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STATISTICAL DISTRIBUTIONS ON WORDS AND q-CALCULUS ON PERMUTATIONS

with a complement by
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FOREWORD

In those ten papers ([1]) bound in a single volume (five `[1--5]` in the series "Signed words and permutations," three `[6--8]` in the series "Fix-Mahonian Calculus" and two isolated papers, `[9]` "Decreases and descents on words," `[10]` "Permutations with extremal number of fixed points") our main goal has been to calculate *multivariable* statistical distributions on the hyperoctahedral group B_n and the symmetric group $\mathfrak{S}_n$. As indicated by the title of this memoir, those calculations first involve the studies of *statistical distributions* on words, then the *standardizations* of those words by permutations, signed or plain, suitably coded, finally *q-calculus methods* to derive generating functions for those permutations by desired multivariable statistics.

1. Word statistical study. This first step is so fundamental that our results could be regarded as contributions to Combinatorial Theory of Words with applications to Statistical Theory of Weyl groups. The words in question are linear sequences $w = x_1x_2\cdots x_n$, whose letters $x_1, x_2, \ldots,$ x_n belong to the finite alphabet $[0,r] = \{0,1,\ldots,r\}$. This set of words is denoted by $[0,r]^*$, often referred to as the free monoid generated by $[0,r]$.

By way of illustration, we should like to mention two derivations developed in `[2]` and `[8]`. Consider the sequence $H_r(u)$ $(r = 0,1,2\ldots)$ of rational functions defined by the recurrence

$$H_0(u) = \frac{1}{1-uX}; \quad H_1(u) = \frac{1-uqZ(1-Y)}{1-u(X+qZ)}; \quad \text{and for } r \ge 1$$

$$H_{2r}(u) = \frac{1-u(q(1-Y)Z+q^2+q^3Z+\cdots+q^{2r-1}Z+q^{2r})}{1-u(X+qZ+q^2+q^3Z+\cdots+q^{2r-1}Z+q^{2r})}H_{2r-2}(uq^2);$$

$$H_{2r+1}(u)=\frac{1-u(q(1-Y)Z+q^2+q^3Z+\cdots+q^{2r}+q^{2r+1}Z)}{1-u(X+qZ+q^2+q^3Z+\cdots+q^{2r}+q^{2r+1}Z)}H_{2r-1}(uq^2).$$

Each rational function $H_r(u)$ $(r \ge 0)$ can be interpreted as a generating function for $[0,r]^*$ by a certain vector $(\lambda, \mathrm{tot}, \mathrm{evenlower}, \mathrm{oddlower}, \mathrm{odd})$, say,

$$H_r(u) = \sum_{c\in[0,r]^*} u^{|c|} q^{\mathrm{tot}\, c} X^{\mathrm{evenlower}\, c} Y^{\mathrm{oddlower}\, c} Z^{\mathrm{odd}\, c}, \tag{1.1}$$

([1]) The references to those papers are written in typewriter font: `[1-10]`. We use the usual Roman font for the other papers: [KiZe01], etc.

but what are the five components of that vector? First, λ is just the *length* of each word. For $c = c_1c_2\cdots c_n$ we have $\lambda\, c = n$. Then, $\operatorname{tot} c = c_1+c_2+\cdots+c_n$. A letter c_i $(1 \le i \le n)$ is said to be an *even lower record* (resp. *odd lower record*) of c, if c_i is even (resp. odd) and if $c_j \ge c_i$ (resp. $c_j > c_i$) for all j such that $1 \le j \le i-1$. For instance, the even (resp. odd) lower records of the word $c = \mathit{5}\,5\,\mathbf{4}\,\mathbf{4}\,\mathit{1}\,2\,1\,\mathbf{0}\,4\,\mathbf{0}\,3$ are reproduced in boldface (resp. in italic). Let $\operatorname{evenlower} c$ (resp. $\operatorname{oddlower} c$) be the number of even (resp. odd) lower records of c and $\operatorname{odd} c$ be the number of *odd* letters in c. With the above example we have $\lambda c = 11$, $\operatorname{tot} c = 29$, $\operatorname{evenlower} c = 4$, $\operatorname{oddlower} c = 2$, $\operatorname{odd} c = 5$. See [2], theorem 2.2 for the proof of (1.1).

Now, let

$$\begin{aligned}(a;q)_n &:= \begin{cases} 1, & \text{if } n = 0;\\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & \text{if } n \ge 1;\end{cases}\\ (a;q)_\infty &:= \prod_{n\ge 0}(1-aq^n);\end{aligned}$$

be the traditional q-ascending factorials. The next step is to form the series $\sum_{r\ge 0} t^r H_r(u)$ and show that it can be expanded as a factorial series

$$\sum_{r\ge 0} t^r H_r(u) = \sum_{n\ge 0}(1+t)B_n(t,q,X,Y,Z)\frac{u^n}{(t^2;q^2)_{n+1}},$$

where each coefficient $B_n(t,q,X,Y,Z)$ is a polynomial with nonnegative integral coefficients [2, § 3]. The final step is to prove that $B_n(t,q,X,Y,Z)$ is the generating polynomial for the hyperoctahedral group B_n by a well-defined statistic $(\operatorname{fdes}, \operatorname{fmaj}, \operatorname{lowerp}, \operatorname{lowern}, \operatorname{neg})$. The proof is based on a *standardization* process, the so-called *MacMahon Verfahren*, that maps each word $c \in [0,r]^*$ onto a pair (w,b), where w is an element of the group B_n and b an increasing word (see [2], theorem 4.1).

In the second example the rational fraction

$$C(r;u,s,q,Y) := \frac{(1-sq)\,(u;q)_r\,(usq;q)_r}{((u;q)_r - sq(usq;q)_r)(uY;q)_{r+1}} \quad (r \ge 0) \tag{1.2}$$

can be shown [8] to be the generating function

$$C(r;u,s,q,Y) = \sum_{c\in[0,r]^*} u^{\lambda c} s^{\operatorname{dec} c} q^{\operatorname{tot} c} Y^{\operatorname{inrec} c}, \tag{1.3}$$

where "dec" and "inrec" are two word statistics defined as follows. Let $c = x_1x_2\cdots x_n$ be an *arbitrary* word. We say that the letter x_i is a *decrease value* (resp. *increase value*) of c if $1 \le i \le n-1$ and $x_i = x_{i+1} = \cdots = x_j > x_{j+1}$ (resp. $x_i = x_{i+1} = \cdots = x_j < x_{j+1}$) for some j such that $i \le j \le n-1$. By convention, $x_{n+1} = +\infty$. The *number of decrease values* of c is denoted by $\operatorname{dec} c$. Furthermore, x_i is a *record value* of c if $x_j \le x_i$ for all $j \le i-1$. Let $\operatorname{inrec} c$ be the number of letters of c which are both record and increase values. For instance, the decrease values (resp. letters which are both record and increase values) of the word $c = \mathbf{3}\,2\,\underline{5}\,\mathbf{6}\,\mathbf{4}\,2\,3\,\mathbf{6}\,\mathbf{6}\,3\,\underline{6}\,\underline{6}$ are in boldface (resp. underlined), so that $\operatorname{dec} c = 5$ and $\operatorname{inrec} c = 3$. Identity (1.3) is proved in [8], Corollary 2.2 and in [9] by a different method. In [8] we make use of the word-analog of the Kim-Zeng transformation [KiZe01]; in [9] identity (1.3) is a consequence of a more general result involving other geometric properties on words.

As above, the second step is to show that we have the expansion

$$\sum_{r\ge 0} t^r C(r;u,s,q,Y) = \sum_{n\ge 0} A_n(s,t,q,Y)\frac{u^n}{(t;q)_{n+1}},$$

where $A_n(s,t,q,Y)$ is the generating polynomial for the symmetric group $\mathfrak{S}_n$ by a well-defined statistic $(\mathrm{exc}, \mathrm{des}, \mathrm{maj}, \mathrm{fix})$. This is achieved by means of another standardization, introduced by Gessel-Reutenauer [GeRe93], that maps each word c onto a pair (σ, b), where this time σ is a (plain) permutation.

2. Length function and major index. As is well-known, the length function of the symmetric group $\mathfrak{S}_n$, when regarded as a Weyl group, is the usual *inversion number* "inv," defined for each permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ by

$$\operatorname{inv}\sigma := \sum_{1\le i\le n-1}\ \sum_{i<j} \chi(\sigma(i) > \sigma(j)),$$

making use of the χ-notation that maps each statement A to the value $\chi(A) = 1$ or 0 depending on whether A is true or not. The inversion number is a true q-analog maker, in the sense that each generating function for $\mathfrak{S}_n$ by some statistic "stat" is liable to have a true q-extension if we know how to calculate the generating function by the pair $(\mathrm{stat}, \mathrm{inv})$. The *major index* "maj", defined by

$$\operatorname{maj}\sigma := \sum_{1\le i\le n-1} i\,\chi(\sigma(i) > \sigma(i+1)),$$

plays an analogous role. Although the two generating polynomials $\sum_\sigma q^{\operatorname{inv}\sigma}$ and $\sum_\sigma q^{\operatorname{maj}\sigma}$ $(\sigma \in \mathfrak{S}_n)$ are equal (as a matter of fact to $(q;q)_n/(1-q)^n$), the q-extensions using either "inv," or "maj" can be very much different.

In the early nineties Reiner [Re93a, b, c, d, Re95] has calculated several q-generating functions for the hyperoctahedral group B_n using the length function. The results he obtained were natural extensions of what was known for $\mathfrak{S}_n$ when "inv" was used instead of "maj." There remained to do a parallel work for B_n with the q-formulas derived for $\mathfrak{S}_n$ by means of the *major index*. What was needed was an appropriate extension of "maj" for B_n. It was found by Adin and Roichman [AR01] with their *flag-major index*, whose definition will be given shortly, together with the *flag-descent number* and *flag-inversion number*.

By *signed word* we mean a word $w = x_1x_2 \dots x_m$, whose letters are positive or negative integers. If $\mathbf{m} = (m_1, m_2, \dots, m_r)$ is a sequence of nonnegative integers such that $m_1 + m_2 + \cdots + m_r = m$, let $B_{\mathbf{m}}$ be the set of all rearrangements $w = x_1x_2 \dots x_m$ of the sequence $1^{m_1}2^{m_2} \dots r^{m_r}$, with the convention that some letters i may be replaced by their opposite values $-i$. For typographical reasons we shall use the notation $\bar{i} := -i$. When $m_1 = m_2 = \cdots = m_r = 1$, $m = r$, the class $B_{\mathbf{m}}$ is simply the *hyperoctahedral group* B_m of the signed permutations of order m.

The *flag-major index*, $\operatorname{fmaj} w$, and the *flag-descent number*, $\operatorname{fdes} w$, of a signed word $w = x_1x_2 \cdots x_m$ are defined by

$$\begin{aligned} \operatorname{fmaj} w &:= 2 \operatorname{maj} w + \operatorname{neg} w; \\ \operatorname{fdes} w &:= 2 \operatorname{des} w + \chi(x_1 < 0); \end{aligned}$$

where "maj" and "des" are the usual *major index* $\operatorname{maj} w := \sum_i i\chi(x_i > x_{i+1})$ and *number of descents* $\operatorname{des} w := \sum_i \chi(x_i > x_{i+1})$, and where $\operatorname{neg} w := \sum_j \chi(x_j < 0)$ $(1 \le j \le m)$.

To define the *flag-inversion number*, $\operatorname{finv} w$, of a signed word $w = x_1x_2 \cdots x_m$ we use the traditional *inversion number*

$$\operatorname{inv} w := \sum_{1 \le i \le m-1} \sum_{i<j} \chi(x_i > x_j),$$

together with

$$\overline{\operatorname{inv}}\, w := \sum_{1 \le i \le m-1} \sum_{i<j} \chi(\overline{x}_i > x_j)$$

and set

$$\operatorname{finv} w := \operatorname{inv} w + \overline{\operatorname{inv}}\, w + \operatorname{neg} w.$$

It is important to note that the restriction $\operatorname{finv}|_{B_n}$ of the flag-inversion number to each hyperoctahedral group B_n is the *length function* ℓ.

Adin, Brenti and Roichman [ABR01] proved that "fmaj" and "ℓ" were equidistributed on each hyperoctahedral group B_n:

$$\sum_{w \in B_n} q^{\operatorname{fmaj} w} = \sum_{w \in B_n} q^{\ell\, w} = \frac{(q^2; q^2)_n}{(1-q)^n}.$$

It was then very tantalizing to construct a bijection Ψ of B_n onto itself such that $\operatorname{fmaj} w = \ell\,\Psi(w)$. However, let

$$ {}^{\rm fmaj}B_n(q,Z) := \sum_{w\in B_n} q^{\operatorname{fmaj} w} Z^{\operatorname{neg} w}, \qquad {}^{\rm finv}B_n(q,Z) := \sum_{w\in B_n} q^{\operatorname{finv} w} Z^{\operatorname{neg} w}. $$

Then, we can calculate those polynomials in the form:

$$ \begin{aligned} {}^{\rm finv}B_n(q,Z) &= (-Zq;q)_n \frac{(q;q)_n}{(1-q^n)};\\ {}^{\rm fmaj}B_n(q,Z) &= (q^2;q^2)_n \Bigl(\frac{1+qZ}{1-q^2}\Bigr)^n. \end{aligned} $$

We see that ${}^{\rm finv}B_n(q,Z) \neq {}^{\rm fmaj}B_n(q,Z)$ as soon as $n\geq 2$. For instance, the coefficient of q in the first polynomial is $(n-1)$ and 0 in the second. However, ${}^{\rm finv}B_n(q,1) = {}^{\rm fmaj}B_n(q,1) = (q^2;q^2)_n/(1-q)^n$, as done above. This shows that the transformation Ψ to be constructed cannot preserve “neg” and will not be a banal extension of the second fundamental transformation [Lo83] (chap. 10) valid for unsigned words.

