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SIGNED WORDS AND PERMUTATIONS, V; A SEXTUPLE DISTRIBUTION

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Abstract

We calculate the distribution of the sextuple statistic over the hyperoctahedral group B_n that involves the flag-excedance and flag-descent numbers "fexc" and "fdes," the flag-major index "fmaj," the positive and negative fixed point numbers "fix^+" and "fix^-" and the negative letter number "neg." Several specializations are considered. In particular, the joint distribution for the pair $(\mathrm{fexc}, \mathrm{fdes})$ is explicitly derived.

1. Introduction

As has been shown in our series of four papers ([FoHa07], [FoHa05], [FoHa06], [FoHa07a]), the *length function* "ℓ" (see [Bo68], p. 7, or [Hu90], p. 12) and the *flag major index* "fmaj" introduced by Adin and Roichman [AR01] have become the true q-analog makers for the calculation of various multivariable distributions on the hyperoctahedral group B_n of the signed permutations. The elements of B_n may be viewed as words $w = x_1x_2\cdots x_n$, where each x_i belongs to the set $\{-n, \ldots, -1, 1, \ldots, n\}$ and $|x_1||x_2|\cdots|x_n|$ is a permutation of $12\ldots n$. The *set* (resp. the *number*) of *negative* letters among the x_i's is denoted by $\mathrm{Neg}\, w$ (resp. $\mathrm{neg}\, w$). A *positive fixed point* of the signed permutation $w = x_1x_2\cdots x_n$ is a (positive) integer i such that $x_i = i$. It is convenient to write $\bar{i} := -i$ for each integer i. If $x_i = \bar{i}$ with i positive, we say that $\bar{i}$ is a *negative fixed point* of w. The set of all positive (resp. negative) fixed points of w is denoted by $\mathrm{Fix}^+ w$ (resp. $\mathrm{Fix}^- w$). Notice that $\mathrm{Fix}^- w \subset \mathrm{Neg}\, w$. Also let

$$\text{(1.1)} \qquad \mathrm{fix}^+ w := \#\,\mathrm{Fix}^+ w; \quad \mathrm{fix}^- w := \#\,\mathrm{Fix}^- w.$$

There are $2^n n!$ signed permutations of order n. The symmetric group $\mathfrak{S}_n$ may be considered as the subset of all w from B_n such that $\mathrm{Neg}\, w = \emptyset$. When w is an (ordinary) permutation from $\mathfrak{S}_n$, then $\mathrm{Fix}^- w = \emptyset$, so that we define $\mathrm{Fix}\, w := \mathrm{Fix}^+ w$ and $\mathrm{fix}\, w = \#\,\mathrm{Fix}\, w$.

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Now, for each statement A let $\chi(A) = 1$ or 0 depending on whether A is true or not. Besides the integer-valued statistics "fix^+," "fix^-" and "neg" the now classical *flag-descent number* "fdes", *flag-major index* "fmaj" and *flag-excedance number* "fexc" are also needed in the following study. They are defined for each signed permutation $w = x_1 x_2 \cdots x_n$ by

$$\begin{aligned} \mathrm{fdes}\, w &:= 2\,\mathrm{des}\, w + \chi(x_1 < 0);\\ \mathrm{fmaj}\, w &:= 2\,\mathrm{maj}\, w + \mathrm{neg}\, w;\\ \mathrm{fexc}\, w &:= 2\,\mathrm{exc}\, w + \mathrm{neg}\, w; \end{aligned}$$

where "des" is the *number of descents* $\mathrm{des}\, w := \sum_{i=1}^{n-1} \chi(x_i > x_{i+1})$, "maj" the *major index* $\mathrm{maj}\, w := \sum_{i=1}^{n-1} i\,\chi(x_i > x_{i+1})$ and "exc" the *number of excedances* $\mathrm{exc}\, w := \sum_{i=1}^{n-1} \chi(x_i > i)$.

Our intention is to calculate the distribution of the sextuple statistic $(\mathrm{fexc}, \mathrm{fdes}, \mathrm{fmaj}, \mathrm{fix}^+, \mathrm{fix}^-, \mathrm{neg})$ on the group B_n. This means that for each $n \ge 0$ the generating polynomial

$$(1.2)\quad B_n(s,t,q,Y_0,Y_1,Z) := \sum_{w\in B_n} s^{\mathrm{fexc}\, w} t^{\mathrm{fdes}\, w} q^{\mathrm{fmaj}\, w}\, Y_0^{\mathrm{fix}^+ w}\, Y_1^{\mathrm{fix}^- w}\, Z^{\mathrm{neg}\, w}$$

can be considered and, with a suitable normalization, the generating function for those polynomials be summed. Furthermore, the summation is to yield an appropriate closed form. Finally, the calculation must be compatible with the symmetric group $\mathfrak{S}_n$ in the sense that when the variable Z is given the zero value, earlier results derived for that group are to be recovered. As is shown below, this goal will be achieved by working in the algebra of q-series.

Our derivation has been motivated by the following recent statistical studies on B_n and $\mathfrak{S}_n$. In [FoHa07a] and [FoHa07b] we have respectively calculated the generating functions for the polynomials $B_n(1,t,q,Y_0,Y_1,Z)$ and $A_n(s,t,q,Y_0)$, where

$$(1.3)\qquad A_n(s,t,q,Y) := \sum_{\sigma\in\mathfrak{S}_n} s^{\mathrm{exc}\,\sigma} t^{\mathrm{des}\,\sigma} q^{\mathrm{maj}\,\sigma} Y^{\mathrm{fix}\,\sigma}$$

and shown that by giving certain variables specific values the former statistical results on $\mathfrak{S}_n$ and B_n could be reobtained. Note that, using the previous definitions of "fexc," "fdes" and "fmaj," the latter polynomial is nothing but $B_n(s^{1/2}, t^{1/2}, q^{1/2}, Y, 0, 0)$.

In the diagram of Fig. 1 the polynomials on the top (resp. bottom) level are specializations of the polynomial $B_n(s,t,q,Y_0,Y_1,Z)$ (resp. $A_n(s,t,q,Y)$). Each vertical arrow is given a $(Z = 0)$-label. This means that when Z is given the 0-value, each polynomial $B_n(\cdots)$ is transformed into the corresponding polynomial $A_n(\cdots)$. The other arrows labelled s, t and Z indicate that each target polynomial is mapped onto the source polynomial when s (resp. t, resp. Z) is given the 1-value.

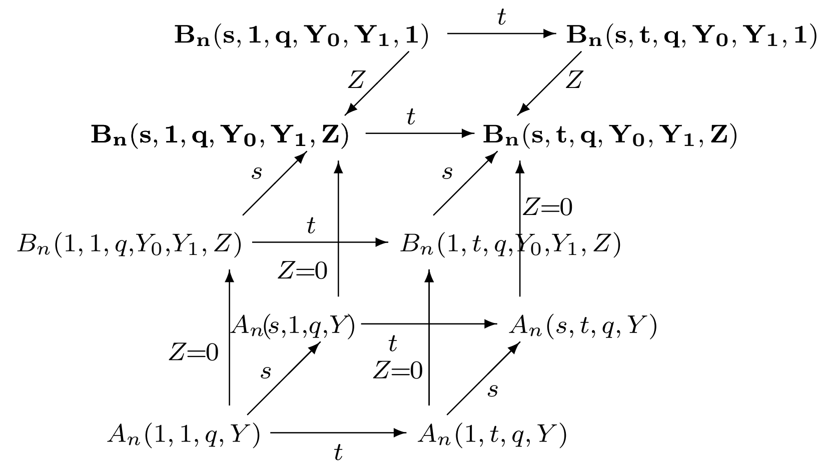

Fig. 1

The polynomials appearing on the top are all generating polynomials for the group B_n. Four of them are reproduced in boldface. Their factorial generating polynomials are derived in the present paper. The generating functions for the other six polynomials have been explicitly determined in earlier papers: $A_n(1,1,q,Y)$ and $A_n(1,t,q,Y)$ by Gessel and Reutenauer [GeRe03], then $A_n(s,1,q,Y)$ by Shareshian and Wachs [ShWa06], furthermore $A_n(s,t,q,Y)$ in [FoHa07b], finally $B_n(1,1,q,Y_0,Y_1,Z)$ and $B_n(1,t,q,Y_0,Y_1,Z)$ in [FoHa07a].

The classical notation on q-series will be used. First, the q-ascending factorials

$$(a;q)_n := \begin{cases} 1, & \text{if } n=0; \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & \text{if } n\ge 1. \end{cases}$$

$$(a;q)_\infty := \prod_{n\ge 1}(1-aq^{n-1});$$

then, the two q-exponentials (see [GaRa90, chap. 1])

$$e_q(u)=\sum_{n\ge 0}\frac{u^n}{(q;q)_n}=\frac{1}{(u;q)_\infty};\quad E_q(u)=\sum_{n\ge 0}q^{\binom{n}{2}}\frac{u^n}{(q;q)_n}=(-u;q)_\infty.$$

The main purpose of this paper is to prove the following theorem.

Theorem 1.1. *For each $n\ge 0$ let $B_n(s,t,q,Y_0,Y_1,Z)$ be the generating polynomial for the hyperoctahedral group B_n by the six-variable statistic* $(\mathrm{fexc},\mathrm{fdes},\mathrm{fmaj},\mathrm{fix}^+,\mathrm{fix}^-,\mathrm{neg})$ *as defined in* (1.2). *Then, the factorial generating function for the polynomials $B_n(s,t,q,Y_0,Y_1,Z)$ is given by:*

$$(1.4)\qquad \sum_{n\ge 0}(1+t)B_n(s,t,q,Y_0,Y_1,Z)\frac{u^n}{(t^2;q^2)_{n+1}} = \sum_{r\ge 0}t^r\frac{(u;q^2)_{\lfloor r/2\rfloor+1}}{(uY_0;q^2)_{\lfloor r/2\rfloor+1}}\frac{(-usqY_1Z;q^2)_{\lfloor (r+1)/2\rfloor}}{(-usqZ;q^2)_{\lfloor (r+1)/2\rfloor}}F_r(u;s,q,Z),$$

where

$$F_r(u;s,q,Z) = \frac{(us^2q^2;q^2)_{\lfloor r/2\rfloor}\,(1-s^2q^2)\,(u;q^2)_{\lfloor (r+1)/2\rfloor}\,(u;q^2)_{\lfloor r/2\rfloor}}{(u;q^2)_{\lfloor r/2\rfloor+1}\Big((u;q^2)_{\lfloor (r+1)/2\rfloor}\big((u;q^2)_{\lfloor r/2\rfloor}-s^2q^2\,(us^2q^2;q^2)_{\lfloor r/2\rfloor}\big) + sqZ\,(u;q^2)_{\lfloor r/2\rfloor}\big((u;q^2)_{\lfloor (r+1)/2\rfloor}-(us^2q^2;q^2)_{\lfloor (r+1)/2\rfloor}\big)\Big)}. \tag{1.5}$$

The factorial generating functions for the other polynomials written in boldface in the diagram of Fig. 1 are shown to be specializations of the factorial generating function for the six-variable polynomials $B_n(s,t,q,Y_0,Y_1,Z)$, as stated in the next three Corollaries.

