathbf{F}_3(w))^{-1} = (\mathrm{DW} \circ \mathrm{Pos}\,w)^{-1};\\ \mathrm{Zero} \circ \mathbf{F}_3 \circ \mathrm{dw}(w) &= \mathrm{Zero} \circ \mathbf{F}_3(w);\\ \mathrm{Pos} \circ \mathbf{F}_3 \circ \mathrm{dw}(w) &= \mathrm{Pos} \circ \mathrm{dw}(w) = (\mathrm{DW} \circ \mathrm{Pos}\,w)^{-1}.\end{aligned}$$

The word $\mathrm{dw}\circ\mathbf{F}_3(w)$ is characterized by the pair

$$(\mathrm{Zero}\circ\mathrm{dw}\circ\mathbf{F}_3(w),\mathrm{Pos}\circ\mathrm{dw}\circ\mathbf{F}_3(w)),$$

which is equal to the pair

$$(\mathrm{Zero}\circ\mathbf{F}_3\circ\mathrm{dw}(w),\mathrm{Pos}\circ\mathbf{F}_3\circ\mathrm{dw}(w)),$$

which corresponds itself to the word $\mathbf{F}_3(w)\circ\mathrm{dw}(w)$. Hence,

$$\mathrm{dw}\circ\mathbf{F}_3=\mathbf{F}_3\circ\mathrm{dw}\,. \tag{3.10}$$

This shows that the top square in Fig. 2 is a commutative diagram, so is the bottom one. ∎

Example 3.4. Noting that $\mathrm{DW}(4\,3\,1\,2)=3\,2\,4\,1$ we have the commutative diagram

$$\begin{array}{ccc}
7\,4\,3\,1\,5\,6\,2 & \xrightarrow{\mathrm{DW}^{\mathrm{loc}}} & 3\,5\,6\,4\,2\,7\,1\\
{\scriptstyle \mathrm{F}_3}\Big\uparrow & & {\scriptstyle \mathrm{F}'_3}\Big\uparrow\\
1\,7\,3\,5\,2\,6\,4 & \xrightarrow{\mathrm{DW}^{\mathrm{loc}}} & 1\,3\,6\,5\,4\,7\,2
\end{array}$$

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