Still, as done in [1], such a transformation Ψ could be constructed, having the following properties:

(a) $\operatorname{fmaj} w = \operatorname{finv} \Psi(w)$ for every signed word w;

(b) the restriction of Ψ to each rearrangement class $B_{\mathbf m}$ of signed words is a bijection of $B_{\mathbf m}$ onto itself, so that “fmaj” and “finv” are equidistributed over each class $B_{\mathbf m}$.

It is also shown that Ψ preserves the *lower records* of each signed word (also called “strict right-to-left minima”), a property that is fully exploited in the subsequent paper [2]. It also preserves the so-called *inverse ligne of route.*

Let

$$ Z_{ij} := \begin{cases} Z, & \text{if } i \text{ and } j \text{ are both odd;}\\ 1, & \text{if } i \text{ and } j \text{ are both even;}\\ 0, & \text{if } i \text{ and } j \text{ have different parity.} \end{cases} $$

This allowed us to calculate the expansion of the infinite product (see [3], formula (4.3))

$$ \prod_{i\geq 0, j\geq 0} \frac{1}{1-uZ_{ij}q_1^i q_2^j} = \sum_{n\geq 0} \frac{u^n}{(q_1^2;q_1^2)_n\,(q_2^2;q_2^2)_n)} B_n(q_1,q_2,Z), $$

where the coefficient $B_n(q_1,q_2,Z)$ is the generating polynomial

$$ B_n(q_1,q_2,Z) = \sum_{w\in B_n} q_1^{\operatorname{fmaj} w} q_2^{\operatorname{fmaj} w^{-1}} Z^{\operatorname{neg} w}, $$

together with the expansion of the graded form of that infinite product (see [3], Theorem 1.1), namely,

$$\sum_{r\ge 0, r\ge 0} t_1^r t_2^s \prod_{0\le i\le r,\, 0\le j\le s} \frac{1}{1-uZ_{ij}q_1^i q_2^j}.$$

3. Fixed points. It is readily seen that the polynomials $B_n(Y_0, Y_1, Z)$ $(n \ge 0)$ defined by

$$\sum_{n\ge 0} \frac{u^n}{n!} B_n(Y_0, Y_1, Z) = \bigl(1-u(1+Z)\bigr)^{-1} \times \frac{\exp(u(Y_0+Y_1Z))}{\exp(u(1+Z))} \tag{3.1}$$

are generating functions for the hyperoctahedral groups B_n by a certain three-variable statistic. When $Z = 0$, the variable Y_1 vanishes and we recover the generating function for the symmetric groups by the number of fixed points "fix":

$$\sum_{n\ge 0} \frac{u^n}{n!} \sum_{\sigma\in\mathfrak{S}_n} Y_0^{\operatorname{fix}\sigma} = (1-u)^{-1} \frac{\exp(uY_0)}{\exp(u)}. \tag{3.2}$$

Accordingly, the variable Y_1 must take another kind of fixed point into account. If $w = x_1x_2\cdots x_n$ is a signed permutation, let $\operatorname{fix}^+ w$ (resp. $\operatorname{fix}^- w$) be the number of i such that $1 \le i \le n$ and $x_i = i$ (resp. $x_i = \bar{i} = -i$). It is rather easy to prove

$$B_n(Y_0, Y_1, Z) = \sum_{w\in B_n} Y_0^{\operatorname{fix}^+ w}\, Y_1^{\operatorname{fix}^- w}\, Z^{\operatorname{neg} w}. \tag{3.3}$$

As the q-analog of (3.2) has already been derived by Gessel and Reutenauer [GeRe93] using the major index "maj," it matters to q-analoguize (3.1) by means of the flag-major index "fmaj" and also the length function "ℓ." The question is then to calculate the factorial generating functions for the polynomials

$$^{\operatorname{fmaj}}B_n(q, Y_0, Y_1, Z) = \sum_{w\in B_n} q^{\operatorname{fmaj} w} Y_0^{\operatorname{fix}^+ w}\, Y_1^{\operatorname{fix}^- w}\, Z^{\operatorname{neg} w}.$$

In [4] we obtain

$$\begin{aligned} \sum_{n\ge 0} \frac{u^n}{(q^2;q^2)_n} {}^{\operatorname{fmaj}}B_n(q, Y_0, Y_1, Z) \qquad\qquad & \\ = \Bigl(1-u\frac{1+qZ}{1-q^2}\Bigr)^{-1} \times \frac{(u;q^2)_\infty}{(uY_0;q^2)_\infty} \frac{(-uqY_1Z;q^2)_\infty}{(-uqZ;q^2)_\infty}, & \end{aligned} \tag{3.4}$$

together with its graded from (see [4] (1.9)).

The q-analog of (3.1) using "ℓ" as a q-maker requires another combinatorial interpretation for the polynomial $B_n(Y_0, Y_1, Z)$. Let $w = x_1 x_2 \cdots x_n$ be a word, all letters of which are integers without any repetitions. Say that w is a *desarrangement* if $x_1 > x_2 > \cdots > x_{2k}$ and $x_{2k} < x_{2k+1}$ for some $k \geq 1$. By convention, $x_{n+1} = \infty$. The notion was introduced by Désarménien [De84]. Let $w = x_1 x_2 \cdots x_n$ be a signed permutation. Unless w is increasing, there is always a nonempty right factor of w which is a desarrangement. It then makes sense to define w^d as the *longest* such a right factor. Hence, w admits a unique factorization $w = w^- w^+ w^d$, called its *pixed factorization*, where w^- and w^+ are both *increasing*, the letters of w^- being *negative*, those of w^+ *positive* and where w^d is the longest right factor of w which is a desarrangement. For example, the pixed factorization of the following signed permutation is materialized by vertical bars: $w = \overline{5}\,\overline{3} \mid 1 \mid 4\,2$. Let $\operatorname{pix}^- w := \lambda w^-$ and $\operatorname{pix}^+ w := \lambda w^+$.

If ${}^{\ell}B_n(q, Y_0, Y_1, Z) := \sum_{w \in B_n} q^{\ell\, w} Y_0^{\operatorname{pix}^+ w} Y_1^{\operatorname{pix}^- w} Z^{\operatorname{neg} w}$, the second q-analog of (3.1) by the q-maker "ℓ" reads ([4], Theorem 1.2)

$$(3.5) \quad \sum_{n \geq 0} \frac{u^n}{(-Zq;q)_n\,(q;q)_n}\, {}^{\ell}B_n(q, Y_0, Y_1, Z) = \Bigl(1 - \frac{u}{1-q}\Bigr)^{-1} \times (u;q)_\infty \Bigl(\sum_{n \geq 0} \frac{(-qY_0^{-1}Y_1 Z;q)_n\,(uY_0)^n}{(-Zq;q)_n\,(q;q)_n}\Bigr).$$

One specialization of (3.4) is worth mentioning. Let d_n be the number of permutations $\sigma \in \mathfrak{S}_n$ such that $\operatorname{fix}\sigma = 0$, also called *derangements*. Then

$$d_n = \sum_{2 \leq 2k \leq n-1} (2k)(2k+2)_{n-2k-1} + \chi(n \text{ even}).$$

Thus, d_n is an explicit sum of *positive* integers. To the best of the authors' knowledge such a formula has not appeared elsewhere.

4. The statistics "maf" and "maz". If $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ is a permutation, let $(j_1, j_2, \ldots, j_m)$ be the increasing sequence of the integers k such that $1 \leq k \leq n$ and $\sigma(k) \neq k$ and "red" be the increasing bijection of $\{j_1, j_2, \ldots, j_m\}$ onto $\{1, 2, \ldots, m\}$. The word $w = x_1 x_2 \cdots x_n$ derived from $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ by replacing each fixed point by 0 and each other letter $\sigma(j_k)$ by $\operatorname{red}\sigma(j_k)$ will be denoted by $\operatorname{ZDer}(\sigma)$. Also let

$$\begin{aligned}
\operatorname{DES}\sigma &:= \{i : 1 \leq i \leq n-1, \sigma(i) > \sigma(i+1)\};\\
\operatorname{DEZ}\sigma &:= \{i : 1 \leq i \leq n-1, x_i > x_{i+1}\};\\
\operatorname{Der}\sigma &:= \operatorname{red}\sigma(j_1)\ \operatorname{red}\sigma(j_2)\ \cdots\ \operatorname{red}\sigma(j_m);
\end{aligned}$$

so that $\mathrm{Der}\,\sigma$ is the word derived from $\mathrm{ZDer}(\sigma)$ by deleting all the zeros. Besides "maf" that was introduced in [CHZ97] we found it convenient to have another statistic "maz". Their definitions are the following:

$$\begin{aligned}\mathrm{maf}\,\sigma &:= \sum_{i,\sigma(i)=i} i - \sum_{1\le i\le \mathrm{fix}\,\sigma} i + \mathrm{maj}\,\mathrm{Der}\,\sigma;\\ \mathrm{maz}\,\sigma &:= \mathrm{maj}\,\mathrm{ZDer}\,\sigma; \quad \mathrm{dez}\,\sigma := \mathrm{des}\,\mathrm{ZDer}\,\sigma.\end{aligned}$$

For instance, for $\sigma = 8\,\mathbf{2}\,1\,3\,\mathbf{5}\,\mathbf{6}\,4\,9\,7$, we have $\mathrm{ZDer}(\sigma) = 5\,0\,1\,2\,0\,0\,3\,6\,4$, $\mathrm{Der}\,\sigma = 5\,1\,2\,3\,6\,4$, $\mathrm{fix}\,\sigma = 3$ and $\mathrm{maf}\,\sigma = (2+5+6) - (1+2+3) + \mathrm{maj}(512364) = 7+6 = 13$. Also $\mathrm{maz}\,\sigma = 1+4+8 = 13$.

It is quite unexpected that "maf," "maz" and "maj" are equidistributed over $\mathfrak{S}_n$. Much more is proved in [6], where we construct two bijections Φ and F_3 of $\mathfrak{S}_n$ onto itself having the following properties:

$$\begin{aligned}(\mathrm{fix}, \mathrm{DEZ}, \mathrm{exc})\,\sigma &= (\mathrm{fix}, \mathrm{DES}, \mathrm{exc})\,\Phi(\sigma);\\ (\mathrm{fix}, \mathrm{maz}, \mathrm{exc})\,\sigma &= (\mathrm{fix}, \mathrm{maf}, \mathrm{exc})\,\mathrm{F}_3(\sigma);\end{aligned}$$

for every σ from $\mathfrak{S}_n$.

In [7] composing several other bijections with the new ones Φ and F_3 we show that several multivariable statistics are equidistributed either with the triplet $(\mathrm{fix}, \mathrm{des}, \mathrm{maj})$, or the pair $(\mathrm{fix}, \mathrm{maj})$.

In [10] the equidistribution of $(\mathrm{fix}, \mathrm{DEZ}, \mathrm{exc})$ with $(\mathrm{fix}, \mathrm{DES}, \mathrm{exc})$ over $\mathfrak{S}_n$ enables the two authors to give an immediate proof of a conjecture by Stanley [St06] on alternating permutations. In fact, they prove a stronger result dealing with permutations with a prescribed descent set.

5. A quadruple distribution. The distribution of the vector $(\mathrm{exc}, \mathrm{des}, \mathrm{maj}, \mathrm{fix})$ over $\mathfrak{S}_n$ had not been calculated before. Several marginal distributions were known. Let

$$A_n(s,t,q,Y) := \sum_{\sigma\in\mathfrak{S}_n} s^{\mathrm{exc}\,\sigma} t^{\mathrm{des}\,\sigma} q^{\mathrm{maj}\,\sigma} Y^{\mathrm{fix}\,\sigma} \quad (n\ge 0). \tag{5.1}$$

The distribution of the three-variable statistic $(\mathrm{des}, \mathrm{maj}, \mathrm{fix})$ had been calculated by Gessel and Reutenauer [GeRe93] in the form

$$\sum_{n\ge 0} A_n(1,t,q,Y)\frac{u^n}{(t;q)_{n+1}} = \sum_{r\ge 0} t^r \Bigl(1 - u\sum_{i=0}^{r} q^i\Bigr)^{-1} \frac{(u;q)_{r+1}}{(uY;q)_{r+1}}. \tag{5.2}$$

Shareshian and Wachs [ShWa07] obtained the distribution of the three-vector $(\mathrm{exc}, \mathrm{maj}, \mathrm{fix})$ in the form

$$\sum_{n\ge 0} A_n(s,1,q,Y)\frac{u^n}{(q;q)_n} = \frac{(1-sq)e_q(Yu)}{e_q(squ) - sq e_q(u)}, \tag{5.3}$$

where $e_q(u)$ is the q-exponential $\sum_{n\ge 0} u^n/(q;q)_n$. We were then convinced that a graded form of (5.3) was to be discovered, which would also be an extension of (5.2). This was achieved by calculating the distribution of the vector $(\mathrm{exc}, \mathrm{des}, \mathrm{maj}, \mathrm{fix})$ over $\mathfrak{S}_n$, which reads [8]:

$$(5.4)\qquad \sum_{n\ge 0} A_n(s,t,q,Y)\frac{u^n}{(t;q)_{n+1}} = \sum_{r\ge 0} t^r \frac{(1-sq)\,(u;q)_r\,(usq;q)_r}{((u;q)_r - sq(usq;q)_r)(uY;q)_{r+1}}.$$

As already explained in Section 1 the derivation was made in two steps, first showing that the fraction on the right-hand side was a generating function for words by an appropriate statistic, then using the Gessel-Reutenauer standardization.