Corollary 1.2. *The factorial generating function for the polynomials $B_n(s,1,q,Y_0,Y_1,Z)$ is given by*

$$\sum_{n\ge 0} B_n(s,1,q,Y_0,Y_1,Z)\frac{u^n}{(q^2;q^2)_n} = \frac{e_{q^2}(uY_0)\,E_{q^2}(usqY_1Z)}{E_{q^2}(usqZ)} \times \frac{(1-s^2q^2)}{e_{q^2}(us^2q^2)-s^2q^2e_{q^2}(u)+sqZ(e_{q^2}\big(us^2q^2)-e_{q^2}(u)\big)}. \tag{1.6}$$

Corollary 1.3. *The factorial generating function for the polynomials $B_n(s,t,q,Y_0,Y_1,1)$ is given by*

$$\sum_{n\ge 0}(1+t)B_n(s,t,q,Y_0,Y_1,1)\frac{u^n}{(t^2;q^2)_{n+1}} = \sum_{r\ge 0} t^r\,\frac{(u;q^2)_{\lfloor r/2\rfloor+1}}{(uY_0;q^2)_{\lfloor r/2\rfloor+1}}\,\frac{(-usqY_1;q^2)_{\lfloor (r+1)/2\rfloor}}{(-usq;q^2)_{\lfloor (r+1)/2\rfloor}}\,F_r(u;s,q,1), \tag{1.7}$$

where $F_r(u;s,q,1)$ is given by

$$F_r(u;s,q,1) = \frac{(us^2q^2;q^2)_{\lfloor r/2\rfloor}}{(u;q^2)_{\lfloor r/2\rfloor+1}} \times \frac{(1-sq)\,(u;q^2)_{\lfloor (r+1)/2\rfloor}\,(usq;q^2)_{\lfloor r/2\rfloor}}{(u;q^2)_{\lfloor (r+1)/2\rfloor}\,(usq;q^2)_{\lfloor r/2\rfloor}-sq\,(usq;q^2)_{\lfloor (r+1)/2\rfloor}\,(us^2q^2;q^2)_{\lfloor r/2\rfloor}}. \tag{1.8}$$

Corollary 1.4. *The factorial generating function for the polynomials $B_n(s,1,q,Y_0,Y_1,1)$ is given by*

$$\sum_{n\ge 0} B_n(s,1,q,Y_0,Y_1,1)\frac{u^n}{(q^2;q^2)_n} = \frac{e_{q^2}(uY_0)\,E_{q^2}(usqY_1)}{E_{q^2}(usq)}\,\frac{(1-sq)}{e_{q^2}(us^2q^2)-sq\,e_{q^2}(u)}. \tag{1.9}$$

The proof of Theorem 1.1 requires several steps. First, the factorial generating function for the polynomials $B_n(s,t,q,Y_0,Y_1,Z)$, as it appears on the left-hand side of (1.4), is shown to be equal to a series $\sum_{r\ge 0} t^r a_r$, where each a_r is the product of the rational function

$$\frac{(u;q^2)_{\lfloor r/2\rfloor+1}}{(uY_0;q^2)_{\lfloor r/2\rfloor+1}}\frac{(-usqY_1Z;q^2)_{\lfloor (r+1)/2\rfloor}}{(-usqZ;q^2)_{\lfloor (r+1)/2\rfloor}}$$

by the generating series $\sum u^{\lambda w} s^{\operatorname{fexc} w} q^{\operatorname{tot} c} Z^{\operatorname{neg} w}$ for the set $\mathrm{WSP}(r)$ of the so-called *weighted signed permutations* by a certain four-variable statistic $(\lambda,\operatorname{tot},\operatorname{fexc},\operatorname{neg})$ (see Theorem 4.2). The combinatorics of the weighted signed permutations was introduced in our previous paper [FoHa07a], but this time the extra variable "s" is to be added to make the calculation. All this derivation is developed in Sections 2 and 3. Noticeably, the variables Y_0 and Y_1 that carry the information on fixed points occur only in the above fraction, but not in the latter generating series. Also, the positive fixed point counter Y_0 occurs in the *denominator* of that fraction, while the negative fixed point counter Y_1 only in the *numerator*.

The next step is to show that this generating series for weighted signed permutations is also equal to the generating series for *words*, whose letters belong to the interval $[0,r]$, by another four-variable statistic $(\lambda,\operatorname{tot} v,\operatorname{evdec}+\operatorname{odd},\operatorname{odd})$. This is achieved by means of the construction of a *bijection* of the set of weighted signed permutations onto the set of those words having the adequate properties (see Theorem 5.1).

The crucial calculation is to evaluate the latter generating function and show that it is equal to the expression $F_r(u;s,q,Z)$ displayed in (1.5). Two proofs are given, the first one using a V-word *decomposition* theorem (see Theorem 6.1) derived in [FoHa07b] together with the traditional q-series *telescoping* technique, the second one taking advantage of a *word factorization*, which consists of cutting each word after every odd letter (see Section 8).

Formula (1.6) (resp. (1.9)) is deduced from (1.4) (resp. from (1.7)) by the traditional token that consists of multiplying the latter one by $(1-t)$ and letting r tend to $+\infty$, so that Corollaries 1.2 and 1.4 are easy consequences of Theorem 1.1 and Corollary 1.3.

We prove Corollary 1.3 in two different ways: first, as a specialization of Theorem 1.1 by an evaluation of the fraction $F_r(u;s,q,Z)$ for $Z=1$ (this requires some manipulations on q-series), second, by showing directly that $F_r(u;s,q,1)$ is the generating function for words by the two-variable statistic $(\operatorname{tot},2\operatorname{evdec}+\operatorname{odd})$, using a bijection constructed in our previous paper [FoHa07b]. We end the paper by deriving several specializations of Theorem 1.1, in particular, the joint distribution of the pair $(\operatorname{fexc},\operatorname{fdes})$ over the group B_n.

2. Weighted signed permutations

This section will appear to be an updated version of Section 4 of our previous paper [FoHa07a], where the notion of *weighted signed permutation* was introduced. With the addition of “fexc” it was essential to ascertain how that statistic behaved in the underlying combinatorial construction.

We use the following notations: if $c = c_1 c_2 \cdots c_n$ is a word, whose letters are nonnegative integers, let $\lambda(c) := n$ be the *length* of c, $\operatorname{tot} c := c_1 + c_2 + \cdots + c_n$ the *sum* of its letters and $\operatorname{odd} c$ the number of its *odd* letters. Furthermore, NIW_n (resp. $\mathrm{NIW}_n(r)$) designates the set of all *nonincreasing* monotonic words of length n, whose letters are nonnegative integers (resp. nonnegative integers at most equal to r). Also let $\mathrm{NIW}_n^e(r)$ (resp. $\mathrm{DW}_n^o(r)$) be the subset of $\mathrm{NIW}_n(r)$ of the monotonic nonincreasing (resp. strictly decreasing) words all letters of which are *even* (resp. *odd*).

Next, each pair $\binom{c}{w}$ is called a *weighted signed permutation* of order n if the four properties (*wsp1*)–(*wsp4*) hold:

(*wsp1*) c is a word $c_1 c_2 \cdots c_n$ from NIW_n;
(*wsp2*) w is a signed permutation $x_1 x_2 \cdots x_n$ from B_n;
(*wsp3*) $c_k = c_{k+1} \Rightarrow x_k < x_{k+1}$ for all $k = 1, 2, \ldots, n-1$;
(*wsp4*) x_k is positive (resp. negative) whenever c_k is even (resp. odd).

When w has no fixed points, either negative or positive, we say that $\binom{c}{w}$ is a *weighted signed derangement*. The set of weighted signed permutations (resp. derangements) $\binom{c}{w} = \binom{c_1 c_2 \cdots c_n}{x_1 x_2 \cdots x_n}$ of order n is denoted by WSP_n (resp. by WSD_n). The subset of all those weighted signed permutations (resp. derangements) such that $c_1 \le r$ is denoted by $\mathrm{WSP}_n(r)$ (resp. by $\mathrm{WSD}_n(r)$).

For example, the following pair

$$\binom{c}{w} = \left(\begin{array}{cc|c|ccc|ccc|c|cc|c} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ 10 & 10 & 9 & 7 & 7 & 7 & 4 & 4 & 4 & 3 & 2 & 2 & 1 \\ \mathbf{1} & \mathbf{2} & \overline{7} & \overline{6} & \overline{\mathbf{5}} & \overline{4} & 3 & \mathbf{8} & \mathbf{9} & \overline{\mathbf{10}} & \underline{12} & \underline{13} & \overline{11} \end{array}\right)$$

is a weighted signed permutation of order 13. It has four positive fixed points $(\mathbf{1}, \mathbf{2}, \mathbf{8}, \mathbf{9})$, two negative fixed points $(\overline{\mathbf{5}}, \overline{\mathbf{10}})$ and two excedances $(\underline{12}, \underline{13})$.

Proposition 2.1. *With each weighted signed permutation $\binom{c}{w}$ from the set $\mathrm{WSP}_n(r)$ can be associated a unique sequence $(i, j, k, \binom{c'}{w'}, v^e, v^o)$ such that*

(1) *i, j, k are nonnegative integers of sum n;*
(2) *$\binom{c'}{w'}$ is a weighted signed derangement from the set $\mathrm{WSD}_i(r)$;*
(3) *v^e is a nonincreasing word with even letters from the set $\mathrm{NIW}_j^e(r)$;*
(4) *v^o is a decreasing word with odd letters from the set $\mathrm{DW}_k^o(r)$;*

having the following properties:

$$
\begin{aligned}
&\operatorname{tot} c = \operatorname{tot} c' + \operatorname{tot} v^e + \operatorname{tot} v^o; \quad \operatorname{neg} w = \operatorname{neg} w' + \lambda(v^o);\\
&\operatorname{fix}^+ w = \lambda(v^e); \quad \operatorname{fix}^- w = \lambda(v^o); \quad \operatorname{fexc} w = \operatorname{fexc} w' + \lambda(v^0).
\end{aligned}
\tag{2.1}
$$

The bijection $\binom{c}{w} \mapsto (\binom{c'}{w'}, v^e, v^o)$ is quite natural to define. Only its reverse requires some attention. To get the latter three-term sequence from $\binom{c}{w}$ proceed as follows:

(a) let $l_1, \ldots, l_\alpha$ (resp. $m_1, \ldots, m_\beta$) be the increasing sequence of the integers l_i (resp. m_i) such that x_{l_i} (resp. x_{m_i}) is a positive (resp. negative) fixed point of w;

(b) define: $v^e := c_{l_1} \cdots c_{l_\alpha}$ and $v^o := c_{m_1} \cdots c_{m_\beta}$;

(c) remove all the columns $\binom{c_{l_1}}{x_{l_1}}, \ldots, \binom{c_{l_\alpha}}{x_{l_\alpha}}$, $\binom{c_{m_1}}{x_{m_1}}, \ldots, \binom{c_{m_\beta}}{x_{m_\beta}}$ from $\binom{c}{w}$ and let c' be the nonincreasing word derived from c after the removal;

(d) once the letters $x_{l_1}, \ldots, x_{l_\alpha}, x_{m_1}, \ldots, x_{m_\beta}$ have been removed from the signed permutation w the remaining ones form a signed permutation of a subset A of $[n]$, of cardinality $n - \alpha - \beta$. Using the unique increasing bijection ϕ of A onto the interval $[n - \alpha - \beta]$ replace each remaining letter x_i by $\phi(x_i)$ if $x_i > 0$ or by $-\phi(-x_i)$ if $x_i < 0$. Let w' be the signed derangement of order $n - \alpha - \beta$ thereby obtained. There is no difficulty verifying that the properties listed in (2.1) hold.

For instance, with the above weighted signed permutation we have: $v^e = 10, 10, 4, 4$ and $v^o = 7, 3$. After removing the fixed point columns we obtain:

$$
\left(\begin{array}{c|cc|c|cc|c}
3 & 4 & 6 & 7 & 11 & 12 & 13\\
9 & 7 & 7 & 4 & 2 & 2 & 1\\
\overline{7} & \overline{6} & \overline{4} & 3 & \underline{12} & \underline{13} & \overline{11}
\end{array}\right)
\text{ and then }
\begin{pmatrix} c' \\ \\ w' \end{pmatrix} =
\left(\begin{array}{c|cc|c|cc|c}
1 & 2 & 3 & 4 & 5 & 6 & 7\\
9 & 7 & 7 & 4 & 2 & 2 & 1\\
\overline{4} & \overline{3} & \overline{2} & 1 & \underline{6} & \underline{7} & \overline{5}
\end{array}\right).
$$

The signed permutation w (resp. w') has two excedances $\underline{12}, \underline{13}$ (resp. $\underline{6}, \underline{7}$) and six (resp. four) negative letters. Furthermore, $v^0 = 7, 3$ is of length 2. Hence $\operatorname{fexc} w = 2 \operatorname{exc} w + \operatorname{neg} w = 2 \times 2 + 6 = 10 = 2 \times 2 + 4 + 2 = \operatorname{fexc} w' + \lambda(v^0)$.

For reconstructing $\binom{c}{w}$ from the sequence $(\binom{c'}{w'}, v^e, v^o)$ consider the nonincreasing rearrangement of the juxtaposition product $v^e v^o$ in the form $b_1^{h_1} \cdots b_m^{h_m}$, where $b_1 > \cdots > b_m$ and $h_i \geq 1$ (resp. $h_i = 1$) if b_i is even (resp. odd). The pair $\binom{c'}{w'}$ being decomposed into matrix blocks, as shown in the example, each letter b_i indicates where the h_i fixed point columns are to be inserted. We do not give more details and simply illustrate the construction with the running example.

With the previous example $b_1^{h_1} \cdots b_m^{h_m} = 10^2\, 7\, 4^2\, 3$. First, implement 10^2:

$$
\left(\begin{array}{cc|c|cc|c|cc|c}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9\\
\mathbf{10} & \mathbf{10} & 9 & 7 & 7 & 4 & 2 & 2 & 1\\
1 & 2 & \overline{6} & \overline{5} & \overline{4} & 3 & 8 & 9 & \overline{7}
\end{array}\right);
$$

then 7:

$$\begin{pmatrix} 1 & 2 & \vert & 3 & \vert & 4 & 5 & 6 & \vert & 7 & \vert & 8 & 9 & \vert & 10 \\ 10 & 10 & \vert & 9 & \vert & 7 & \mathbf{7} & 7 & \vert & 4 & \vert & 2 & 2 & \vert & 1 \\ 1 & 2 & \vert & \overline{7} & \vert & \overline{6} & \overline{5} & \overline{4} & \vert & 3 & \vert & 9 & 10 & \vert & \overline{8} \end{pmatrix};$$

notice that because of condition (*wsp3*) the letter **7** is to be inserted in *second* position in the third block; then insert 4^2:

$$\begin{pmatrix} 1 & 2 & \vert & 3 & \vert & 4 & 5 & 6 & \vert & 7 & 8 & 9 & \vert & 10 & 11 & \vert & 12 \\ 10 & 10 & \vert & 9 & \vert & 7 & 7 & 7 & \vert & 4 & \mathbf{4} & \mathbf{4} & \vert & 2 & 2 & \vert & 1 \\ 1 & 2 & \vert & \overline{7} & \vert & \overline{6} & \overline{5} & \overline{4} & \vert & 3 & 8 & 9 & \vert & 11 & 12 & \vert & \overline{10} \end{pmatrix}.$$

The implementation of 3 gives back the original weighted signed permutation $\binom{c}{w}$.

3. A summation on weighted signed permutations

For $0 \le k \le n$ let $\begin{bmatrix} n \\ k \end{bmatrix}_q := \dfrac{(q;q)_n}{(q;q)_k\,(q;q)_{n-k}}$ be the usual q-binomial coefficient. It is q-routine (see, e.g., [An76, chap. 3]) to prove the following identities, where v_1 is the first letter of v:

$$\frac{1}{(u;q)_N} = \sum_{n\ge 0} \begin{bmatrix} N+n-1 \\ n \end{bmatrix}_q u^n; \qquad \begin{bmatrix} N+n \\ n \end{bmatrix}_q = \sum_{v\in \mathrm{NIW}_n(N)} q^{\mathrm{tot}\, v};$$

$$\frac{1}{(u;q)_{N+1}} = \sum_{n\ge 0} u^n \sum_{v\in \mathrm{NIW}_n(N)} q^{\mathrm{tot}\, v} = \frac{1}{1-u} \sum_{v\in \mathrm{NIW}_n} q^{\mathrm{tot}\, v} u^{v_1};$$

$$\frac{1}{(u;q^2)_{\lfloor r/2\rfloor+1}} = \sum_{n\ge 0} u^n \sum_{v^e\in \mathrm{NIW}^e_n(r)} q^{\mathrm{tot}\, v^e}; \tag{3.1}$$

$$(-uq;q^2)_{\lfloor (r+1)/2\rfloor} = \sum_{n\ge 0} u^n \sum_{v^o\in \mathrm{DW}^o_n(r)} q^{\mathrm{tot}\, v^o}. \tag{3.2}$$

The last two formulas and Proposition 2.1 are now used to calculate the generating function for the weighted signed permutations. The symbols $\mathrm{NIW}^e(r)$, $\mathrm{DW}^o(r)$, $\mathrm{WSP}(r)$, $\mathrm{WSD}(r)$ designate the unions for $n\ge 0$ of the corresponding symbols with an n-subscript.

Proposition 3.1. *The following identity holds:*

$$\sum_{n\ge 0} u^n \sum_{\binom{c}{w}\in \mathrm{WSP}_n(r)} q^{\mathrm{tot}\, c} s^{\mathrm{fexc}\, w} Y_0^{\mathrm{fix}^+ w} Y_1^{\mathrm{fix}^- w} Z^{\mathrm{neg}\, w} \tag{3.3}$$
$$= \frac{(u;q^2)_{\lfloor r/2\rfloor+1}}{(uY_0;q^2)_{\lfloor r/2\rfloor+1}} \frac{(-usqY_1Z;q^2)_{\lfloor (r+1)/2\rfloor}}{(-usqZ;q^2)_{\lfloor (r+1)/2\rfloor}} \times \sum_{n\ge 0} u^n \sum_{\binom{c}{w}\in \mathrm{WSP}_n(r)} q^{\mathrm{tot}\, c} s^{\mathrm{fexc}\, w} Z^{\mathrm{neg}\, w}.$$

Proof. First, summing over $(w^e, w^o, \binom{c}{w}) \in \mathrm{NIW}^e(r) \times \mathrm{DW}^o(r) \times \mathrm{WSP}(r)$, we have

$$\sum_{w^e,w^o,\binom{c}{w}} u^{\lambda(w^e)} q^{\operatorname{tot} w^e} \times (usZ)^{\lambda(w^o)} q^{\operatorname{tot} w^o} \times u^{\lambda(c)} q^{\operatorname{tot} c} s^{\operatorname{fexc} w} Y_0^{\operatorname{fix}^+ w} Y_1^{\operatorname{fix}^- w} Z^{\operatorname{neg} w}$$

$$= \frac{(-usqZ;q^2)_{\lfloor (r+1)/2 \rfloor}}{(u;q^2)_{\lfloor r/2 \rfloor + 1}} \times \sum_{\binom{c}{w}} u^{\lambda(c)} q^{\operatorname{tot} c} s^{\operatorname{fexc} w} Y_0^{\operatorname{fix}^+ w} Y_1^{\operatorname{fix}^- w} Z^{\operatorname{neg} w} \tag{3.4}$$

by (3.1) and (3.2). As $\operatorname{fexc} w = \operatorname{fexc} w' + \lambda(v^o)$, Proposition 2.1 implies that the initial expression can also be summed over five-term sequences $(\binom{c'}{w'}, v^e, v^o, w^e, w^o)$ from $\mathrm{WSD}(r) \times \mathrm{NIW}^e(r) \times \mathrm{DW}^o(r) \times \mathrm{NIW}^e(r) \times \mathrm{DW}^o(r)$ in the form

$$\begin{aligned}
&\sum_{\binom{c'}{w'},v^e,v^o,w^e,w^o} u^{\lambda(c')} q^{\operatorname{tot} c'} s^{\operatorname{fexc} w'} Z^{\operatorname{neg} w'} \times (uY_0)^{\lambda(v^e)} q^{\operatorname{tot} v^e} \times (usY_1 Z)^{\lambda(v^o)} q^{\operatorname{tot} v^o} \\
&\qquad\qquad\qquad\qquad \times u^{\lambda(w^e)} q^{\operatorname{tot} w^e} \times (usZ)^{\lambda(w^o)} q^{\operatorname{tot} w^o} \\
&\quad = \sum_{v^e,v^o} (uY_0)^{\lambda(v^e)} q^{\operatorname{tot} v^e} \times (usY_1 Z)^{\lambda(v^o)} q^{\operatorname{tot} v^o} \\
&\quad \times \sum_{\binom{c'}{w'},w^e,w^o} u^{\lambda(c')} q^{\operatorname{tot} c'} s^{\operatorname{fexc} w'} Z^{\operatorname{neg} w'} \times u^{\lambda(w^e)} q^{\operatorname{tot} w^e} \times (usZ)^{\lambda(w^o)} q^{\operatorname{tot} w^o}.
\end{aligned}$$