An interesting specialization yields the distribution of the pair of the two *Eulerian* statistics "exc" and "des" over $\mathfrak{S}_n$ in the form:

$$\sum_{n\ge 0} A_n(s,t,1,1)\frac{u^n}{(1-t)^{n+1}} = \sum_{r\ge 0} t^r \frac{1-s}{(1-u)^{r+1}(1-us)^{-r} - s(1-u)}.$$

6. A sextuple distribution. Since the distribution of the vector $(\mathrm{fix}, \mathrm{exc}, \mathrm{des}, \mathrm{maj})$ over $\mathfrak{S}_n$ could be derived, it was natural to look for an extension of this result for the hyperoctahedral group B_n. The polynomial

$$B_n(s,t,q,Y_0,Y_1,Z) := \sum_{w\in B_n} s^{\mathrm{fexc}\,w} t^{\mathrm{fdes}\,w} q^{\mathrm{fmaj}\,w}\, Y_0^{\mathrm{fix}^+\,w}\, Y_1^{\mathrm{fix}^-\,w}\, Z^{\mathrm{neg}\,w}$$

had to be considered, introducing a new variable "fexc" defined for each signed permutation $w = x_1x_2\cdots x_n$ by

$$\mathrm{fexc}\,w := 2\sum_{1\le i\le n-1} \chi(x_i > i) + \mathrm{neg}\,w,$$

which is known to be equidistributed with "fdes."

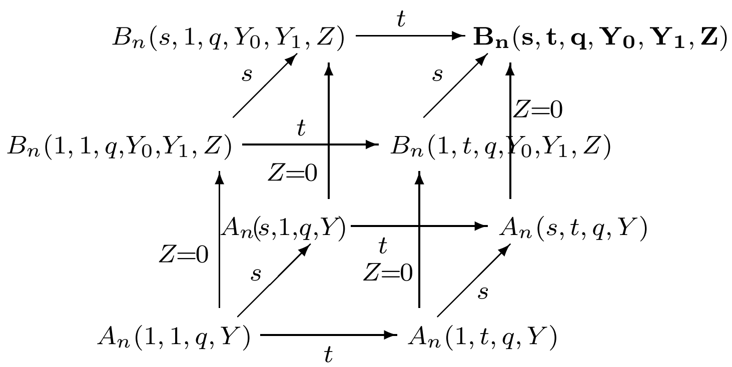

Fig. 1

The diagram of Fig. 1 shows that the polynomials written in roman can be obtained from the polynomial $\mathbf{B_n(s,t,q,Y_0,Y_1,Z)}$ written in boldface by giving certain variables specific values. For instance, the polynomial $A_n(s,t,q,Y)$ (defined in (5.1)) is simply $B_n(s^{1/2},t^{1/2},q^{1/2},Y,0,0)$. More importantly, all the generating functions for those polynomials are true specializations of the generating function derived for the polynomials $\mathbf{B_n(s,t,q,Y_0,Y_1,Z)}$.

The factorial generating function in question is the following

$$\sum_{n\ge 0}(1+t)B_n(s,t,q,Y_0,Y_1,Z)\frac{u^n}{(t^2;q^2)_{n+1}} = \sum_{r\ge 0} t^r\,\frac{(u;q^2)_{\lfloor r/2\rfloor+1}}{(uY_0;q^2)_{\lfloor r/2\rfloor+1}}\,\frac{(-usqY_1Z;q^2)_{\lfloor (r+1)/2\rfloor}}{(-usqZ;q^2)_{\lfloor (r+1)/2\rfloor}}\,F_r(u;s,q,Z),$$

where

$$F_r(u;s,q,Z) = \frac{(us^2q^2;q^2)_{\lfloor r/2\rfloor}\,(1-s^2q^2)\,(u;q^2)_{\lfloor (r+1)/2\rfloor}\,(u;q^2)_{\lfloor r/2\rfloor}}{(u;q^2)_{\lfloor r/2\rfloor+1}\Big((u;q^2)_{\lfloor (r+1)/2\rfloor}\big((u;q^2)_{\lfloor r/2\rfloor}-s^2q^2\,(us^2q^2;q^2)_{\lfloor r/2\rfloor}\big) + sqZ\,(u;q^2)_{\lfloor r/2\rfloor}\big((u;q^2)_{\lfloor (r+1)/2\rfloor}-(us^2q^2;q^2)_{\lfloor (r+1)/2\rfloor}\big)\Big)}.$$

This derivation is the main result of our paper [5].

7. Our Ur-result. Our last paper [9] may be regarded as our fundamental result, as we explicitly calculate a generating function for words by a statistic that involves *six* sets of variables (X_i), (Y_i), (Z_i), (T_i), (Y_i'), (T_i') $(i\ge 0)$. Without getting into details, say that the weight $\psi(w)$ of each word w includes variables referring to *decreases*, *descents*, *increases*, *rises* and *records*. Let C be the $(r+1)\times(r+1)$ matrix

$$C=\begin{pmatrix}
0 & \frac{X_1}{1-Z_1} & \frac{X_2}{1-Z_2} & \cdots & \frac{X_{r-1}}{1-Z_{r-1}} & \frac{X_r}{1-Z_r}\\
\frac{Y_0}{1-T_0} & 0 & \frac{X_2}{1-Z_2} & \cdots & \frac{X_{r-1}}{1-Z_{r-1}} & \frac{X_r}{1-Z_r}\\
\frac{Y_0}{1-T_0} & \frac{Y_1}{1-T_1} & 0 & \cdots & \frac{X_{r-1}}{1-Z_{r-1}} & \frac{X_r}{1-Z_r}\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
\frac{Y_0}{1-T_0} & \frac{Y_1}{1-T_1} & \frac{Y_2}{1-T_2} & \cdots & 0 & \frac{X_r}{1-Z_r}\\
\frac{Y_0}{1-T_0} & \frac{Y_1}{1-T_1} & \frac{Y_2}{1-T_2} & \cdots & \frac{Y_{r-1}}{1-T_{r-1}} & 0
\end{pmatrix}.$$

FOREWORD

We prove that the generating function for the set $[0,r]^*$ by the weight ψ is given by

$$\sum_{w\in[0,r]^*} \psi(w) = \frac{\displaystyle\prod_{0\le j\le r}\Bigl(1+\frac{Y'_j}{1-T'_j}\Bigr)}{\det(I-C)},$$

where I is the identity matrix of order $(r+1)$.

The basic techniques of proof are the traditional MacMahon Master Theorem, together with the first fundamental transformation for words. The specializations are numerous, in particular in q-series Calculus, for instance,

$$\sum_{w\in[0,r]^*} s^{\operatorname{dec} w} R^{\operatorname{inrec} w} u^{\lambda w} q^{\operatorname{tot} w} = \frac{\dfrac{(us;q)_{r+1}}{(uR;q)_{r+1}}}{1-\displaystyle\sum_{0\le l\le r}\frac{(us;q)_l}{(u;q)_l}usq^l}.$$

We can then use the q-telescoping argument provided by Krattenthaler

$$\frac{(us;q)_l}{(u;q)_l}usq^l = \frac{sq}{1-sq}\Bigl(\frac{(us;q)_{l+1}}{(u;q)_l} - \frac{(us;q)_l}{(u;q)_{l-1}}\Bigr) \quad (1\le l\le r)$$

to reprove identity (1.3).

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SIGNED WORDS AND PERMUTATIONS, I; A FUNDAMENTAL TRANSFORMATION

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This paper is dedicated to the memory of Percy Alexander MacMahon.

Abstract. The statistics major index and inversion number, usually defined on ordinary words, have their counterparts in signed words, namely the so-called flag-major index and flag-inversion number. We give the construction of a new transformation on those signed words that maps the former statistic onto the latter one. It is proved that the transformation also preserves two other set-statistics: the inverse ligne of route and the lower records.

1. Introduction

The *second fundamental transformation*, as it was called later on (see [16], chap. 10 or [15], ex. 5.1.1.19), was described in these proceedings [8]. Let $w = x_1x_2\ldots x_m$ be a (finite) word, whose letters $x_1, x_2, \ldots, x_m$ are integers. The integer-valued statistics *Inversion Number* "inv" and *Major Index* "maj" attached to the word w are defined by

$$\operatorname{inv} w := \sum_{1\le i\le m-1} \sum_{i<j} \chi(x_i > x_j\};\tag{1.1}$$

$$\operatorname{maj} w := \sum_{1\le i\le m-1} i\,\chi(x_i > x_{i+1});\tag{1.2}$$

making use of the χ-notation that maps each statement A to the value $\chi(A) = 1$ or 0 depending on whether A is true or not.

If $\mathbf{m} = (m_1, m_2, \ldots, m_r)$ is a sequence of r nonnegative integers, the rearrangement class of the nondecreasing word $1^{m_1}2^{m_2}\ldots r^{m_r}$, that is, the class of all the words than can be derived from $1^{m_1}2^{m_2}\ldots r^{m_r}$ by permutation of the letters, is denoted by $R_{\mathbf{m}}$. The second fundamental transformation, denoted by Φ, maps each word w on another word $\Phi(w)$ and has the following properties:

(a) $\operatorname{maj} w = \operatorname{inv} \Phi(w)$;
(b) $\Phi(w)$ is a rearrangement of w and the restriction of Φ to each rearrangement class $R_{\mathbf{m}}$ is a bijection of $R_{\mathbf{m}}$ onto itself.

Further properties were proved later on by Foata, Schützenberger [10] and Björner, Wachs [5], in particular, when the transformation is restricted to act on rearrangement classes $R_{\mathbf{m}}$ such that $m_1 = \cdots = m_r = 1$, that is, on symmetric groups $\mathfrak{S}_r$.

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The purpose of this paper is to construct an analogous transformation not simply on words, but on *signed words*, so that new equidistribution properties on classical statistics, such as the (Coxeter) *length function* (see [7, p. 9], [14, p. 12]), defined on the group B_n of the *signed permutations* can be derived. By *signed word* we understand a word $w = x_1x_2\ldots x_m$, whose letters are positive or negative integers. If $\mathbf{m} = (m_1, m_2, \ldots, m_r)$ is a sequence of nonnegative integers such that $m_1+m_2+\cdots+m_r = m$, let $B_{\mathbf{m}}$ be the set of all rearrangements $w = x_1x_2\ldots x_m$ of the sequence $1^{m_1}2^{m_2}\ldots r^{m_r}$, with the convention that some letters i may be replaced by their opposite values $-i$. For typographical reasons we shall use the notation $\overline{i} := -i$ in the sequel. The class $B_{\mathbf{m}}$ contains $2^m\binom{m}{m_1,m_2,\ldots,m_r}$ signed words. When $m_1 = m_2 = \cdots = m_r = 1$, $m = r$, the class $B_{\mathbf{m}}$ is simply the group B_m of the signed permutations of order m.

Next, the statistics "inv" and "maj" must be adapted to signed words and correspond to classical statistics when applied to signed permutations. Let

$$(\omega;q)_n := \begin{cases} 1, & \text{if } n = 0; \\ (1-\omega)(1-\omega q)\ \ldots\ (1-\omega q^{n-1}), & \text{if } n \geq 1; \end{cases}$$

denote the usual *q-ascending factorial in a ring element* ω and

$$\begin{bmatrix} m_1+\cdots+m_r \\ m_1,\ldots,m_r \end{bmatrix}_q := \frac{(q;q)_{m_1+\cdots+m_r}}{(q;q)_{m_1}\cdots(q;q)_{m_r}}$$

be the *q-multinomial coefficient*. Back to MacMahon [17, 18, 19] it was known that the above q-multinomial coefficient, which is the true q-analog of the cardinality of $R_{\mathbf{m}}$, was the generating function for the class $R_{\mathbf{m}}$ by either one of the statistics "inv" or "maj." Consequently, the generating function for $B_{\mathbf{m}}$ by the new statistics that are to be introduced on $B_{\mathbf{m}}$ must be a *plausible* q-analog of the cardinality of $B_{\mathbf{m}}$. The most natural q-analog we can think of is certainly $(-q;q)_m \begin{bmatrix} m_1+\cdots+m_r \\ m_1,\ldots,m_r \end{bmatrix}_q$, that tends to $2^m\binom{m_1+\cdots+m_r}{m_1,\ldots,m_r}$ when q tends to 1. As a substitute for "inv" we are led to introduce the following statistic "finv," called the *flag-inversion number*, which will be shown to meet our expectation, that is,

$$(-q;q)_m \begin{bmatrix} m_1+\cdots+m_r \\ m_1,\ldots,m_r \end{bmatrix}_q = \sum_{w\in B_{\mathbf{m}}} q^{\operatorname{finv} w}. \tag{1.3}$$

This identity is easily proved by induction on r. Let $w = x_1x_2\ldots x_m$ be a signed word from the class $B_{\mathbf{m}}$. To define $\operatorname{finv} w$ we use "inv" defined in (1.1), together with

$$\overline{\operatorname{inv}}\, w := \sum_{1\leq i\leq m-1}\ \sum_{i<j} \chi(\overline{x}_i > x_j) \tag{1.4}$$

and define

$$\operatorname{finv} w := \operatorname{inv} w + \overline{\operatorname{inv}}\, w + \sum_{1\leq j\leq m} \chi(x_j < 0). \tag{1.5}$$

The salient feature of this definition of "finv" is the fact that it does *not* involve the *values* of the letters, but only the *comparisons* between letters, so that it can be applied to each *arbitrary* signed word. Moreover, the definition of "finv" is similar to that of "fmaj" given below in (1.7). Finally, its restriction to the group B_m of the *signed permutations* is the traditional *length function*:

$$\operatorname{finv}|_{B_m} = \ell. \tag{1.6}$$

This is easily shown, for instance, by using the formula derived by Brenti [6] for the length function ℓ over B_n, that reads.

$$\ell\, w = \operatorname{inv} w + \sum_{1\le j\le m} |x_j|\, \chi(x_j<0).$$

Next, the statistic "maj" is to be replaced by "fmaj", the *flag-major index*, introduced by Adin and Roichman [1] for *signed permutations*. The latter authors (see also [2]) showed that "fmaj" was equidistributed with the length function ℓ over B_n. Their definition of "fmaj" can be used *verbatim* for signed *words*, as well as their definition of "fdes." For a signed word $w = x_1x_2\ldots x_m$ those definitions read:

$$\operatorname{fmaj} w := 2 \operatorname{maj} w + \sum_{1\le j\le m} \chi(x_j<0); \tag{1.7}$$

$$\operatorname{fdes} w := 2 \operatorname{des} w + \chi(x_1<0); \tag{1.8}$$

where "des" is the usual *number of descents* $\operatorname{des} w := \sum_i \chi(x_i > x_{i+1})$. We postpone the construction of our transformation Ψ on signed words to the next section. The main purpose of this paper is to prove the following theorem.