The first summation can be evaluated by (3.1) and (3.2), while by Proposition 2.1 again the second sum can be expressed as a sum over weighted signed permutations $\binom{c}{w} \in \mathrm{WSP}(r)$. Therefore, the initial sum is also equal to

$$\frac{(-usqY_1Z;q^2)_{\lfloor (r+1)/2 \rfloor}}{(uY_0;q^2)_{\lfloor r/2 \rfloor + 1}} \times \sum_{\binom{c}{w} \in \mathrm{WSP}(s)} u^{\lambda(c)} q^{\operatorname{tot} c} s^{\operatorname{fexc} w} Z^{\operatorname{neg} w}. \tag{3.5}$$

Identity (3.3) follows by equating (3.4) with (3.5). ∎

4. A further evaluation

Proposition 4.1. *Let $B_n(s,t,q,Y_0,Y_1,Z)$ denote the generating polynomial for the group B_n by the statistic* $(\operatorname{fexc}, \operatorname{fdes}, \operatorname{maj}, \operatorname{fix}^+, \operatorname{fix}^-, \operatorname{neg})$. *Then*

$$\begin{aligned}
&\frac{1+t}{(t^2;q^2)_{n+1}} B_n(s,t,q,Y_0,Y_1,Z) \\
&\qquad\qquad = \sum_{r \geq 0} t^r \sum_{\binom{c}{w} \in \mathrm{WSP}_n(r)} q^{\operatorname{tot} c} s^{\operatorname{fexc} w} Y_0^{\operatorname{fix}^+ w} Y_1^{\operatorname{fix}^- w} Z^{\operatorname{neg} w}.
\end{aligned} \tag{4.1}$$

Proof. A very similar calculation has been made in the proof of Theorem 4.1 in [FoHa05]. We also make use of the identities on the q-ascending factorials that were recalled in the previous section. First,

$$\begin{aligned}\frac{1+t}{(t^2;q^2)_{n+1}} &= \sum_{r'\geq 0}(t^{2r'}+t^{2r'+1})\begin{bmatrix} n+r' \\ r' \end{bmatrix}_{q^2} \\ &= \sum_{r\geq 0} t^r \begin{bmatrix} n+\lfloor r/2\rfloor \\ \lfloor r/2\rfloor \end{bmatrix}_{q^2} = \sum_{r\geq 0} t^r \sum_{b\in \mathrm{NIW}_n(\lfloor r/2\rfloor)} q^{2\,\mathrm{tot}\, b}.\end{aligned}$$

Then,

$$\begin{aligned}&\frac{1+t}{(t^2;q^2)_{n+1}}\, B_n(s,t,q,Y_0,Y_1,Z) \\ &\quad = \sum_{r\geq 0} t^r \sum_{\substack{b\in \mathrm{NIW}_n,\\ 2b_1\leq r}} q^{2\,\mathrm{tot}\, b} \sum_{w\in B_n} s^{\mathrm{fexc}\, w} t^{\mathrm{fdes}\, w} q^{\mathrm{fmaj}\, w} Y_0^{\mathrm{fix}^+ w} Y_1^{\mathrm{fix}^- w} Z^{\mathrm{neg}\, w} \\ &\quad = \sum_{r\geq 0} t^r \sum_{\substack{b\in \mathrm{NIW}_n,\, w\in B_n\\ 2b_1+\mathrm{fdes}\, w\leq r}} s^{\mathrm{fexc}\, w} q^{2\,\mathrm{tot}\, b+\mathrm{fmaj}\, w} Y_0^{\mathrm{fix}^+ w} Y_1^{\mathrm{fix}^- w} Z^{\mathrm{neg}\, w}.\end{aligned}$$

As proved in [FoHa05, § 4] to each $\binom{c}{w} = \binom{c_1\cdots c_n}{x_1\cdots x_n} \in \mathrm{WSP}_n(s)$ there corresponds a unique $b = b_1\cdots b_n \in \mathrm{NIW}_n$ such that $2b_1+\mathrm{fdes}\, w = c_1$ and $2\,\mathrm{tot}\, b+\mathrm{fmaj}\, w = \mathrm{tot}\, c$. Moreover, the mapping $\binom{c}{w} \mapsto (b,w)$ is a bijection of $\mathrm{WSP}_n(r)$ onto the set of all pairs (b,w) such that $b = b_1\cdots b_n \in \mathrm{NIW}_n$ and $w\in B_n$ with the property that $2b_1+\mathrm{fdes}\, w\leq r$.

The word b is determined as follows: write the signed permutation w as a linear word $w = x_1x_2\ldots x_n$ and for each $k = 1,2,\ldots,n$ let z_k be the number of descents ($x_i > x_{i+1}$) in the right factor $x_kx_{k+1}\cdots x_n$ and let ϵ_k be equal to 0 or 1 depending on whether x_k is positive or negative. Also for each $k = 1,2,\ldots,n$ define $a_k := (c_k-\epsilon_k)/2$, $b_k := (a_k - z_k)$ and form the word $b = b_1\cdots b_n$.

For example,

$$\begin{array}{rl}\mathrm{Id} &= 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10 \\ c &= 9\ 7\ 7\ 4\ 4\ 4\ 2\ 2\ 1\ \ 1 \\ w &= \overline{4}\ \overline{3}\ \overline{2}\ 1\ 5\ 6\ 8\ 9\ \overline{10}\ \overline{7} \\ z &= 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ \ 0 \\ \epsilon &= 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 1\ \ 1 \\ a &= 4\ 3\ 3\ 2\ 2\ 2\ 1\ 1\ 0\ \ 0 \\ b &= 3\ 2\ 2\ 1\ 1\ 1\ 0\ 0\ 0\ \ 0\end{array}$$

Pursuing the above calculation we get (4.1). ∎

The next theorem is then a consequence of Propositions 3.1 and 4.1.

Theorem 4.2. *The following identity holds:*

$$\sum_{n\geq 0}(1+t)B_n(s,t,q,Y_0,Y_1,Z)\frac{u^n}{(t^2;q^2)_{n+1}} \tag{4.2}$$

$$=\sum_{r\geq 0}t^r\frac{(u;q^2)_{\lfloor r/2\rfloor+1}}{(uY_0;q^2)_{\lfloor r/2\rfloor+1}}\frac{(-usqY_1Z;q^2)_{\lfloor (r+1)/2\rfloor}}{(-usqZ;q^2)_{\lfloor (r+1)/2\rfloor}}\sum_{\binom{c}{w}\in \mathrm{WSP}(r)}u^{\lambda w}q^{\mathrm{tot}\, c}s^{\mathrm{fexc}\, w}Z^{\mathrm{neg}\, w}.$$

In view of the statements of Theorems 1.1 and 4.2 we see that the former theorem will be proved if we can show that the following identity holds:

$$\sum_{\binom{c}{w}\in \mathrm{WSP}(r)}u^{\lambda w}q^{\mathrm{tot}\, c}s^{\mathrm{fexc}\, w}Z^{\mathrm{neg}\, w}=F_r(u;s,q,Z), \tag{4.3}$$

with $F_r(u;s,q,Z)$ given by (1.5). In our paper [FoHa06a] the sum $\sum q^{\mathrm{tot}\, c}Z^{\mathrm{neg}\, w}$ $(\binom{c}{w}\in \mathrm{WSP}_n(r))$ has been calculated, by setting up a bijection of $\mathrm{WSP}_n(r)$ onto the set $[0,r]^n$ of the words of length n, whose letters are taken from the interval $[0,r]$. Because of the presence of the new statistic "fexc" *another* bijection is to be constructed. If v is the image of $\binom{c}{w}$ under the new bijection, then the statistic "fexc" on signed permutations must have a counterpart on words. This role is played by the so-called *number of even decreases*, as shown in the next section.

5. Even decreases on words

Say that a letter y_i of a word $v=y_1y_2\cdots y_n$ from $[0,r]^n$ is a *decrease* in v if $1\leq i\leq n-1$ and $y_i\geq y_{i+1}\geq\cdots\geq y_j>y_{j+1}$ for some j such that $i\leq j\leq n-1$. Let $\mathrm{dec}\, v$ (resp. $\mathrm{evdec}\, v$) denote the *number* of decreases (resp. of *even* decreases) in v. Notice that $\mathrm{dec}\, v\geq \mathrm{des}\, v$ (the number of *descents* in v) and $\mathrm{dec}\, v=\mathrm{des}\, v$ whenever v is a permutation (without repetitions). Also, let $\mathrm{odd}\, v$ be the number of odd letters in v.

Recall that a nonempty word $v=y_1y_2\cdots y_n$ is a *Lyndon word*, if either $n=1$, or $n\geq 2$ and, with respect to the lexicographic order, the inequality $y_1y_2\cdots y_n>y_iy_{i+1}\cdots y_ny_1\cdots y_{i-1}$ holds for every i such that $2\leq i\leq n$. Let v, v' be two nonempty primitive words (none of them can be written as v_0^a for $a\geq 2$ and some word v_0). We write $v\preceq v'$ if and only if $v^a\leq v'^a$ with respect to the lexicographic order for an integer a large enough. As shown for instance in [Lo83, Theorem 5.1.5] (also see [Ch58], [Sch65]) each nonempty word v can be written uniquely as a product $l_1l_2\cdots l_k$, called its *Lyndon factorization*, where each l_i is a Lyndon word and $l_1\preceq l_2\preceq\cdots\preceq l_k$. In the example below the Lyndon factorization of v has been materialized by vertical bars. The essential property of the Lyndon factorization needed in the sequel is the following:

$$\mathrm{dec}\, v=\mathrm{dec}\, l_1+\mathrm{dec}\, l_2+\cdots+\mathrm{dec}\, l_k.$$

Now, start with the Lyndon factorization $l_1 l_2 \cdots l_k$ of a word v from $[0,r]^n$. With such a v we associate a permutation σ from $\mathfrak{S}_n$ by means of a procedure developed by Gessel-Reutenauer [GeRe93]: each letter y_i of v belongs to a Lyndon word factor l_h, so that $l_h = v' y_i v''$. Then, form the infinite word $A(y_i) := y_i v'' v' y_i v'' v' \cdots$ If y_i and $y_{i'}$ are two letters of v, say that y_i *precedes* $y_{i'}$ if $A(y_i) > A(y_{i'})$ for the lexicographic order, or if $A(y_i) = A(y_{i'})$ and y_i is to the right of $y_{i'}$ in the word v. This precedence determines a total order on the n letters of v. The letter that precedes all the other ones is given label 1, the next one label 2, and so on. When each letter y_i of v is replaced by its label, say, $\mathrm{lab}(y_i)$, each Lyndon word factor l_j becomes a new word τ_j. The essential property is that each τ_j starts with its *minimum* element and those minimum elements read from left to right are in *decreasing order*. We can then interpret each τ_j as the *cycle* of a permutation and the (juxtaposition) product $\tau_1 \tau_2 \cdots \tau_k$ as the (functional) product of *disjoint* cycles. This product, said to be written *in canonical form*, defines a unique permutation σ from $\mathfrak{S}_n$ ([Lo83], § 10.2).

For example,

$$\begin{array}{l} v = 2 \mid 3\ \ 2\ \ 1\ \ 1 \mid 3 \mid 5 \mid 6\ 4\ 2\ \ 1\ \ 3\ \ 2\ 3 \mid 6\ 6\ 3\ \ 1\ 6\ 6\ 2 \mid 6 \\ \sigma = 16 \mid 12\ 18\ 22\ 21 \mid 10 \mid 7 \mid 4\ 8\ 17\ 20\ 11\ 15\ 9 \mid 2\ 5\ 13\ 19\ 3\ 6\ 14 \mid 1 \end{array}$$

The labels on the second row are obtained as follows: read the letters equal to 6 (the maximal letter) from left to right and form their associated infinite words: $64213236421\cdots$, $66316626631\cdots$, $6316621131\cdots$, $662663166\cdots$, $62663166\cdots$, $66666\cdots$ Those letters 6 read from left to right will be given the labels 4, 2, 5, 3, 6, 1. We continue the labellings by reading the letters equal to 5, then 4, ... in the above word v.

No decrease y_i in v can be the rightmost letter of a Lyndon word factor l_h. We have then $l_h = \cdots y_i y_{i+1} \cdots y_j y_{j+1} \cdots$ with $y_i \ge y_{i+1} \ge \cdots \ge y_j > y_{j+1}$. Consequently, $A(y_i) > A(y_{i+1})$ and $\mathrm{lab}(y_i) < \mathrm{lab}(y_{i+1})$. Conversely, if $\mathrm{lab}(y_i) < \mathrm{lab}(y_{i+1})$ and y_i, y_{i+1} belong to the same Lyndon factor, then y_i is a decrease in v. To each decrease y_i in v there corresponds a unique cycle τ_h of σ and a pair $\mathrm{lab}(y_i)\,\mathrm{lab}(y_{i+1})$ of *successive* letters of τ_h such that $\mathrm{lab}(y_i) < \mathrm{lab}(y_{i+1})$ and $\mathrm{lab}(y_{i+1}) = \sigma(\mathrm{lab}(y_i))$.

Consider the monotonic nonincreasing rearrangement $c = c_1 c_2 \cdots c_n$ of v and form the three-row matrix

$$\begin{array}{cccc} 1 & 2 & \cdots & n \\ c_1 & c_2 & \cdots & c_n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{array}$$

Then, if y_i is a decrease in v, the $\mathrm{lab}(y_i)$-th column of the previous matrix is of the form

$$\begin{array}{c} \mathrm{lab}(y_i) \\ y_i \\ \mathrm{lab}(y_{i+1}) \end{array} \quad \text{with} \quad \mathrm{lab}(y_i) < \mathrm{lab}(y_{i+1}).$$

At this stage we have $\operatorname{dec} v = \operatorname{exc} \sigma$. We then have to transform σ into a *signed* permutation w in such a way that only the excedances corresponding to the even decreases of v are preserved. We proceed as follows. The word c can be expressed as $a_1^{m_1} a_2^{m_2} \cdots a_k^{m_k}$ where $a_1 > a_2 > \cdots > a_k \ge 0$ and $m_1 \ge 1$, $m_2 \ge 1$, … , $m_k \ge 1$. We then define

$$
\begin{aligned}
x_1 \cdots x_{m_1} &:= \begin{cases} \sigma(1) \cdots \sigma(m_1), & \text{if } a_1 \text{ is even;} \\ \overline{\sigma}(m_1) \cdots \overline{\sigma}(1), & \text{if } a_1 \text{ is odd;} \end{cases} \\
x_{m_1+1} \cdots x_{m_1+m_2} &:= \begin{cases} \sigma(m_1+1) \cdots \sigma(m_1+m_2), & \text{if } a_2 \text{ is even;} \\ \overline{\sigma}(m_1+m_2) \cdots \overline{\sigma}(m_1+1), & \text{if } a_2 \text{ is odd;} \end{cases} \\
\cdots \quad & \quad \cdots \\
x_{m_1+\cdots+m_{k-1}+1} \cdots x_n &:= \begin{cases} \sigma(m_1+\cdots+m_{k-1}+1) \cdots \sigma(n), & \text{if } a_k \text{ is even;} \\ \overline{\sigma}(n) \cdots \overline{\sigma}(m_1+\cdots+m_{k-1}+1), & \text{if } a_k \text{ is odd.} \end{cases}
\end{aligned}
$$

The word $w := x_1 x_2 \cdots x_n$ is then a *signed* permutation. When going from σ to w, the excedances $\operatorname{lab}(y_i) < \operatorname{lab}(y_{i+1})$ of σ such that y_i is odd have vanished. The other ones have been preserved. Hence, $2 \operatorname{evdec} v + \operatorname{odd} v = 2 \operatorname{exc} w + \operatorname{neg} w = \operatorname{fexc} w$. Finally, as $\sigma(i) > \sigma(i+1) \Rightarrow c_i > c_{i+1}$, the pair $\binom{c}{w}$ is a *weighted signed permutation*. We then have the desired bijection $[0,r]^n \to \binom{c}{w}$.

Theorem 5.1. *The mapping $v \to \binom{c}{w}$ is a bijection of $[0,r]^n$ onto $\mathrm{WSP}_n(r)$ having the following properties:*

$$\operatorname{tot} c = \operatorname{tot} v; \quad \operatorname{fexc} w = 2 \operatorname{evdec} v + \operatorname{odd} v; \quad \operatorname{neg} w = \operatorname{odd} v.$$

With the running example we have

$$
\begin{array}{r|c|c|c|c|c}
\mathrm{Id} = & 1\ 2\ 3\ 4\ 5\ 6 & 7 & 8 & 9\ 10\ 11\ 12\ 13 & 14\ 15\ 16\ 17\ 18 & 19\ 20\ 21\ 22 \\
c = & 6\ 6\ 6\ 6\ 6\ 6 & 5 & 4 & 3\ 3\ 3\ 3\ 3 & 2\ 2\ 2\ 2\ 2 & 1\ 1\ 1\ 1 \\
\sigma = & 1\ 5\ 6\ 8\ 13\ 14 & 7 & 17 & 4\ 10\ 15\ 18\ 19 & 2\ 9\ 16\ 20\ 22 & 3\ 11\ 12\ 21 \\
w = & 1\ \mathbf{5}\ \mathbf{6}\ \mathbf{8}\ \mathbf{13}\ \mathbf{14} & \overline{7} & \mathbf{17} & \overline{19}\ \overline{18}\ \overline{15}\ \overline{10}\ \overline{4} & 2\ 9\ 16\ \mathbf{20}\ \mathbf{22} & \overline{21}\ \overline{12}\ \overline{11}\ \overline{3}
\end{array}
$$

The signed permutation w has eight excedances (reproduced in boldface) and ten negative letters. Therefore, $\operatorname{fexc} w = 2 \times 8 + 10 = 26$. There are eight even decreases of the word v: $2 \to 1$, $6 \to 4$, $4 \to 2$, $2 \to 1$, $6 \to 6$, $6 \to 3$, $6 \to 6$, $6 \to 2$. Moreover, $\operatorname{odd} v = 10$. Hence $2 \operatorname{evdec} v + \operatorname{odd} v = 26 = \operatorname{fexc} w$ and of course $\operatorname{neg} w = \operatorname{odd} v = 10$.

In view of (4.3) and Theorem 5.1 the proof of Theorem 1.1 will be completed if the identy

$$\sum_{v \in [0,r]^*} u^{\lambda v} q^{\operatorname{tot} v} s^{2 \operatorname{evdec} v + \operatorname{odd} v} Z^{\operatorname{odd} v} = F_r(u; s, q, Z) \tag{5.1}$$

holds, the sum being over all words whose letters are in $[0,r]$.

6. The calculation of the latter sum

As introduced in our previous paper [FoHa07b] a word $w = x_1x_2\ldots x_n$ of length $n \geq\ 2$ is said to be a *V-word*, if for some integer i such that $1 \le i \le n-1$ we have $x_1 \ge x_2 \ge \cdots \ge x_i > x_{i+1}$ and $x_{i+1} \le x_{i+2} \le \cdots \le x_n < x_i$ whenever $i+1 < n$. The pair (x_i, x_{i+1}) is called the *critical biletter* of w. The V-word decomposition for permutations was introduced by Kim and Zeng [KiZe01]. The following theorem is a simple consequence of Theorem 3.4 proved in our previous paper [FoHa07b].