Theorem 1.1. *The transformation Ψ constructed in section 2 has the following properties:*

(a) $\operatorname{fmaj} w = \operatorname{finv} \Psi(w)$ *for every signed word w;*

(b) *the restriction of Ψ to each rearrangement class $B_{\mathbf{m}}$ of signed words is a bijection of $B_{\mathbf{m}}$ onto itself, so that "fmaj" and "finv" are equidistributed over each class $B_{\mathbf{m}}$.*

Definition 1.1. Let $w = x_1x_2\ldots x_m \in B_{\mathbf{m}}$ be a signed word. We say that a *nonnegative* integer i belongs to the *inverse ligne of route*, $\operatorname{Iligne} w$, of w, if one of the following two conditions holds:

(1) $i=0$, $m_1\ge 1$ and the rightmost letter x_k satisfying $|x_k|=1$ is equal to $\overline{1}$;

(2) $i\ge 1$, $m_i = m_{i+1} = 1$ and the rightmost letter that belongs to $\{i, \overline{i}, i+1, \overline{i+1}\}$ is equal to i or $\overline{i+1}$.

For example, with $w = \overline{4}\,4\,\overline{1}\,3\,2\,5\,\overline{5}\,6\,\overline{7}$ we have: $\operatorname{Iligne} w = \{0, 2, 6\}$.

Remark. The expression "line of route" was used by Foulkes [11,12]. We have added the letter "g" making up "ligne of route," thus bringing a slight touch of French. Notice that 0 may or may not belong to the inverse ligne of route. The *ligne of route* of a signed word $w = x_1x_2\ldots x_m$ is defined to be the *set*, denoted by $\operatorname{Ligne} w$, of all the i's such that either $1\le i\le m-1$ and $x_i > x_{i+1}$, or $i=0$ and $x_1<0$. In particular, $\operatorname{maj} w = \sum\limits_{0\le i\le m-1} i\,\chi(i\in \operatorname{Ligne} w)$. Finally, if w is a *signed permutation*, then $\operatorname{Iligne} w = \operatorname{Ligne} w^{-1}$. For *ordinary permutations*, some authors speak of *descent set* and *descent set of the inverse*, instead of ligne of route and inverse ligne of route, respectively.

Theorem 1.2. *The transformation Ψ constructed in section 2 preserves the inverse ligne of route:*

(c) $\operatorname{Iligne} \Psi(w) = \operatorname{Iligne} w$ *for every signed word w.*

Definition 1.2. Let $w = x_1x_2\dots x_m$ be a signed word of length m. A letter x_i is said to be a *lower record* of w, if either $i = m$, or $1 \le i \le m-1$ and $|x_i| < |x_j|$ for all j such that $i+1 \le j \le m$. When reading the lower records of w from left to right, we get a *signed* subword $x_{i_1}x_{i_2}\dots x_{i_k}$, called the *lower record subword*, denoted by $\operatorname{Lower} w$, which has the property that: $\min_i x_i = |x_{i_1}| < |x_{i_2}| < \cdots < |x_{i_k}| = |x_m|$. The notion of lower record is classical in the statistical literature. In combinatorics the expression "strict right-to-left minimum" is also used.

With our previous example $w = \overline{4}\,4\,\overline{1}\,3\,2\,5\,\overline{5}\,6\,\overline{7}$ we get $\operatorname{Lower} w = \overline{1}\,2\,\overline{5}\,6\,\overline{7}$. Our third goal is to prove the following result.

Theorem 1.3. *The transformation Ψ constructed in section 2 preserves all the lower records:*

(d) $\operatorname{Lower} \Psi(w) = \operatorname{Lower} w$ *for every signed word w.*

For each signed permutation $w = x_1x_2\dots x_m$ let

$$\operatorname{ifmaj} w := 2 \sum_{1\le j\le m} i\,\chi(j \in \operatorname{Iligne} w) + \sum_{1\le j\le m} \chi(x_j < 0); \tag{1.11}$$

$$\operatorname{ifdes} w := 2 \sum_{1\le j\le m} \chi(j \in \operatorname{Iligne} w) + \chi(x_i = -1 \text{ for some } i). \tag{1.12}$$

It is immediate to verify that

$$\operatorname{finv} w = \operatorname{finv} w^{-1}, \quad \operatorname{ifmaj} w = \operatorname{fmaj} w^{-1}, \quad \operatorname{ifdes} w = \operatorname{fdes} w^{-1},$$

where w^{-1} denotes the inverse of the signed permutation w (written as a linear word $w^{-1} = w^{-1}(1)\dots w^{-1}(m)$).

Let $\mathbf{i}\, w := w^{-1}$; then the chain

$$\begin{matrix} w & \xrightarrow{\ \mathbf{i}\ } & w_1 & \xrightarrow{\ \Psi\ } & w_2 & \xrightarrow{\ \mathbf{i}\ } & w_3 \\ \begin{pmatrix}\text{fdes}\\ \text{ifmaj}\end{pmatrix} & & \begin{pmatrix}\text{ifdes}\\ \text{fmaj}\end{pmatrix} & & \begin{pmatrix}\text{ifdes}\\ \text{finv}\end{pmatrix} & & \begin{pmatrix}\text{fdes}\\ \text{finv}\end{pmatrix} \end{matrix}$$

shows that the four generating polynomials $\sum t^{\operatorname{fdes} w} q^{\operatorname{ifmaj} w}$, $\sum t^{\operatorname{ifdes} w} q^{\operatorname{fmaj} w}$, $\sum t^{\operatorname{ifdes} w} q^{\operatorname{finv} w}$ and $\sum t^{\operatorname{fdes} w} q^{\operatorname{finv} w}$ $(w \in B_m)$ are identical. Their analytic expression will be derived in a forthcoming paper [9].

2. The construction of the transformation

For each signed word $w = x_1x_2\dots x_m$ the first or leftmost (resp. last or rightmost) letter x_1 (resp. x_m) is denoted by $F(w)$ (resp. $L(w)$). Next, define $\mathbf{s_1}\, w := \overline{x}_1 x_2\dots x_m$. The transformation $\mathbf{s_1}$ changes the sign of the first letter. Together with $\mathbf{s_1}$ the main ingredients of our transformation are the bijections γ_x and δ_x defined for each integer x, as follows.

If $L(w) \le x$ (resp. $L(w) > x$), then w admits the unique factorization

$$(v_1y_1, v_2y_2, \dots, v_py_p),$$

called its *x-right-to-left factorisation* having the following properties:

(i) each y_i $(1 \le i \le p)$ is a *letter* verifying $y_i \le x$ (resp. $y_i > x$);

(ii) each v_i $(1 \le i \le p)$ is a factor which is either empty or has all its letters greater than (resp. smaller than or equal to) x.

Then, γ_x is defined to be the bijection that maps $w = v_1y_1v_2y_2\ldots v_py_p$ onto the signed word

$$\gamma_x(w) := y_1v_1y_2v_2\ldots y_pv_p. \tag{2.1}$$

In a dual manner, if $F(w) \geq x$ (resp. $F(w) < x$) the signed word w admits the unique factorization

$$(z_1w_1, z_2w_2, \ldots, z_qw_q)$$

called its *x-left-to-right factorisation* having the following properties:
(i) each z_i $(1 \leq i \leq q)$ is a *letter* verifying $z_i \geq x$ (resp. $z_i < x$);
(ii) each w_i $(1 \leq i \leq q)$ is a factor which is either empty or has all its letters less than (resp. greater than or equal to) x.

Then, δ_x is defined to be the bijection that sends $w = z_1w_1z_2w_2\ldots z_qw_q$ onto the signed word

$$\delta_x(w) := w_1z_1w_2z_2\ldots w_qz_q. \tag{2.2}$$

Next, if $(v_1y_1, v_2y_2, \ldots, y_pv_p)$ is the x-right-to-left factorization of w, we define

$$\beta_x(w) := \begin{cases} \delta_{\overline{x}}\,\gamma_x(w), & \text{if either } \overline{x} \leq y_1 \leq x, \text{or } x < y_1 < \overline{x}; \\ \delta_{\overline{x}}\,\mathbf{s_1}\,\gamma_x(w), & \text{otherwise.} \end{cases} \tag{2.3}$$

The fundamental transformation Ψ on signed words that is the main object of this paper is defined as follows: if w is a one-letter signed word, let $\Psi(w) := w$; if it has more than one letter, write the word as wx, where x is the last letter. By induction determine $\Psi(w)$, then apply β_x to $\Psi(w)$ and define $\Psi(wx)$ to be the juxtaposition product:

$$\Psi(wx) := \beta_x(\Psi(w))\,x. \tag{2.4}$$

The proof of Theorem 1.1 is given in section 3. It is useful to notice the following relation

$$y_1 \leq x \Leftrightarrow L(w) \leq x \tag{2.5}$$

and the identity

$$\Psi(wx) = \Psi(w)x, \quad \text{whenever } x < -\max\{|x_i|\} \text{ or } x \geq \max\{|x_i|\}. \tag{2.6}$$

Example. Let $w = 3\,\overline{2}\,1\,\overline{3}\,\overline{4}\,3$. The factorizations used in the definitions of γ_x and $\delta_{\overline{x}}$ are indicated by vertical bars. First, $\Psi(3) = 3$. Then

$|\,3\,| \xrightarrow{\gamma_{\overline{2}}} 3 \xrightarrow{\mathbf{s_1}} |\,\overline{3}\,| \xrightarrow{\delta_2} \overline{3}$, so that $\Psi(3\,\overline{2}) = \overline{3}\,\overline{2}$;

$|\,\overline{3}\,|\,\overline{2}\,| \xrightarrow{\gamma_1} \overline{3}\,\overline{2} \xrightarrow{\mathbf{s_1}} |\,3\,\overline{2}\,| \xrightarrow{\delta_{\overline{1}}} \overline{2}\,3$, so that $\Psi(3\,\overline{2}\,1) = \overline{2}\,3\,1$;

$|\,\overline{2}\,|\,3\,|\,1\,| \xrightarrow{\gamma_{\overline{3}}} |\,\overline{2}\,3\,|\,1\,| \xrightarrow{\delta_3} 3\,\overline{2}\,1$, so that $\Psi(3\,\overline{2}\,1\,\overline{3}) = 3\,\overline{2}\,1\,\overline{3}$;

and $\Psi(3\,\overline{2}\,1\,\overline{3}\,\overline{4}) = 3\,\overline{2}\,1\,\overline{3}\,\overline{4}$, because of (2.6);

$|\,3\,|\,\overline{2}\,|\,1\,|\,\overline{3}\,|\,\overline{4}\,| \xrightarrow{\gamma_3} |\,3\,|\,\overline{2}\,|\,1\,|\,\overline{3}\,\overline{4}\,| \xrightarrow{\delta_{\overline{3}}} 3\,\overline{2}\,1\,\overline{4}\,\overline{3}$.

Thus, with $w = 3\,\overline{2}\,1\,\overline{3}\,\overline{4}\,3$ we get $\Psi(w) = 3\,\overline{2}\,1\,\overline{4}\,\overline{3}\,3$. We verify that $\operatorname{fmaj} w = \operatorname{finv} \Psi(w) = 19$, $\operatorname{Iligne} w = \operatorname{Iligne} \Psi(w) = \{1\}$, $\operatorname{Lower} w = \operatorname{Lower} \Psi(w) = 1\,3$.

3. Proof of Theorem 1.1

Before proving the theorem we state a few properties involving the above statistics and transformations. Let $|w|$ be the number of letters of the signed word w and $|w|_{>x}$ be the number of its letters greater than x with analogous expressions involving ths subscripts "$\geq x$", "$< x$" and "$\leq x$". We have:

$$\operatorname{fmaj} wx = \operatorname{fmaj} w + \chi(x<0) + 2\,|w|\,\chi(L(w) > x); \tag{3.1}$$
$$\operatorname{finv} wx = \operatorname{finv} w + |w|_{>x} + |w|_{<\overline{x}} + \chi(x<0); \tag{3.2}$$
$$\operatorname{finv} \gamma_x(w) = \operatorname{finv} w + |w|_{\leq x} - |w|\,\chi(L(w) \leq x); \tag{3.3}$$
$$\operatorname{finv} \delta_x(w) = \operatorname{finv} w + |w|_{\geq x} - |w|\,\chi(F(w) \geq x). \tag{3.4}$$

Next, let y_1 denote the first letter of the signed word w''. Then

$$\operatorname{finv} \mathbf{s_1}\, w'' = \operatorname{finv} w'' + \chi(y_1>0) - \chi(y_1<0); \tag{3.5}$$
$$\begin{aligned}|\mathbf{s_1}\, w''|_{>x} = |w''|_{>x} &+ \chi(x>0)(\chi(y_1<\overline{x}) - \chi(x<y_1))\\ &+ \chi(x<0)(\chi(y_1\leq x) - \chi(\overline{x}\leq y_1)).\end{aligned} \tag{3.6}$$

Theorem 1.1 is now proved by induction on the word length. Assume that $\operatorname{fmaj} w = \operatorname{finv} \Psi(w)$ for a given w. Our purpose is to show that

$$\operatorname{fmaj} wx = \operatorname{finv} \Psi(wx) \tag{3.7}$$

holds for all letters x. Let $w' = \Psi(w)$, so that by (2.4) the words w and w' have the same rightmost letter. Denote the x-right-to-left factorization of w' by $(v_1y_1,\ldots,v_py_p)$. By (2.3) the signed word $v := \beta_x(w')$ is defined by the chain

$$\begin{aligned} w' = v_1y_1\ldots v_py_p &\xrightarrow{\gamma_x} w'' = y_1v_1\ldots y_pv_p = z_1w_1\ldots z_qw_q\\ &\xrightarrow{\delta_{\overline{x}}} v = w_1z_1\ldots w_qz_q \end{aligned}\tag{3.8}$$

if either $\overline{x}\leq y_1\leq x$, or $x<y_1<\overline{x}$, and by the chain

$$\begin{aligned} w' = v_1y_1\ldots v_py_p &\xrightarrow{\gamma_x} w'' = y_1v_1\ldots y_pv_p\\ &\xrightarrow{\mathbf{s_1}} w''' = \overline{y}_1v_1\ldots y_pv_p = z_1w_1\ldots z_qw_q\\ &\xrightarrow{\delta_{\overline{x}}} v = w_1z_1\ldots w_qz_q, \end{aligned}\tag{3.9}$$

otherwise. Notice that $(z_1w_1,\ldots,z_qw_q)$ designates the $\overline{x}$-left-to-right factorization of w'' in chain (3.8) and of w''' in chain (3.9).