Theorem 6.1 (V-word decomposition)**.** *To each word $v = y_1y_2\cdots y_n$ whose letters are nonnegative integers there corresponds a unique sequence $(v_0, v_1, v_2, \ldots, v_k)$, where v_0 is a monotonic nondecreasing word and v_1, v_2, ... , v_k are V-words with the further property that $v_0v_1v_2\cdots v_k$ is a rearrangement of v and*

$$\operatorname{evdec} v = \operatorname{evdec} v_1 + \operatorname{evdec} v_2 + \cdots + \operatorname{evdec} v_k. \tag{6.1}$$

Let X_0, X_1, ... , X_r be $(r+1)$ commuting variables. The *even weight*, $\operatorname{evweight} v$, of each word $v = y_1y_2\ldots y_n$ is defined to be

$$\operatorname{evweight} v := X_{y_1}X_{y_2}\cdots X_{y_n}\, s^{2\operatorname{evdec} v}.$$

Now, consider the infinite series:

$$\begin{aligned} U(l) &:= \prod_{l\le j\le r} (1 - s^{2\chi(j\ \text{even})}X_j)^{-1}, \quad (1 \le l \le r);\\ M(k,l) &:= \prod_{k\le j\le l-1} (1 - X_j)^{-1}, \quad (0 \le k < l). \end{aligned}$$

Then, the generating function for V-words, whose critical biletter is (l,k) $(0 \le k < l \le r)$, by "evweight" is equal to:

$$U(l)\, s^{2\chi(l\ \text{even})}\, X_lX_k\, M(k,l).$$

The following theorem is then a consequence of Theorem 6.1.

Theorem 6.2. *We have the identity:*

$$\sum_v \operatorname{evweight} v = \frac{M(0,r+1)}{1 - \sum\limits_{0\le k<l\le r} U(l)\, s^{2\chi(l\ \text{even})}\, X_lX_k\, M(k,l)},$$

where the sum is over all words whose letters belong to the interval $[0, r]$.

Consider the homomorphism ϕ generated by:

$$\phi(X_k) := uq^k(sZ)^{\chi(k\ \text{odd})} \quad (0 \le k \le r).$$

If $v = y_1y_2\cdots y_n$, then $\phi(\operatorname{evweight} v) = u^{\lambda v}q^{\operatorname{tot} v}s^{2\operatorname{evdec} v + \operatorname{odd} v}Z^{\operatorname{odd} v}$.

Also

$$\phi(U(l)) = \prod_{l\le j\le r} (1 - uq^j s^{2\chi(j\,\mathrm{even})}(sZ)^{\chi(j\ \mathrm{odd})})^{-1};$$
$$\phi(M(k,l)) = \prod_{k\le j\le l-1} (1 - uq^j (sZ)^{\chi(j\ \mathrm{odd})})^{-1}.$$

Now define

$$(u)'_k := \begin{cases} 1, & \text{if } k=0;\\ \prod\limits_{0\le j\le k-1} (1-uq^j(sZ)^{\chi(j\ \mathrm{odd})}), & \text{if } k\ge 1;\end{cases}$$
$$(u)''_l := \begin{cases} 1, & \text{if } l=0;\\ \prod\limits_{1\le j\le l} (1-uq^j s^{2\chi(j\,\mathrm{even})}(sZ)^{\chi(j\ \mathrm{odd})}), & \text{if } l\ge 1.\end{cases}$$

Then $\phi(U(l)) = \dfrac{(u)''_{l-1}}{(u)''_r}$ $(1 \le l)$; $\phi(M(k,l)) = \dfrac{(u)'_k}{(u)'_l}$ $(0 \le k < l)$. Now, take the image of the identity of Theorem 6.2 under ϕ:

$$\sum_{w\in[0,r]^*} u^{\lambda w} q^{\mathrm{tot}\, w} s^{(2\,\mathrm{evdec}\,+\,\mathrm{odd})\, w} Z^{\mathrm{odd}\, w} = \sum_{w\in[0,r]^*} \phi(\mathrm{evweight}\, w) = \frac{1}{(u)'_{r+1}}\frac{1}{S},$$

where

$$S = 1 - \sum_{0\le k<l\le r} \frac{(u)''_{l-1}}{(u)''_r} s^{2\chi(l\ \mathrm{even})} uq^l (sZ)^{\chi(l\ \mathrm{odd})} uq^k (sZ)^{\chi(k\ \mathrm{odd})} \frac{(u)'_k}{(u)'_l}.$$

As $-uq^k(sZ)^{\chi(k\ \mathrm{odd})}(u)'_k = (u)'_{k+1} - (u)'_k$, we have:

$$S = 1 + \sum_{1\le l\le r} \frac{(u)''_{l-1}}{(u)''_r} s^{2\chi(l\ \mathrm{even})} uq^l (sZ)^{\chi(l\ \mathrm{odd})} \frac{1}{(u)'_l}((u)'_l - 1).$$

Now, $-uq^l s^{2\chi(l\ \mathrm{even})}(sZ)^{\chi(l\ \mathrm{odd})}(u)''_{l-1} = (u)''_l - (u)''_{l-1}$. Hence,

$$S = \frac{1}{(u)''_r} - \frac{1}{(u)''_r}\sum_{1\le l\le r} \frac{(u)''_{l-1}}{(u)'_l} uq^l s^{2\chi(l\ \mathrm{even})}(sZ)^{\chi(l\ \mathrm{odd})}.$$

We are then left to prove:

$$\text{(6.2)}\quad \frac{(u)''_r}{(u)'_{r+1}}\Bigl(1 - \sum_{1\le l\le r} \frac{(u)''_{l-1}}{(u)'_l} uq^l s^{2\chi(l\ \mathrm{even})}(sZ)^{\chi(l\ \mathrm{odd})}\Bigr)^{-1} = F_r(u;s,q,Z)$$

We can also write:

$$(u)'_r = (u;q^2)_{\lfloor (r+1)/2\rfloor}\,(usqZ;q^2)_{\lfloor r/2\rfloor},$$
$$(u)''_r = (usqZ;q^2)_{\lfloor (r+1)/2\rfloor}\,(us^2q^2;q^2)_{\lfloor r/2\rfloor},$$

so that

$$\frac{(u)''_r}{(u)'_{r+1}} = \frac{(us^2q^2;q^2)_{\lfloor r/2\rfloor}}{(u;q^2)_{\lfloor r/2\rfloor+1}} \quad\text{and}\quad \frac{(u)''_{l-1}}{(u)'_l} = \frac{(us^2q^2;q^2)_{\lfloor (l-1)/2\rfloor}}{(u;q^2)_{\lfloor (l+1)/2\rfloor}}. \tag{6.3}$$

Identity (6.2) may be rewritten as

$$\frac{(us^2q^2;q^2)_{\lfloor r/2\rfloor}}{(u;q^2)_{\lfloor r/2\rfloor+1}}(1-T)^{-1} = F_r(u;s,q,Z), \tag{6.4}$$

where

$$\begin{aligned}T &:= \sum_{1\le l\le r} \frac{(us^2q^2;q^2)_{\lfloor (l-1)/2\rfloor}}{(u;q^2)_{\lfloor (l+1)/2\rfloor}} uq^l s^{2\chi(l\text{ even})}(sZ)^{\chi(l\text{ odd})}\\ &= \sum_{0\le l\le \lfloor (r-1)/2\rfloor} \frac{(us^2q^2;q^2)_l}{(u;q^2)_{l+1}} uq^{2l}sqZ + \sum_{0\le l\le \lfloor r/2\rfloor-1} \frac{(us^2q^2;q^2)_l}{(u;q^2)_{l+1}} uq^{2l}s^2q^2.\end{aligned}$$

We can then introduce

$$G(m) := \sum_{0\le l\le m} \frac{(us^2q^2;q^2)_l}{(u;q^2)_{l+1}} uq^{2l}$$

and try to sum it. As

$$\frac{(us^2q^2;q^2)_{l+1}}{(u;q^2)_{l+1}} - \frac{(us^2q^2;q^2)_l}{(u;q^2)_l} = \frac{(us^2q^2;q^2)_l}{(u;q^2)_{l+1}} uq^{2l}(1-s^2q^2),$$

we get

$$G(m) = \frac{1}{1-s^2q^2}\Bigl(\frac{(us^2q^2;q^2)_{m+1}}{(u;q^2)_{m+1}} - 1\Bigr),$$

so that

$$\begin{aligned}T &= sqZ\,G(\lfloor (r-1)/2\rfloor) + s^2q^2\,G(\lfloor r/2\rfloor - 1)\\ &= \frac{1}{1-s^2q^2}\Bigl(sqZ\frac{(us^2q^2;q^2)_{\lfloor (r+1)/2\rfloor}}{(u;q^2)_{\lfloor (r+1)/2\rfloor}} - sqZ\\ &\qquad\qquad + s^2q^2\frac{(us^2q^2;q^2)_{\lfloor r/2\rfloor}}{(u;q^2)_{\lfloor r/2\rfloor}} - s^2q^2\Bigr).\end{aligned}$$

By reporting this value of T into (6.4) we exactly find the expression of $F_r(u;s,q,Z)$ displayed in (1.5). ∎

7. Second proof of (5.1)

When $Z=0$ in $B_n(s,t,q,Y_0,Y_1,Z)$, then Y_1 is also null. Furthermore, each polynomial $B_n(s,t,q,Y_0,0,0)$ is a polynomial in s^2, t^2, q^2, Y_0. The summation on the right-hand side of (1.6) only involves even powers of t. We then have

$$F_{2r}(u;s,q,0)=\frac{(us^2q^2;q^2)_r\,(1-s^2q^2)\,(u;q^2)_r}{(u;q^2)_{r+1}\big((u;q^2)_r-s^2q^2\,(us^2q^2;q^2)_r\big)} \tag{7.1}$$

and

$$\sum_{n\ge 0}B_n(s,t,q,Y_0,0,0)\frac{u^n}{(t^2;q^2)_{n+1}}=\sum_{r\ge 0}t^{2r}\frac{(u;q^2)_{r+1}}{(uY_0;q^2)_{r+1}}F_{2r}(u;s,q,0). \tag{7.2}$$

But $B_n(s^{1/2},t^{1/2},q^{1/2},Y,0,0)$ is the generating polynomial $A_n(s,t,q,Y)$ for the symmetric group $\mathfrak{S}_n$ by the statistic $(\mathrm{exc},\mathrm{des},\mathrm{maj},\mathrm{fix})$ and it was proved in our previous paper [FoHa06b] that

$$\sum_{n\ge 0}A_n(s,t,q,Y)\frac{u^n}{(t;q)_{n+1}}=\sum_{r\ge 0}t^r\frac{(u;q)_{r+1}}{(uY;q)_{r+1}}\,\frac{(usq;q)_r\,(1-sq)\,(u;q)_r}{(u;q)_{r+1}\big((u;q)_r-sq(usq;q)_r\big)}.$$

Accordingly, (1.4) holds when $Z=0$ and consequently (5.1) holds when r is even and Z null. To show that identity (5.1), which we shall rewrite as

$$\sum_{v\in[0,r]^*}\mathrm{EV}(v)=F_r(u;s,q,Z), \tag{7.3}$$

is true for all values of r and when Z is not necessarily 0 we proceed as follows.

Cut each word $v\in[0,r]^*$ in the (5.1) summation *after* every occurrence of an *odd* letter. This defines a unique factorization:

$$v=p_0i_0p_1i_1\cdots p_ki_kp_{k+1} \tag{7.4}$$

of v having the following properties

(C1) $k\ge 0$,
(C2) $p_0,\,p_1,\,\ldots\,,\,p_k,\,p_{k+1}$ all from P_r^* with $P_r:=\{0,2,4,\ldots,2\lfloor r/2\rfloor\}$,
(C3) $i_0,\,i_1,\,\ldots,\,i_k$ all from I_r with $I_r:=\{1,3,5,\ldots,2\lfloor (r+1)/2\rfloor-1\}$.

The following properties of this factorization are easy to verify:

(P1) $\mathrm{tot}\,v=\mathrm{tot}(p_0i_0)+\mathrm{tot}(p_1i_1)+\cdots+\mathrm{tot}(p_ki_k)+\mathrm{tot}(p_{k+1})$,
(P2) $\mathrm{evdec}\,v=\mathrm{evdec}(p_0i_0)+\mathrm{evdec}(p_1i_1)+\cdots+\mathrm{evdec}(p_ki_k)+\mathrm{evdec}(p_{k+1})$,
(P3) $\mathrm{odd}\,v=\mathrm{odd}(p_0i_0)+\mathrm{odd}(p_1i_1)+\cdots+\mathrm{odd}(p_ki_k)+\mathrm{odd}(p_{k+1})$,
(P4) $\lambda v=\lambda(p_0i_0)+\lambda(p_1i_1)+\cdots+\lambda(p_ki_k)+\lambda(p_{k+1})$,

(P5) $\mathrm{EV}(p_0 i_0 p_1 i_1 \cdots p_k i_k p_{k+1}) = \mathrm{EV}(p_0 i_0)\,\mathrm{EV}(p_1 i_1) \cdots \mathrm{EV}(p_k i_k)\,\mathrm{EV}(p_{k+1})$,
(P6) $\mathrm{EV}(pi) = sqZ\,\mathrm{EV}(p(i-1))$ for $p \in P_r^*$ and $i \in I_r$,
(P7) $\mathrm{EV}(p(2k)) = uq^{2k}\,\mathrm{EV}(p)$ for $p \in P_{2k}^*$.

Hence

$$\sum_{v\in[0,r]^*} v = \Bigl(1 - \sum_{p\in P_r^*, i\in I_r} pi\Bigr)^{-1} \sum_{p\in P_r^*} p;$$

$$\sum_{v\in[0,r]^*} \mathrm{EV}(v) = \frac{1}{1-\sum_{p\in P_r^*, i\in I_r} \mathrm{EV}(pi)} \sum_{p\in P_r^*} \mathrm{EV}(p). \tag{7.5}$$

On the other hand,

$$\sum_{p\in P_r^*, i\in I_r} \mathrm{EV}(pi) = sqZ \sum_{p\in P_r^*, i\in I_r} \mathrm{EV}(p(i-1)) = sqZ \sum_{p\in P_r^*, j\in P_{r-1}} \mathrm{EV}(pj).$$

If $r = 2k+1$, then $P_r = P_{r-1} = P_{2k}$, so that

$$\sum_{p\in P_r^*, j\in P_{r-1}} \mathrm{EV}(pj) = \sum_{p\in P_{2k}^*, \lambda(p)\ge 1} \mathrm{EV}(p) = F_{2k}(u;s,q,0) - 1,$$

because (7.3) holds for r even and $Z = 0$. It follows from (7.5) that

$$\sum_{v\in[0,2k+1]^*} \mathrm{EV}(v) = \frac{F_{2k}(u;s,q,0)}{1 - sqZ\bigl(F_{2k}(u;s,q,0)-1\bigr)}. \tag{7.6}$$

If $r = 2k$, then $P_r = P_{2k}$ and $P_{r-1} = P_{2k-2}$, so that

$$\begin{aligned}
\sum_{p\in P_r^*, j\in P_{r-1}} \mathrm{EV}(pj) &= \sum_{p\in P_{2k}^*, j\in P_{2k-2}} \mathrm{EV}(pj)\\
&= \sum_{p\in P_{2k}^*, \lambda(p)\ge 1} \mathrm{EV}(p) - \sum_{p\in P_{2k}^*} \mathrm{EV}(p(2k))\\
&= (F_{2k}(u;s,q,0) - 1) - uq^{2k} F_{2k}(u;s,q,0)\\
&= (1-uq^{2k})F_{2k}(u;s,q,0) - 1.
\end{aligned}$$

Hence (7.5) implies

$$\sum_{v\in[0,2k]^*} \mathrm{EV}(v) = \frac{F_{2k}(u;s,q,0)}{1 - sqZ\bigl((1-uq^{2k})F_{2k}(u;s,q,0)-1\bigr)}. \tag{7.7}$$

We can then report the value of $F_{2k}(u;s,q,0)$ obtained in (7.1) in both expressions (7.6) and (7.7). By combining them in a single formula we exactly get the formula displayed in (1.5) for $F_r(u;s,q,Z)$. ∎

8. Two proofs of Corollary 1.3

For the first proof we proceed as follows. Let $Z = 1$ in (1.5) and write the formula as:

$$\frac{(us^2q^2;q^2)_{\lfloor r/2\rfloor}\,(1-s^2q^2)\,(u;q^2)_{\lfloor (r+1)/2\rfloor}}{(u;q^2)_{\lfloor r/2\rfloor+1}F_r(u;s,q,1)}$$
$$= (u;q^2)_{\lfloor (r+1)/2\rfloor} - s^2q^2\,\frac{(u;q^2)_{\lfloor (r+1)/2\rfloor}}{(u;q^2)_{\lfloor r/2\rfloor}}(us^2q^2;q^2)_{\lfloor r/2\rfloor}$$
$$+ sq\,(u;q^2)_{\lfloor (r+1)/2\rfloor} - sq(us^2q^2;q^2)_{\lfloor (r+1)/2\rfloor}.$$

We have

$$s^2q^2\,\frac{(u;q^2)_{\lfloor (r+1)/2\rfloor}}{(u;q^2)_{\lfloor r/2\rfloor}}(us^2q^2;q^2)_{\lfloor r/2\rfloor} + sq(us^2q^2;q^2)_{\lfloor (r+1)/2\rfloor}$$
$$= sq(us^2q^2;q^2)_{\lfloor r/2\rfloor}\big(sq(1-uq^{2\lfloor (r-1)/2\rfloor}) + 1 - us^2q^{2\lfloor (r-1)/2\rfloor+2}\big)$$
$$= sq(us^2q^2;q^2)_{\lfloor r/2\rfloor}(1+sq)(1-usq^{2\lfloor (r-1)/2\rfloor+1})$$
$$= sq(us^2q^2;q^2)_{\lfloor r/2\rfloor}(1+sq)\frac{(usq;q^2)_{\lfloor (r+1)/2\rfloor}}{(usq;q^2)_{\lfloor r/2\rfloor}},$$

so that

$$\frac{(us^2q^2;q^2)_{\lfloor r/2\rfloor}\,(1-s^2q^2)\,(u;q^2)_{\lfloor (r+1)/2\rfloor}}{(u;q^2)_{\lfloor r/2\rfloor+1}F_r(u;s,q,1)}$$
$$= (1+sq)\big((u;q^2)_{\lfloor (r+1)/2\rfloor} - sq(us^2q^2;q^2)_{\lfloor r/2\rfloor}\,\frac{(usq;q^2)_{\lfloor (r+1)/2\rfloor}}{(usq;q^2)_{\lfloor r/2\rfloor}}.$$

By dividing both sides by $(1+sq)$ we recover (1.8). $\square$

For the second proof we again use the notations introduced in Section 7, namely,

$$(u)'_r := \begin{cases} 1, & \text{if } r=0;\\ \prod\limits_{0\le j\le r-1}(1-uq^j(sZ)^{\chi(j\ \mathrm{odd})}), & \text{if } r\ge 1;\end{cases}$$
$$= (u;q^2)_{\lfloor (r+1)/2\rfloor}\,(usqZ;q^2)_{\lfloor r/2\rfloor},$$

and directly prove the identity

$$\sum_{v\in[0,r]^n} q^{\mathrm{tot}\,v}s^{2\,\mathrm{evdec}\,v+\mathrm{odd}\,v} = F_r(u;s,q,1),$$

where $F_r(u;s,q,1)$ is given by (1.8), an identity that may be rewritten as:

$$\sum_{v\in[0,r]^n} q^{\mathrm{tot}\,v}s^{2\,\mathrm{evdec}\,v+\mathrm{odd}\,v} = \frac{1}{(u)'_{r+1}}\times\frac{(1-sq)(u)'_r(usq)'_r}{(u)'_r - sq(usq)'_r}. \tag{8.1}$$

For each $r \ge 0$ let $D(r)$ be the set of all pairs (w,i) such that $w \in \mathrm{NIW}(r-1)$, $1 \le i \le \lambda w - 1$, $\lambda w \ge 2$. In our previous paper ([FoHa07b], Theorem 2.1) we have constructed a bijection mapping each word v onto a sequence

$$(w_0, (w_1, i_1), (w_2, i_2), \ldots, (w_k, i_k)),$$

such that $w_0 \in \mathrm{NIW}(r)$ and each pair $(w_l, i_l) \in D(r)$ $(1 \le l \le k)$ and $w_0 w_1 \cdots w_k$ is a rearrangement of v. Furthermore, the bijection has the following property:

$$2 \operatorname{evdec} v + \operatorname{odd} v = i_1 + i_2 + \cdots i_k + \operatorname{odd} w_0 + \operatorname{odd} w_1 + \cdots + \operatorname{odd} w_k. \tag{8.2}$$