(i) Suppose that one of the conditions $\overline{x}\leq y_1\leq x$, $x<y_1<\overline{x}$ holds, so that (3.8) applies. We have

$$\begin{aligned}
\operatorname{finv}\Psi(wx) &= \operatorname{finv} vx = \operatorname{finv} v + |v|_{>x} + |v|_{<\overline{x}} + \chi(x<0) && \text{[by (3.2)]}\\
\operatorname{finv} v &= \operatorname{finv}\delta_{\overline{x}}(w'') = \operatorname{finv} w'' + |w''|_{\geq\overline{x}} - |w''|\,\chi(F(w'')\geq\overline{x}) && \text{[by (3.4)]}\\
\operatorname{finv} w'' &= \operatorname{finv}\gamma_x(w') = \operatorname{finv} w' + |w'|_{\leq x} - |w'|\,\chi(L(w')\leq x) && \text{[by (3.3)]}\\
\operatorname{finv} w' &= \operatorname{fmaj} w && \text{[by induction]}\\
\operatorname{fmaj} w &= \operatorname{fmaj} wx - \chi(x<0) - 2\,|w|\,\chi(L(w)>x). && \text{[by (3.1)]}
\end{aligned}$$

By induction,

$$L(w) = L(w') \text{ and } \chi(L(w')>x) = 1 - \chi(L(w')\leq x). \tag{3.10}$$

Also $F(w'') = y_1$. As w', w'', v are true rearrangements of each other, we have $|v|_{>x} + |w'|_{\le x} = |w|$, $|v|_{<\overline{x}} + |w''|_{\ge\overline{x}} = |w|$. Hence,

$$\operatorname{finv}\Psi(wx) = \operatorname{fmaj} wx + |w|[\chi(L(w') \le x) - \chi(y_1 \ge \overline{x})].$$

By (2.5), if $\overline{x} \le y_1 \le x$ holds, then $L(w') \le x$ and the expression between brackets is null. If $x < y_1 < \overline{x}$ holds, then $L(w') > x$ and the same expression is also null. Thus (3.7) holds.

(ii) Suppose that none of the conditions $\overline{x} \le y_1 \le x$, $x < y_1 < \overline{x}$ holds, so that (3.9) applies. We have

$$\begin{aligned}
\operatorname{finv}\Psi(wx) &= \operatorname{finv} vx = \operatorname{finv} v + |v|_{>x} + |v|_{<\overline{x}} + \chi(x<0) && \text{[by (3.2)]}\\
\operatorname{finv} v &= \operatorname{finv} \delta_{\overline{x}}(w''') = \operatorname{finv} w''' + |w'''|_{\ge\overline{x}} - |w'''|\,\chi(F(w''') \ge \overline{x}) && \text{[by (3.4)]}\\
\operatorname{finv} w''' &= \operatorname{finv} \mathbf{s_1}\, w'' = \operatorname{finv} w'' + \chi(y_1 > 0) - \chi(y_1 < 0) && \text{[by (3.5)]}\\
\operatorname{finv} w'' &= \operatorname{finv} \gamma_x(w') = \operatorname{finv} w' + |w'|_{\le x} - |w'|\,\chi(L(w') \le x) && \text{[by (3.3)]}\\
\operatorname{finv} w' &= \operatorname{fmaj} w && \text{[by induction]}\\
\operatorname{fmaj} w &= \operatorname{fmaj} wx - \chi(x<0) - 2\,|w|\,\chi(L(w) > x). && \text{[by (3.1)]}
\end{aligned}$$

Moreover, $F(w''') = \overline{y}_1$, so that $\chi(F(w''') \ge \overline{x}) = \chi(\overline{y}_1 \ge \overline{x}) = \chi(y_1 \le x) = \chi(L(w') \le x) = \chi(L(w) \le x)$. As v and w''' are rearrangements of each other, we have $|v|_{<\overline{x}} + |w'''|_{\ge\overline{x}} = |w|$. Using (3.6) since $|v|_{>x} = |w'''|_{>x} = |\mathbf{s_1} w''|_{>x}$ we have:

$$\begin{aligned}
\operatorname{finv}\Psi(wx) &= |w''|_{>x} + \chi(x>0)(\chi(y_1 < \overline{x}) - \chi(x < y_1))\\
&\qquad + \chi(x<0)(\chi(y_1 \le x) - \chi(\overline{x} \le y_1)) + \chi(x<0)\\
&\qquad + |w| - |w|\,\chi(y_1 \le x) + \chi(y_1 > 0) - \chi(y_1 < 0)\\
&\qquad + |w'|_{\le x} - |w'|\,\chi(y_1 \le x)\\
&\qquad + \operatorname{fmaj} wx - \chi(x<0) - 2\,|w| + 2|w|\,\chi(y_1 \le x)\\
&= \operatorname{fmaj} wx + \chi(x>0)(\chi(y_1 < \overline{x}) - \chi(x < y_1))\\
&\qquad + \chi(x<0)(\chi(y_1 \le x) - \chi(\overline{x} \le y_1))\\
&\qquad + \chi(y_1 > 0) - \chi(y_1 < 0)\\
&= \operatorname{fmaj} wx,
\end{aligned}$$

for, if none of the conditions $\overline{x} \le y_1 \le x$, $x < y_1 < \overline{x}$ holds, then one of the following four ones holds: (a) $y_1 > x > 0$; (b) $\overline{y}_1 > x > 0$; (c) $y_1 \le x < 0$; (d) $\overline{y}_1 \le x < 0$; and in each case the sum of the factors in the above sum involving χ is zero.

The construction of Ψ is perfectly reversible. First, note that $\mathbf{s_1}$ is an involution and the maps γ_x, $\delta_{\overline{x}}$ send each class $B_{\mathbf{m}}$ onto itself, so that their inverses are perfectly defined. They can also be described by means of left-to-right and right-to-left factorizations. Let us give the construction of the inverse Ψ^{-1} of Ψ. Of course, $\Psi^{-1}(v) := v$ if v is a one-letter word. If vx is a signed word, whose last letter is x, determine $v' := \delta_{\overline{x}}^{-1}(v)$ and let z_1 be its first letter. If one of the conditions $\overline{x} \le z_1 \le x$ or $x < z_1 < \overline{x}$ holds, the chain (3.8) is to be used in reverse order, so that $\Psi^{-1}(vx) := (\Psi^{-1}\,\gamma_x^{-1}\,\delta_{\overline{x}}^{-1}(v))x$. If none of those two conditions holds, then $\Psi^{-1}(vx) := (\Psi^{-1}\,\gamma_x^{-1}\,\mathbf{s_1}\,\delta_{\overline{x}}^{-1}(v))x$. $\square$

4. Proofs of Theorems 1.2 and 1.3

Before proving Theorem 1.2 we note the following two properties.

Property 4.1. *Let $I_x = \{i \in \mathbb{Z} : i < -|x|\}$ (resp. $J_x = \{i \in \mathbb{Z} : -|x| < i < |x|\}$, resp. $K_x = \{i \in \mathbb{Z} : |x| < i\}$) and w be a signed word. Then, the bijections γ_x and $\delta_{\overline{x}}$ do not modify the mutual order of the letters of w that belong to I_x (resp. J_x, resp. K_x).*

Proof. Let $y \in I_x$ and $z \in I_x$ (resp. $y \in J_x$ and $z \in J_x$, resp. $y \in K_x$ and $z \in K_x$) be two letters of w with y to the left of z. In the notations of (2.1) (resp. of (2.2)) both y, z are, either among the y_i's (resp. the z_i's), or letters of the v_i's (resp. the w_i's). Accordingly, y remains to the left of z when γ_x (resp. $\delta_{\overline{x}}$) is applied to w. $\square$

Property 4.2. *Let $w = x_1x_2\ldots x_m \in B_{\mathbf{m}}$ be a signed word and i be a positive integer such that $m_i = m_{i+1} = 1$. Furthermore, let x be an integer such that $x \notin \{i, \overline{i}, i+1, \overline{i+1}\}$. Then the following conditions are equivalent:*
(a) $i \in \operatorname{Iligne} w$; (b) $i \in \operatorname{Iligne} \mathbf{s_1} w$; (c) $i \in \operatorname{Iligne} \gamma_x w$; (d) $i \in \operatorname{Iligne} \delta_{\overline{x}} w$.

Proof. (a)$\Leftrightarrow$(b) holds by definition 1.1, because $\mathbf{s_1}$ has no action on the rightmost letter belonging to $\{i, \overline{i}, i+1, \overline{i+1}\}$. For the other equivalences we can say the following. If the two letters of w that belong to $\{i, \overline{i}, i+1, \overline{i+1}\}$ are in I_x (resp. J_x, resp. K_x), Property 4.1 applies. Otherwise, if $i \in \operatorname{Iligne} w$, then w is either of the form $\ldots\overline{i+1}\ldots i\ldots$ or $\ldots i\ldots\overline{i+1}\ldots$ and the order of those two letters is immaterial. $\square$

Theorem 1.2 holds for each one-letter signed word. Let $w = x_1x_2\ldots x_m \in B_{\mathbf{m}}$ be a signed word, x a letter and i a positive integer. Assume that $\operatorname{Iligne} w = \operatorname{Iligne} \Psi(w)$.

If $x = i$, then $i \in \operatorname{Iligne} wx$ if and only if w contains no letter equal to $\pm i$ and exactly one letter equal to $\pm(i+1)$. As $\beta_x\Psi(w)$ is a rearrangement of w with possibly sign changes for some letters, the last statement is equivalent to saying that $\beta_x\Psi(w)$ has no letter equal to $\pm i$ and exactly one letter equal to $\pm(i+1)$. This is also equivalent to saying that $i \in \operatorname{Iligne} \beta_x\Psi(w)x = \operatorname{Iligne} \Psi(wx)$. In the same manner, we can show that

if $x = \overline{i+1}$, then $i \in \operatorname{Iligne} wx$ if and only if $i \in \operatorname{Iligne} \Psi(wx)$;
if $x = \overline{i}$, then $i \notin \operatorname{Iligne} wx$ *and* $i \notin \operatorname{Iligne} \Psi(wx)$;
if $x = i+1$, then $i \notin \operatorname{Iligne} wx$ *and* $i \notin \operatorname{Iligne} \Psi(wx)$.

Now, let i be such that none of the integers i, $\overline{i+1}$, $\overline{i}$, $i+1$ is equal to x. There is nothing to prove if $m_i = m_{i+1} = 1$ does not hold, as i does not belong to any of the sets $\operatorname{Iligne} w$, $\operatorname{Iligne} \beta_x\Psi(w)$. Otherwise, the result follows from Property 4.2 because Ψ is a composition product of β_x, $\mathbf{s_1}$ and $\gamma_{\overline{x}}$. Finally, the equivalence $[0 \in \operatorname{Iligne} \beta_x(w)x] \Leftrightarrow [0 \in \operatorname{Iligne} wx]$ follows from Proposition 4.1 when $|x| > 1$ and the result is evident when $|x| = 1$. $\square$

The proof of Theorem 1.3 also follows from Property 4.1. By definition the lower records of wx, other than x, belong to J_x. As the bijections γ_x and $\delta_{\overline{x}}$ do not modify the mutual order of the letters of w that belong to J_x, we have $\operatorname{Lower} wx = \operatorname{Lower} \beta_x(w)x$ when the chain (3.8) is used. When (3.9) is applied, so that $y_1 \notin J_x$, we also have $z_1 = \overline{y}_1 \notin J_x$. Thus, neither y_1, nor z_1 can be lower records for each word ending with x. Again, $\operatorname{Lower} wx = \operatorname{Lower} \beta_x(w)x$. $\square$

5. Concluding remarks

Since the works by MacMahon, much attention has been given to the study of statistics on the symmetric group or on classes of word rearrangements, in particular by the M.I.T. school ([25, 26, 27, 13, 6]). It was then natural to extend those studies to other classical Weyl groups, as was done by Reiner [20, 21, 22, 23, 24] for the signed permutation group. Today the work has been pursued by the Israeli and Roman schools [1, 2, 3, 4]. The contribution of Adin, Roichman [1] has been essential with their definition of the *flag major index* for signed permutations. In our forthcoming paper [9] we will derive new analytical expressions, in particular for several *multivariable* statistics involving "fmaj," "finv" and the number of lower records.

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SIGNED WORDS AND PERMUTATIONS, II; THE EULER-MAHONIAN POLYNOMIALS

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Dans la théorie de Morse, quand on veut étudier un espace, on introduit une fonction numérique; puis on aplatit cet espace sur l'axe de la valeur de cette fonction. Dans cette opération d'aplatissement, on crée des singularités de la fonction et celles-ci sont en quelque sorte les vestiges de la topologie qu'on a tuée.
René Thom, *Logos et Théorie des catastrophes*, 1982.

Dedicated to Richard Stanley,
on the occasion of his sixtieth birthday.

Abstract

As for the symmetric group of ordinary permutations there is also a statistical study of the group of signed permutations, that consists of calculating multivariable generating functions for this group by statistics involving record values and the length function. Two approaches are here systematically explored, using the flag-major index on the one hand, and the flag-inversion number on the other hand. The MacMahon Verfahren appears as a powerful tool throughout.

1. Introduction

The elements of the hyperoctahedral group B_n $(n \geq 0)$, usually called *signed permutations*, may be viewed as words $w = x_1x_2 \ldots x_n$, where the letters x_i are positive or negative integers and where $|x_1|\,|x_2| \ldots |x_n|$ is a permutation of $1\,2 \ldots n$ (see [Bo68] p. 252–253). For typographical reasons we shall use the notation $\bar{i} := -i$ in the sequel. Using the χ-notation that maps each statement A onto the value $\chi(A) = 1$ or 0

depending on whether A is true or not, we recall that the usual *inversion number*, $\operatorname{inv} w$, of the signed permutation $w = x_1x_2\ldots x_n$ is defined by

$$\operatorname{inv} w := \sum_{1\le j\le n}\ \sum_{i<j} \chi(x_i > x_j).$$

It also makes sense to introduce

$$\overline{\operatorname{inv}}\, w := \sum_{1\le j\le n}\ \sum_{i<j} \chi(\overline{x}_i > x_j),$$

and verify that the *length function* (see [Bo68, p. 7], [Hu90, p. 12]), that will be denoted by "finv" (*flag-inversion number*) in the whole paper, can be defined, using the notation $\operatorname{neg} w := \sum\limits_{1\le j\le n} \chi(x_j < 0)$, by

$$\operatorname{finv} w := \operatorname{inv} w + \overline{\operatorname{inv}}\, w + \operatorname{neg} w.$$

Another equivalent definition will be given in (7.1). The *flag-major index* "fmaj" and the *flag descent number* "fdes" were introduced by Adin and Roichman [AR01] and read:

$$\operatorname{fmaj} w := 2 \operatorname{maj} w + \operatorname{neg} w;$$
$$\operatorname{fdes} w := 2 \operatorname{des} w + \chi(x_1 < 0);$$

where $\operatorname{maj} w := \sum_j j\, \chi(x_j > x_{j+1})$ denotes the usual *major index* of w and $\operatorname{des} w$ the *number of descents* $\operatorname{des} w := \sum_j \chi(x_j > x_{j+1})$.

Another class of statistics needed here is the class of *lower records*. A letter x_i $(1 \le i \le n)$ is said to be a *lower record* of the signed permutation $w = x_1x_2\ldots x_n$, if $|x_i| < |x_j|$ for all j such that $i+1 \le j \le n$. When reading the lower records of w from left to right we get a *signed subword*, called the *lower record subword*, denoted by $\operatorname{Lower} w$. Denote the number of *positive* (resp. *negative*) letters in $\operatorname{Lower} w$ by $\operatorname{lowerp} w$ (resp. $\operatorname{lowern} w$).

In our previous paper [FoHa05] we gave the construction of a transformation Ψ on (arbitrary) signed words, that is, words, whose letters are positive or negative with repetitions allowed. When applied to the group B_n, the transformation Ψ has the following properties:

(a) $\operatorname{fmaj} w = \operatorname{finv} \Psi(w)$ for every signed permutation w;

(b) Ψ is a bijection of B_n onto itself, so that "fmaj" and "finv" are equidistributed over the hyperoctahedral group B_n;

(c) $\operatorname{Lower} w = \operatorname{Lower} \Psi(w)$, so that $\operatorname{lowerp} w = \operatorname{lowerp} \Psi(w)$ and $\operatorname{lowern} w = \operatorname{lowern} \Psi(w)$.

Actually, the transformation Ψ has stronger properties than those stated above, but these restrictive properties will suffice for the following derivation. Having properties (a)–(c) in mind, we see that the two three-variable statistics (fmaj, lowerp, lowern) and (finv, lowerp, lowern) are equidistributed over B_n. Hence, the two generating polynomials

$$^{\mathrm{fmaj}}B_n(q, X, Y) := \sum_{w\in B_n} q^{\mathrm{fmaj}} X^{\operatorname{lowerp} w} Y^{\operatorname{lowern} w}$$
$$^{\mathrm{finv}}B_n(q, X, Y) := \sum_{w\in B_n} q^{\mathrm{finv}} X^{\operatorname{lowerp} w} Y^{\operatorname{lowern} w}$$

are *identical.* To derive the analytical expression for the common polynomial we have two approaches, using the "fmaj" interpretation, on the one hand, and the "finv" geometry, on the other. In each case we will go beyond the three-variable case, as we consider the generating polynomial for the group B_n by the five-variable statistic $(\mathrm{fdes}, \mathrm{fmaj}, \mathrm{lowerp}, \mathrm{lowern}, \mathrm{neg})$

$$ {}^{\mathrm{fmaj}}B_n(t,q,X,Y,Z) := \sum_{w\in B_n} t^{\mathrm{fdes}\,w} q^{\mathrm{fmaj}\,w} X^{\mathrm{lowerp}\,w} Y^{\mathrm{lowern}\,w} Z^{\mathrm{neg}\,w} \tag{1.1} $$

and the generating polynomial for the group B_n by the four-variable statistic $(\mathrm{finv}, \mathrm{lowerp}, \mathrm{lowern}, \mathrm{neg})$

$$ {}^{\mathrm{finv}}B_n(q,X,Y,Z) := \sum_{w\in B_n} q^{\mathrm{finv}\,w} X^{\mathrm{lowerp}\,w} Y^{\mathrm{lowern}\,w} Z^{\mathrm{neg}\,w}. \tag{1.2} $$

Using the usual notations for the q-*ascending factorial*

$$ (a;q)_n := \begin{cases} 1, & \text{if } n=0; \\ (1-a)(1-aq)\,\ldots\,(1-aq^{n-1}), & \text{if } n\ge 1; \end{cases} \tag{1.3} $$

in its *finite* form and

$$ (a;q)_\infty := \lim_n (a;q)_n = \prod_{n\ge 0}(1-aq^n); \tag{1.4} $$

in its *infinite* form, we consider the products

$$ H_\infty(u) := \frac{\Bigl(uq\Bigl(\dfrac{Z+q}{1-q^2}-ZY\Bigr);q^2\Bigr)_\infty}{\Bigl(u\Bigl(\dfrac{q(Z+q)}{1-q^2}+X\Bigr);q^2\Bigr)_\infty}, \tag{1.5} $$

in its infinite version, and

$$ H_{2s}(u) := \frac{1-q^2}{1-q^2+uq^{2s+1}(Z+q)}\,\frac{\Bigl(uq\dfrac{Z+q-ZY(1-q^2)}{1-q^2+uq^{2s+1}(Z+q)};q^2\Bigr)_s}{\Bigl(u\dfrac{q(Z+q)+X(1-q^2)}{1-q^2+uq^{2s+1}(Z+q)};q^2\Bigr)_{s+1}}, \tag{1.6} $$

as well as

$$ H_{2s+1}(u) := \frac{\Bigl(uq\dfrac{Z+q-ZY(1-q^2)}{1-q^2+uq^{2s+2}(Zq+1)};q^2\Bigr)_{s+1}}{\Bigl(u\dfrac{q(Z+q)+X(1-q^2)}{1-q^2+uq^{2s+2}(Zq+1)};q^2\Bigr)_{s+1}}, \tag{1.7} $$

in its graded version under the form $\sum_{s\ge 0} t^s H_s(u)$.

The purpose of this paper is to prove the following two theorems and derive several applications regarding statistical distributions over B_n.

Theorem 1.1 (the "fmaj" approach). *Let* ${}^{\mathrm{fmaj}}B_n(t,q,X,Y,Z)$ *be the generating polynomial for the group* B_n *by the five-variable statistic* (fdes, fmaj, lowerp, lowern, neg) *as defined in* (1.1). *Then*

$$\sum_{n\ge 0}(1+t)\,{}^{\mathrm{fmaj}}B_n(t,q,X,Y,Z)\frac{u^n}{(t^2;q^2)_{n+1}}=\sum_{s\ge 0}t^s H_s(u), \tag{1.8}$$

where $H_s(u)$ *is the finite product introduced in* (1.6) *and* (1.7).

Theorem 1.2 (the "finv" approach). *Let* ${}^{\mathrm{finv}}B_n(q,X,Y,Z)$ *be the generating polynomial for the group* B_n *by the four-variable statistic* (finv, lowerp, lowern, neg), *as defined in* (1.2). *Then*

$$\begin{aligned}{}^{\mathrm{finv}}B_n(q,X,Y,Z)&=(X+q+\cdots+q^{n-1}+q^nZ+\cdots+q^{2n-2}Z+q^{2n-1}YZ)\\&\cdots\times(X+q+q^2+q^3Z+q^4Z+q^5YZ)(X+q+q^2Z+q^3YZ)(X+qYZ).\end{aligned} \tag{1.9}$$

The proofs of those two theorems are very different in nature. For proving Theorem 1.1 we re-adapt the *MacMahon Verfahren* to make it work for *signed* permutations. René Thom's quotation that appears as an epigraph to this paper illustrates the essence of the *MacMahon Verfahren.* The topology of the signed permutations measured by the various statistics, "fdes", "fmaj", ... must be reconstructed when the group of the signed permutations is mapped onto a set of plain words for which the calculation of the associated statistic is easy. There is then a combinatorial bijection between signed permutations and plain words that describes the "flattening" ("aplatissement") process. This is the content of Theorem 4.1.

Another approach might have been to make use of the P-partition technique introduced by Stanley [St72] and successfully employed by Reiner [Re93a, Re93b, Re93c, Re95a, Re95b] in his statistical study of the hyperoctahedral group.

Theorem 1.2 is based upon another definition of the length function for B_n (see formula (7.1)). Notice that in the two theorems we have included a variable Z, which takes the number "neg" of *negative* letters into account. This allows us to re-obtain the classical results on the symmetric group by letting $Z=0$.

In the next section we show that the infinite product $H_s(u)$ first appears as the generating function for a class of *plain* words by a four-variable statistic (see Theorem 2.2). This theorem will be an essential tool in section 4 in the MacMahon Verfahren for signed permutations to handle the five-variable polynomial ${}^{\mathrm{fmaj}}B_n(t,q,X,Y,Z)$. Section 3 contains an axiomatic definition of the *Record-Signed-Euler-Mahonian Polynomials* $B_n(t,q,X,Y,Z)$. They are defined, not only by (1.8) (with B_n replacing ${}^{\mathrm{fmaj}}B_n$), but also by a recurrence relation. The proof of Theorem 1.1 using the MacMahon Verfahren is found in Section 4. In Section 5 we show how to prove that the polynomials ${}^{\mathrm{fmaj}}B_n(t,q,X,Y,Z)$ satisfy the same recurrence as the polynomials $B_n(t,q,X,Y,Z)$, using an insertion technique. The specializations of Theorem 1.1 are numerous and described in section 6. We end the paper with the proof of Theorem 1.2 and its specializations.

2. Lower Records on Words

As mentioned in the introduction, Theorem 2.2 below, dealing with *ordinary words*, appears to be a *preparation lemma* for Theorem 1.1, that takes the geometry of *signed permutations* into account. Consider an ordinary word $c = c_1c_2\dots c_n$, whose letters belong to the alphabet $\{0, 1, \dots, s\}$, that is, a word from the free monoid $\{0, 1, \dots, s\}^*$. A letter c_i ($1 \le i \le n$) is said to be an *even lower record* (resp. *odd lower record*) of c, if c_i is even (resp. odd) and if $c_j \ge c_i$ (resp. $c_j > c_i$) for all j such that $1 \le j \le i-1$. Notice the discrepancy between even and odd letters. Also, to define those even and odd lower records for words the reading is made *from left to right*, while for signed permutations, the lower records are read from *right to left* (see Sections 1 and 4). We could have considered a totally ordered alphabet with two kinds of letters, but playing with the parity of the nonnegative integers is more convenient for our applications. For instance, the even (resp. odd) lower records of the word $c = \mathit{5}\,\mathbf{4}\,\mathbf{4}\,\mathit{1}\,5\,2\,1\,\mathbf{0}\,4\,\mathbf{0}\,3$ are reproduced in boldface (resp. in italic).

For each word c let $\operatorname{evenlower} c$ (resp. $\operatorname{oddlower} c$) be the number of even (resp. odd) lower records of c. Also let $\operatorname{tot} c$ (“tot” stands for “total”) be the *sum* $c_1 + c_2 + \dots + c_n$ of the letters of c and $\operatorname{odd} c$ be the *number* of its odd letters. Also denote its *length* by $|c|$ and let $|c|_k$ be the number of letters in c equal to k. Our purpose is to calculate the generating function for $\{0, 1, \dots, s\}^*$ by the four-variable statistic $(\operatorname{tot}, \operatorname{evenlower}, \operatorname{oddlower}, \operatorname{odd})$.

Say that $c = c_1c_2\dots c_n$ is of *minimal index* k ($0 \le k \le s/2$), if $\min c := \min\{c_1, \dots, c_n\}$ is equal to $2k$ or $2k + 1$. Let c_j be the leftmost letter of c equal to $2k$ or $2k + 1$. Then, c admits a unique factorization

$$c = c' c_j c'', \tag{2.1}$$

having the following properties:

$$c' \in \{2k + 2, 2k + 3, \dots, s\}^*,\ c_j = 2k \text{ or } 2k + 1,\ c'' \in \{2k, 2k + 1, \dots, s\}^*.$$

With the forementioned example we have the factorization $c' = 5\,4\,4$, $c_j = 1$, $c'' = 5\,2\,1\,0\,4\,0\,3$. In this example notice that $c_j = 1 \ne \min c = 0$.