Hence, $\sum_{v \in [0,r]^n} q^{\operatorname{tot} v} s^{2\operatorname{evdec} v + \operatorname{odd} v} = \dfrac{A}{1-B}$, where $A = \sum_{w \in \mathrm{NIW}(r)} q^{\operatorname{tot} w} s^{\operatorname{odd} w} u^{\lambda w}$ and $B = \sum_{(w,i) \in D(r)} (sq)^i q^{\operatorname{tot} w} s^{\operatorname{odd} w} u^{\lambda w}$. But $A = \dfrac{1}{(u)'_{r+1}}$ and

$$\begin{aligned}
B &= \sum_{(w,i)\in D(r)} (sq)^i q^{\operatorname{tot} w} s^{\operatorname{odd} w} u^{\lambda w} \\
&= \sum_{n\ge 2} \sum_{i=1}^{n-1} (sq)^i \sum_{w \in \mathrm{NIW}_n(r-1)} q^{\operatorname{tot} w} s^{\operatorname{odd} w} u^{\lambda w} \\
&= \sum_{n \ge 2} \frac{sq - (sq)^n}{1-sq} \sum_{w \in \mathrm{NIW}_n(r-1)} q^{\operatorname{tot} w} s^{\operatorname{odd} w} u^{\lambda w} \\
&= \frac{sq}{1-sq} \sum_{n\ge 2} \sum_{w \in \mathrm{NIW}_n(r-1)} q^{\operatorname{tot} w} s^{\operatorname{odd} w} u^{\lambda w} \\
&\qquad + \frac{1}{1-sq} \sum_{n\ge 2} \sum_{w \in \mathrm{NIW}_n(r-1)} q^{\operatorname{tot} w} s^{\operatorname{odd} w} (usq)^{\lambda w} \\
&= \frac{sq}{1-sq}\Bigl(\frac{1}{(u)'_r} - (1+cu)\Bigr) + \frac{1}{1-sq}\Bigl(\frac{1}{(usq)'_r} - (1+cusq)\Bigr) \\
&\qquad\qquad [\text{where } c \text{ is the coefficient of } u \text{ in } 1/(u)'_r] \\
&= 1 + \frac{1}{1-sq}\Bigl(\frac{sq}{(u)'_r} - \frac{1}{(usq)'_r}\Bigr).
\end{aligned}$$

Finally, $\displaystyle\sum_{v\in[0,r]^n} q^{\operatorname{tot} v} s^{2\operatorname{evdec} v + \operatorname{odd} v} = \frac{A}{1-B} = \frac{1}{(u)'_{r+1}} \times \frac{(1-sq)(u)'_r (usq)'_r}{(u)'_r - sq(usq)'_r}$,
which is the expression written in (8.1). $\square$

Remark. The method used in the latter proof cannot be applied when $Z \ne 1$. Although identity (8.2) always holds, we do not have

$$\operatorname{odd} v = \operatorname{odd} w_0 + \operatorname{odd} w_1 + \cdots + \operatorname{odd} w_k$$

in general.

9. Specializations

In Section 7 we have seen that the generating function for the polynomials $A_n(s,t,q,Y)$ can be derived from (1.6) by letting $Z=0$ and replace the triplet (s,t,q,Y_0) by $(s^{1/2},t^{1/2},q^{1/2},Y)$. We just comment the specializations $s=1$ and $Y_0=Y_1=Z=1$.

9.1. *Case* $s=1$. We have

$$
\begin{aligned}
&F_r(u;1,q,Z)\\
&\quad=\frac{(uq^2;q^2)_{\lfloor r/2\rfloor}\,(1-q^2)\,(u;q^2)_{\lfloor (r+1)/2\rfloor}\,(u;q^2)_{\lfloor r/2\rfloor}}{(u;q^2)_{\lfloor r/2\rfloor+1}\Big((u;q^2)_{\lfloor (r+1)/2\rfloor}\big((u;q^2)_{\lfloor r/2\rfloor}-q^2\,(uq^2;q^2)_{\lfloor r/2\rfloor}\big)}\\
&\qquad\qquad\overline{\qquad+qZ\,(u;q^2)_{\lfloor r/2\rfloor}\big((u;q^2)_{\lfloor (r+1)/2\rfloor}-(uq^2;q^2)_{\lfloor (r+1)/2\rfloor}\big)\Big)},
\end{aligned}
$$

so that we can divide both numerator and denominator by the product $(u;q^2)_{\lfloor (r+1)/2\rfloor}\,(u;q^2)_{\lfloor r/2\rfloor}$. We then get

$$
\begin{aligned}
F_r(u;1,q,Z)&=\frac{1-q^2}{1-u}\Big(1-q^2\frac{1-uq^{2\lfloor r/2\rfloor}}{1-u}+qZ-qZ\frac{1-uq^{2\lfloor (r+1)/2\rfloor}}{1-u}\Big)^{-1}\\
&=\Big(1-u\frac{1-q^{2\lfloor r/2\rfloor+2}}{1-q^2}-uqZ\frac{1-q^{2\lfloor (r+1)/2\rfloor}}{1-q^2}\Big)^{-1}\\
&=\Big(1-u\sum_{0\le i\le r}q^iZ^{\chi(i\ \mathrm{odd})}\Big)^{-1}.
\end{aligned}
$$

We then recover identity (1.9) from our paper [FoHa07a] in the form:

$$
\begin{aligned}
(9.1)\qquad &\sum_{n\ge 0}(1+t)B_n(1,t,q,Y_0,Y_1,Z)\frac{u^n}{(t^2;q^2)_{n+1}}\\
&=\sum_{r\ge 0}t^r\frac{(u;q^2)_{\lfloor r/2\rfloor+1}}{(uY_0;q^2)_{\lfloor r/2\rfloor+1}}\,\frac{(-uqY_1Z;q^2)_{\lfloor (r+1)/2\rfloor}}{(-uqZ;q^2)_{\lfloor (r+1)/2\rfloor}}\Big(1-u\sum_{i=0}^{r}q^iZ^{\chi(i\ \mathrm{odd})}\Big)^{-1}.
\end{aligned}
$$

9.2. *Case* $Y_0=Y_1=Z=1$. Identity (1.7) simply becomes

$$
(9.2)\qquad \sum_{n\ge 0}(1+t)B_n(s,t,q,1,1,1)\frac{u^n}{(t^2;q^2)_{n+1}}=\sum_{r\ge 0}t^rF_r(u;s,q,1),
$$

where $F_r(u;s,q,1)$ is given by (1.8). Note that $B_n(s,t,q,1,1,1)$ is the generating polynomial for the group B_n by (fexc, fdes, fmaj). Write $B_n(s,t)$ for $B_n(s,t,1,1,1,1)$. From (9.2) we deduce

$$
\begin{aligned}
(9.3)\quad \sum_{n\ge 0}B_n(s,t)\frac{u^n}{(1-t^2)^n}=\sum_{r\ge 0}\Big(&\frac{(1-s)(1-t)t^{2r}}{(1-u)^{r+1}(1-us^2)^{-r}-s(1-u)}\\
&+\frac{(1-s)(1-t)t^{2r+1}}{(1-u)^{r+1}(1-us^2)^{-r}-s(1-us)}\Big).
\end{aligned}
$$

To obtain the marginal distribution $B_n(s,1) = \sum_{w\in B_n} s^{\mathrm{fexc}\, w}$ it is convenient to specialize formula (1.9). We obtain

$$\sum_{n\ge 0} B_n(s,1)\frac{u^n}{n!} = \frac{1-s}{-s+\exp(u(s^2-1))}. \tag{9.4}$$

As for $B_n(1,t) = \sum_{w\in B_n} t^{\mathrm{fdes}\, w}$ we specialize (9.1) and find:

$$\sum_{n\ge 0} B_n(1,t)\frac{u^n}{(1-t^2)^n} = \sum_{r\ge 0} t^r \frac{1-t}{1-u(r+1)}. \tag{9.5}$$

Hence, $B_n(1,t)/(1-t^2)^n = (1-t)\sum_{r\ge 0} t^r(r+1)^n$ and

$$\sum_{n\ge 0} B_n(1,t)\frac{u^n}{n!} = \frac{1-t}{-t+\exp(u(t^2-1))}. \tag{9.6}$$

It follows from (9.4) and (9.6) that $B_n(s,1) = B_n(1,s)$, so that "fexc" and "fdes" are equally distributed on the group B_n. For $k, n \ge 0$ let $B_{n,k}$ be the number of signed permutations w from B_n such that $\mathrm{fdes}\, w = k$. In our paper [FoHa06] we have derived the recurrence formula for the $B_{n,k}$'s in a more general context. The recurrence reads as follows

$$\begin{gathered} B_{0,0} = 1, \quad B_{0,k} = 0 \text{ for all } k \ne 0; \\ B_{1,0} = 1, \quad B_{1,1} = 1, \quad B_{1,k} = 0 \text{ for all } k \ne 0, 1; \\ B_{n,k} = (k+1)B_{n-1,k} + B_{n-1,k-1} + (2n-k)B_{n-1,k-2}; \end{gathered}$$

for $n \ge 2$ and $0 \le k \le 2n-1$. Let $B'_{n,k} := \#\{w \in B_n : \mathrm{fexc}\, w = k\}$. The coefficients $B'_{n,k}$'s satisfy the same recurrence as the $B_{n,k}$'s. The induction is easy, so that we can fabricate a bijection ψ of B_n onto itself such that $\mathrm{fexc}\, w = \mathrm{fdes}\, \psi(w)$.

10. Concluding remarks

The statistical study of the group B_n and some other Weyl groups has been initiated by Reiner ([Re93a], [Re93b], [Re93c], [Re95a], [Re95b]) and continued by the Roman school ([Br94], [Bi03], [BiCa04]). It has been rejuvenated by Adin, Roichman [AR01] with their definition of the *flag major index* for signed permutations and the first proof of the fact that length function and flag-major index were equidistributed over B_n ([ABR01], [ABR05], [ABR06]). In our series "Signed words and permutations; I–V" we have tried to work out analytical expressions for the *multivariable* distributions on the group B_n that were natural extensions of the expressions already derived for the symmetric group $\mathfrak{S}_n$. Other works along those lines are due to Gessel and his school ([ChGe07], [Ch03]). Further algebraic extensions have recently been done by the Minnesota school [BRS07].

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