Lemma 2.1. *The numbers of even and odd lower records of a word c can be calculated by induction as follows:* $\operatorname{evenlower} c = \operatorname{oddlower} c := 0$ *if c is empty; otherwise, let $c = c'c_jc''$ be its minimal index factorization (defined in* (2.1)*). Then*

$$\operatorname{evenlower} c = \operatorname{evenlower} c' + \chi(c_j = 2k) + |c''|_{2k}; \tag{2.2}$$

$$\operatorname{oddlower} c = \operatorname{oddlower} c' + \chi(c_j = 2k + 1). \tag{2.3}$$

Proof. Keep the same notations as in (2.1). If $c_j = 2k$, then c_j is an even lower record, as well as all the letters equal to $2k$ to the right of c_j. On the other hand, there is no even lower record equal to $2k$ to the left of c_j, so that (2.2) holds. If $c_j = 2k + 1$, then c_j is an odd lower record and there is no odd lower record equal to $2k + 1$ to the right of c_j. Moreover, there is no odd lower record to the left of c_j equal to c_j. Again (2.3) holds. □

It is straightforward to verify that the fraction $H_s(u)$ displayed in (1.6) and (1.7) can also be expressed as

$$H_{2s}(u) = \prod_{0\le k\le s} \frac{1-u([q^{2k+1}(1-Y)Z+q^{2k+2}+\cdots+q^{2s-2}+q^{2s-1}Z+q^{2s}])}{1-u(q^{2k}X+[q^{2k+1}Z+q^{2k+2}+\cdots+q^{2s-2}+q^{2s-1}Z+q^{2s}])} \tag{2.4}$$

$$H_{2s+1}(u) = \prod_{0\le k\le s} \frac{1-u(q^{2k+1}(1-Y)Z+[q^{2k+2}+\cdots+q^{2s}+q^{2s+1}Z])}{1-u(q^{2k}X+q^{2k+1}Z+[q^{2k+2}+\cdots+q^{2s}+q^{2s+1}Z])}, \tag{2.5}$$

where the expression between brackets vanishes whenever $k=s$, and that the $H_s(u)$'s satisfy the recurrence formula

$$H_0(u)=\frac{1}{1-uX};\quad H_1(u)=\frac{1-uqZ(1-Y)}{1-u(X+qZ)};\quad \text{and for } s\ge 1 \tag{2.6}$$
$$H_{2s}(u)=\frac{1-u(q(1-Y)Z+q^2+q^3Z+\cdots+q^{2s-1}Z+q^{2s})}{1-u(X+qZ+q^2+q^3Z+\cdots+q^{2s-1}Z+q^{2s})}H_{2s-2}(uq^2);$$
$$H_{2s+1}(u)=\frac{1-u(q(1-Y)Z+q^2+q^3Z+\cdots+q^{2s}+q^{2s+1}Z)}{1-u(X+qZ+q^2+q^3Z+\cdots+q^{2s}+q^{2s+1}Z)}H_{2s-1}(uq^2).$$

Theorem 2.2. *The generating function for the free monoid* $\{0,1,\ldots,s\}^*$ *by the four-variable statistic* (tot, evenlower, oddlower, odd) *is equal to* $H_s(u)$*, that is to say,*

$$\sum_{c\in\{0,1,\ldots,s\}^*} u^{|c|}q^{\mathrm{tot}\,c}X^{\mathrm{evenlower}\,c}Y^{\mathrm{oddlower}\,c}Z^{\mathrm{odd}\,c}=H_s(u). \tag{2.7}$$

Proof. Let $H_s^*(u)$ denote the left-hand side of (2.7). Then,

$$H_0^*(u)=\sum_{c\in\{0\}^*} u^{|c|}q^0X^{|c|}Y^0Z^0=\frac{1}{1-uX}.$$

When $s=1$ the minimal index factorization of each nonempty word c reads $c=c_jc''$, so that

$$H_1^*(u)=1+u(X+qYZ)\sum_{c''\in\{0,1\}^*} u^{|c''|}q^{|c''|_1}X^{|c''|_0}Y^0Z^{|c''|_1}$$
$$=1+u(X+qYZ)\frac{1}{1-u(X+qZ)}=\frac{1-uqZ(1-Y)}{1-u(X+qZ)}.$$

Consequently, $H_s^*(u)=H_s(u)$ for $s=0,1$. For $s\ge 2$ we write

$$H_s^*(u)=\sum_{0\le k\le s/2} H_{s,k}^*(u)$$

with

$$H_{s,k}^*(u):=\sum_{\substack{c\in\{0,1,\ldots,s\}^*\\ \min_i c_i=2k \text{ or } 2k+1}} u^{|c|}q^{\mathrm{tot}\,c}X^{\mathrm{evenlower}\,c}Y^{\mathrm{oddlower}\,c}Z^{\mathrm{odd}\,c}.$$

From Lemma 2.1 it follows that

$$H^*_{s,0}(u) = \sum_{c'\in\{2,\dots,s\}^*} u^{|c'|} q^{\mathrm{tot}\,c'} X^{\mathrm{evenlower}\,c'} Y^{\mathrm{oddlower}\,c'} Z^{\mathrm{odd}\,c'} \times u(X+qYZ)$$

$$\times \sum_{c''\in\{0,\dots,s\}^*} u^{|c''|} q^{\mathrm{tot}\,c''} X^{|c''|_0} Z^{\mathrm{odd}\,c''}$$

$$= \sum_{c'\in\{0,\dots,s-2\}^*} (uq^2)^{|c'|} q^{\mathrm{tot}\,c'} X^{\mathrm{evenlower}\,c'} Y^{\mathrm{oddlower}\,c'} Z^{\mathrm{odd}\,c'} \times u(X+qYZ)$$

$$\times \sum_{c''\in\{0,\dots,s\}^*} (uX)^{|c''|_0} (uqZ)^{|c''|_1} (uq^2)^{|c''|_2} (uq^3Z)^{|c''|_3} (uq^4)^{|c''|_4}\cdots$$

$$= H^*_{s-2}(uq^2)u(X+qYZ)\frac{1}{1-u(X+qZ+q^2+q^3Z+q^4+\cdots)},$$

the polynomial in the denominator ending with $\cdots + q^{s-1}Z + q^s$ or $\cdots + q^{s-1} + q^sZ$ depending on whether s is even or odd.

On the other hand,

$$\sum_{1\le k\le s/2} H^*_{s,k}(u) = \sum_{c\in\{2,3,\dots,s\}^*} u^{|c|} q^{\mathrm{tot}\,c} X^{\mathrm{evenlower}\,c} Y^{\mathrm{oddlower}\,c} Z^{\mathrm{odd}\,c}$$

$$= \sum_{c\in\{0,1,\dots,s-2\}^*} (uq^2)^{|c|} q^{\mathrm{tot}\,c} X^{\mathrm{evenlower}\,c} Y^{\mathrm{oddlower}\,c} Z^{\mathrm{odd}\,c}$$

$$= H^*_{s-2}(uq^2).$$

Hence,

$$H^*_s(u) = \Bigl(1+\frac{u(X+qYZ)}{1-u(X+qZ+q^2+q^3Z+q^4+\cdots)}\Bigr)H^*_{s-2}(uq^2)$$

$$= \frac{1-u(qZ(1-Y)+q^2+q^3Z+q^4+\cdots)}{1-u(X+qZ+q^2+q^3Z+q^4+\cdots)}H^*_{s-2}(uq^2).$$

As the fractions $H^*_s(u)$ satisfy the same induction relation as the $H_s(u)$'s, we conclude that $H^*_s(u) = H_s(u)$ for all s. ∎

When s tends to infinity, then $H_s(u)$ tends to $H_\infty(u)$, whose expression is shown in (1.5). In particular, we have the identity:

$$\sum_{c\in\{0,1,2,\dots\}^*} u^{|c|} q^{\mathrm{tot}\,c} X^{\mathrm{evenlower}\,c} Y^{\mathrm{oddlower}\,c} Z^{\mathrm{odd}\,c} = H_\infty(u). \tag{2.8}$$

3. The Record-Signed-Euler-Mahonian Polynomials

Our next step is to form the series $\sum_{s\ge0} t^s H_s(u)$ and show that the series can be expanded as a series in the variable u in the form

$$\sum_{n\ge0} C_n(t,q,X,Y,Z)\frac{u^n}{(t^2;q^2)_{n+1}} = \sum_{s\ge0} t^s H_s(u), \tag{3.1}$$

where $B_n(t,q,X,Y,Z) := C_n(t,q,X,Y,Z)/(1+t)$ is a *polynomial* with nonnegative integral coefficients such that $B_n(1,1,1,1,1) = 2^n n!$.

Definition. A sequence $\left(B_n(t,q,X,Y,Z) = \sum_{k\ge 0} t^k B_{n,k}(q,X,Y,Z)\right)$ $(n \ge 0)$ of polynomials in five variables t, q, X, Y and Z is said to be *record-signed-Euler-Mahonian*, if one of the following *equivalent* three conditions holds:

(1) The (t^2,q^2)-factorial generating function for the polynomials

$$C_n(t,q,X,Y,Z) := (1+t)B_n(t,q,X,Y,Z) \tag{3.2}$$

is given by identity (3.1).

(2) For $n \ge 2$ the recurrence relation holds:

$$\begin{aligned}
(3.3)\ (1-q^2)&B_n(t,q,X,Y,Z)\\
=&\Big(X(1-q^2)+(Zq+q^2)(1-t^2q^{2n-2})+t^2q^{2n-1}(1-q^2)ZY\Big)B_{n-1}(t,q,X,Y,Z)\\
&-\frac12(1-t)q(1+q)(1+tq)(1+Z)B_{n-1}(tq,q,X,Y,Z)\\
&+\frac12(1-t)q(1-q)(1-tq)(1-Z)B_{n-1}(-tq,q,X,Y,Z),
\end{aligned}$$

while $B_0(t,q,X,Y,Z)=1$, $\quad B_1(t,q,X,Y,Z)=X+tqYZ$.

(3) The recurrence relation holds for the coefficients $B_{n,k}(q,X,Y,Z)$:

$$\begin{gathered}
B_{0,0}(q,X,Y,Z)=1,\quad B_{0,k}(q,X,Y,Z)=0 \text{ for all } k\ne 0;\\
B_{1,0}(q,X,Y,Z)=X,\quad B_{1,1}(q,X,Y,Z)=qYZ,\\
B_{1,k}(q,X,Y,Z)=0 \text{ for all } k\ne 0,1;
\end{gathered} \tag{3.4}$$

$$\begin{aligned}
B_{n,2k}(q,X,Y,Z) &= (X+qZ+q^2+q^3Z+\cdots+q^{2k})B_{n-1,2k}(q,X,Y,Z)\\
&\quad + q^{2k}B_{n-1,2k-1}(q,X,Y,Z)\\
&\quad + (q^{2k}+q^{2k+1}Z+\cdots+q^{2n-1}YZ)B_{n-1,2k-2}(q,X,Y,Z),\\
B_{n,2k+1}(q,X,Y,Z) &= (X+qZ+q^2+\cdots+q^{2k}+q^{2k+1}Z)B_{n-1,2k+1}(q,X,Y,Z)\\
&\quad + q^{2k+1}ZB_{n-1,2k}(q,X,Y,Z)\\
&\quad + (q^{2k+1}Z+q^{2k+2}+\cdots+q^{2n-2}+q^{2n-1}YZ)B_{n-1,2k-1}(q,X,Y,Z),
\end{aligned}$$

for $n\ge 2$ and $0\le 2k+1\le 2n-1$.

Theorem 3.1. *The conditions* (1), (2) *and* (3) *in the previous definition are equivalent.*

Proof. The equivalence (2) ⇔ (3) requires a lengthy but elementary algebraic argument and will be omitted. The other equivalence (1) ⇔ (2) involves a more elaborate q-series technique, which is now developed. Let $G_s(u) := H_s(u^2)$; then

$$G_0(u)=\frac{1}{1-u^2X};\quad G_1(u)=\frac{1-u^2qZ(1-Y)}{1-u^2(X+qZ)};$$

and by (2.6)

$$G_{2s}(u) = \frac{1-u^2(qZ(1-Y)+q^2+q^3Z+\cdots+q^{2s-1}Z+q^{2s})}{1-u^2(X+qZ+q^2+q^3Z+\cdots+q^{2s-1}Z+q^{2s})}G_{2s-2}(uq),$$
$$G_{2s+1}(u) = \frac{1-u^2(qZ(1-Y)+q^2+q^3Z+\cdots+q^{2s}+q^{2s+1}Z)}{1-u^2(X+qZ+q^2+q^3Z+\cdots+q^{2s}+q^{2s+1}Z)}G_{2s-1}(uq),$$

for $s \geq 1$. Working with the series $\sum_{s\geq 0} t^s G_s(u)$ we obtain

$$\begin{aligned}
&\sum_{s\geq 0} t^{2s}G_{2s}(u)\Big(1-u^2\big(X+\frac{Zq+q^2}{1-q^2}-\frac{q^{2s}}{1-q^2}(Zq+q^2)\big)\Big)\\
&\qquad+\sum_{s\geq 0} t^{2s+1}G_{2s+1}(u)\Big(1-u^2\big(X+\frac{Zq+q^2}{1-q^2}-\frac{q^{2s+2}}{1-q^2}(Zq+1)\big)\Big)\\
&\quad=1+t(1-u^2qZ(1-Y))\\
&\qquad+\sum_{s\geq 1} t^{2s}G_{2s-2}(qu)\Big(1-u^2\big(-qZY+\frac{Zq+q^2}{1-q^2}-\frac{q^{2s}}{1-q^2}(Zq+q^2)\big)\Big)\\
&\qquad+\sum_{s\geq 1} t^{2s+1}G_{2s-1}(qu)\Big(1-u^2\big(-qZY+\frac{Zq+q^2}{1-q^2}-\frac{q^{2s+2}}{1-q^2}(Zq+1)\big)\Big),
\end{aligned}$$

which may be rewritten as

$$\begin{aligned}
&\sum_{s\geq 0} t^{s}G_{s}(u)\Big(1-u^2\big(X+\frac{Zq+q^2}{1-q^2}\big)\Big)=1+t(1-u^2qZ(1-Y))\\
&\qquad+\sum_{s\geq 0} t^{s+2}G_{s}(qu)\Big(1-u^2\big(-qZY+\frac{Zq+q^2}{1-q^2}\big)\Big)\\
&\qquad-\sum_{s\geq 0}(tq)^{2s}\big(G_{2s}(u)-t^2q^2G_{2s}(qu)\big)u^2\frac{Zq+q^2}{1-q^2}\\
&\qquad-\sum_{s\geq 0}(tq)^{2s+1}\big(G_{2s+1}(u)-t^2q^2G_{2s+1}(qu)\big)u^2q\frac{Zq+1}{1-q^2}.
\end{aligned}$$

Now let $\sum_{n\geq 0} b_n(t)u^{2n} := \sum_{s\geq 0} t^s G_s(u)$. This gives:

$$\begin{aligned}
&\sum_{n\geq 0} b_n(t)u^{2n}\Big(1-u^2\big(X+\frac{Zq+q^2}{1-q^2}\big)\Big)=1+t(1-u^2qZ(1-Y))\\
&\qquad+\sum_{n\geq 0} b_n(t)t^2q^{2n}u^{2n}\Big(1-u^2\big(-qZY+\frac{Zq+q^2}{1-q^2}\big)\Big)\\
&\qquad-\sum_{n\geq 0}\frac{b_n(tq)+b_n(-tq)}{2}(1-t^2q^{2n+2})u^{2n+2}\frac{Zq+q^2}{1-q^2}\\
&\qquad-\sum_{n\geq 0}\frac{b_n(tq)-b_n(-tq)}{2}(1-t^2q^{2n+2})u^{2n+2}q\frac{Zq+1}{1-q^2}.
\end{aligned}$$

We then have $b_0(t) = \dfrac{1}{1-t}$, $\quad b_1(t) = \dfrac{X + tqYZ}{(1-t)(1-t^2q^2)}$ and for $n \geq 2$

$$b_n(t)(1-t^2q^{2n}) = \Big(X + \frac{Zq+q^2}{1-q^2} + t^2q^{2n-1}ZY - t^2q^{2n-2}\frac{Zq+q^2}{1-q^2}\Big)b_{n-1}(t)$$
$$- \frac{b_{n-1}(tq)}{2(1-q^2)}(1-t^2q^{2n})q(1+q)(1+Z) + \frac{b_{n-1}(-tq)}{2(1-q^2)}(1-t^2q^{2n})q(1-q)(1-Z).$$

Because of the presence of the factors of the form $(1-t^2q^{2n})$ we are led to introduce the coefficients $C_n(t,q,X,Y,Z) := b_n(t)(t^2;q^2)_{n+1}$ $(n \geq 0)$. By multiplying the latter equation by $(t^2;q^2)_n$ we get for $n \geq 2$

(3.5) $$(1-q^2)C_n(t,q,X,Y,Z)$$
$$= \Big(X(1-q^2) + (Zq+q^2)(1-t^2q^{2n-2}) + t^2q^{2n-1}(1-q^2)ZY\Big)C_{n-1}(t,q,X,Y,Z)$$
$$- \frac{1}{2}(1-t^2)q(1+q)(1+Z)C_{n-1}(tq,q,X,Y,Z)$$
$$+ \frac{1}{2}(1-t^2)q(1-q)(1-Z)C_{n-1}(-tq,q,X,Y,Z),$$

while $C_0(t,q,X,Y,Z) = 1+t$, $\quad C_1(t,q,X,Y,Z) = (1+t)(X+tqYZ)$.

Finally, with $C_n(t,q,X,Y,Z) := (1+t)B_n(t,q,X,Y,Z)$ $(n \geq 0)$ we get the recurrence formula (3.3), knowing that the factorial generating function for the polynomials $C_n(t,q,X,Y,Z) = (1+t)B_n(t,q,X,Y,Z)$ is given by (3.1). As all the steps are perfectly reversible, the equivalence holds. $\square$

4. The MacMahon Verfahren

Now having three equivalent definitions for the record-signed-Euler-Mahonian polynomial $B_n(t,q,X,Y,Z)$, our next task is to prove the identity

$$\text{(4.1)} \qquad {}^{\text{fmaj}}B_n(t,q,X,Y,Z) = B_n(t,q,X,Y,Z).$$

Let $\mathbb{N}^n$ (resp. $\text{NIW}(n)$) be the set of all the words (resp. all the nonincreasing words) of length n, whose letters are nonnegative integers. As we have seen in section 2 (Theorem 2.2), we know how to calculate the generating function for words by a certain four-variable statistic. The next step is to map each pair $(b,w) \in \text{NIW}(n) \times B_n$ onto $c \in \mathbb{N}^n$ in such a way that the geometry on w can be derived from the latter statistic on c.

For the construction we proceed as follows. Write the signed permutation w as the linear word $w = x_1x_2\ldots x_n$, where x_k is the image of the integer k $(1 \leq k \leq n)$. For each $k = 1,2,\ldots,n$ let z_k be the number of descents in the right factor $x_kx_{k+1}\ldots x_n$ and ϵ_k be equal to 0 or 1 depending on whether x_k is positive or negative. Next, form the words $z = z_1z_2\ldots z_n$ and $\epsilon = \epsilon_1\epsilon_2\ldots\epsilon_n$.

Now, take a nonincreasing word $b = b_1b_2\dots b_n$ and define $a_k := b_k + z_k$, $c'_k := 2a_k + \epsilon_k$ $(1 \le k \le n)$, then $a := a_1a_2\dots a_n$ and $c' := c'_1c'_2\dots c'_n$. Finally, form the two-matrix $\begin{pmatrix} c'_1 & c'_2 & \dots & c'_n \\ |x_1| & |x_2| & \dots & |x_n| \end{pmatrix}$. Its bottom row is a permutation of $1\,2\,\dots\,n$; rearrange the columns in such a way that the bottom row is precisely $1\,2\,\dots\,n$. Then the word $c = c_1c_2\dots c_n$ which corresponds to the pair (b, w) is defined to be the top row in the resulting matrix.

Example. Start with the pair (b, w) below and calculate all the necessary ingredients:

$$\begin{aligned} b &= 1\ 1\ 1\ 0\ 0\ 0\ 0 \\ w &= 6\ \bar{5}\ \bar{4}\ 1\ 7\ \bar{3}\ \bar{2} \\ z &= 2\ 1\ 1\ 1\ 1\ 0\ 0 \\ \epsilon &= 0\ 1\ 1\ 0\ 0\ 1\ 1 \\ a &= 3\ 2\ 2\ 1\ 1\ 0\ 0 \\ c' &= 6\ 5\ 5\ 2\ 2\ 1\ 1 \\ c &= 2\ 1\ 1\ 5\ 5\ 6\ 2 \end{aligned}$$

Theorem 4.1. *For each nonnegative integer r the above mapping is a bijection of the set of all the pairs $(b, w) = (b_1b_2\dots b_n,\ x_1x_2\dots x_n) \in \mathrm{NIW}(n) \times B_n$ such that $2b_1 + \operatorname{fdes} w = r$ onto the set of the words $c = c_1c_2\dots c_n \in \mathbb{N}^n$ such that $\max c = r$. Moreover,*

$$2b_1 + \operatorname{fdes} w = \max c; \qquad 2\operatorname{tot} b + \operatorname{fmaj} w = \operatorname{tot} c; \tag{4.2}$$

$$\operatorname{lowerp} w = \operatorname{evenlower} c; \quad \operatorname{lowern} w = \operatorname{oddlower} c; \quad \operatorname{neg} w = \operatorname{odd} c. \tag{4.3}$$

Before giving the proof of Theorem 4.1 we derive its analytic consequences. First, it is q-routine to prove the three identities, where b_1 is the first letter of b,

$$\frac{1}{(u;q)_N} = \sum_{n\ge 0} \begin{bmatrix} N+n-1 \\ n \end{bmatrix}_q u^n; \tag{4.4}$$

$$\begin{bmatrix} N+n \\ n \end{bmatrix}_q = \sum_{b\in \mathrm{NIW}(N),\, b_1\le n} q^{\operatorname{tot} b}; \tag{4.5}$$

$$\frac{1}{(u;q)_{N+1}} = \sum_{n\ge 0} u^n \sum_{b\in \mathrm{NIW}(N),\, b_1\le n} q^{\operatorname{tot} b}. \tag{4.6}$$

We then consider

$$\frac{1+t}{(t^2;q^2)_{n+1}}\, {}^{\mathrm{fmaj}}B_n(t,q,X,Y,Z),$$

where

$${}^{\mathrm{fmaj}}B_n(t,q,X,Y,Z) := \sum_{w\in B_n} t^{\operatorname{fdes} w} q^{\operatorname{fmaj} w} X^{\operatorname{lowerp} w} Y^{\operatorname{lowern} w} Z^{\operatorname{neg} w},$$

which we rewrite as

$$\begin{aligned}
&\sum_{r'\geq 0}(t^{2r'}+t^{2r'+1})\begin{bmatrix} n+r' \\ r' \end{bmatrix}_{q^2} {}^{\mathrm{fmaj}}B_n(t,q,X,Y,Z) && \text{[by (4.4)]}\\
&=\sum_{r\geq 0} t^r \begin{bmatrix} n+\lfloor r/2\rfloor \\ \lfloor r/2\rfloor \end{bmatrix}_{q^2} {}^{\mathrm{fmaj}}B_n(t,q,X,Y,Z) && \text{[and by (4.5)]}\\
&=\sum_{r\geq 0} t^r \sum_{b\in \mathrm{NIW}(n),\, 2b_1\leq r} q^{2\,\mathrm{tot}\, b} \sum_{w\in B_n} t^{\mathrm{fdes}\, w} q^{\mathrm{fmaj}\, w} X^{\mathrm{lowerp}\, w} Y^{\mathrm{lowern}\, w} Z^{\mathrm{neg}\, w}\\
&=\sum_{s\geq 0} t^s \sum_{\substack{b\in \mathrm{NIW}(n),\, w\in B_n\\ 2b_1+\mathrm{fdes}\, w\leq s}} q^{2\,\mathrm{tot}\, b+\mathrm{fmaj}\, w} X^{\mathrm{lowerp}\, w} Y^{\mathrm{lowern}\, w} Z^{\mathrm{neg}\, w}.
\end{aligned}$$

By Theorem 4.1 the set $\{(b,w) : b\in \mathrm{NIW}(n),\ w\in B_n,\ 2b_1+\mathrm{fdes}\, w\leq s\}$ is in bijection with the set $\{0,1,\ldots,s\}^n$ and (4.2) and (4.3) hold. Hence,

$$\frac{1+t}{(t^2;q^2)_{n+1}} {}^{\mathrm{fmaj}}B_n(t,q,X,Y,Z)=\sum_{s\geq 0} t^s \sum_{c\in\{0,\ldots,s\}^n} q^{\mathrm{tot}\, c} X^{\mathrm{evenlower}\, c} Y^{\mathrm{oddlower}\, c} Z^{\mathrm{odd}\, c}.$$

Now use Theorem 2.2:

$$\begin{aligned}
\sum_{s\geq 0} t^s H_s(u) &= \sum_{s\geq 0} t^s \sum_{c\in\{0,1,\ldots,s\}^*} u^{|c|} q^{\mathrm{tot}\, c} X^{\mathrm{evenlower}\, c} Y^{\mathrm{oddlower}\, c} Z^{\mathrm{odd}\, c}\\
&= \sum_{n\geq 0} u^n \sum_{s\geq 0} t^s \sum_{c\in\{0,1,\ldots,s\}^n} q^{\mathrm{tot}\, c} X^{\mathrm{evenlower}\, c} Y^{\mathrm{oddlower}\, c} Z^{\mathrm{odd}\, c}\\
&= \sum_{n\geq 0} (1+t)\, {}^{\mathrm{fmaj}}B_n(t,q,X,Y,Z) \frac{u^n}{(t^2;q^2)_{n+1}}.
\end{aligned}$$

Thus, identity (4.1) is proved, as well as Theorem 1.1.

Let us now complete the proof of Theorem 4.1. First, $\max c = c'_1 = 2a_1+\epsilon_1 = 2b_1+2z_1+\epsilon_1 = 2b_1+\mathrm{fdes}\, w$; also $\mathrm{tot}\, c = \mathrm{tot}\, c' = 2\,\mathrm{tot}\, a+\mathrm{tot}\,\epsilon = 2\,\mathrm{tot}\, b+2\,\mathrm{tot}\, z+\mathrm{tot}\,\epsilon = 2\,\mathrm{tot}\, b+\mathrm{fmaj}\, w$. Hence (4.2) holds.

Next, prove that $(b,w)\mapsto c$ is bijective. As both sequences b and z are nonincreasing, $a=b+z$ is also nonincreasing. If $x_k > x_{k+1}$, then $z_k = z_{k+1}+1$ and $a_k\geq a_{k+1}+1$, then $c'_k = 2a_k+\epsilon_k \geq 2a_{k+1}+2+\epsilon_k \geq 2a_{k+1}+2 > 2a_{k+1}+\epsilon_{k+1} = c'_{k+1}$. Thus,

$$x_k > x_{k+1} \Rightarrow c'_k > c'_{k+1}. \tag{4.7}$$

To construct the reverse bijection we proceed as follows. Start with a sequence $c=c_1c_2\ldots c_n$; form the word $\delta=\delta_1\delta_2\ldots\delta_n$, where $\delta_i := \chi(c_i \text{ even}) - \chi(c_i \text{ odd})$ $(1\leq i\leq n)$ and the two-row matrix

$$\begin{pmatrix} c_1 & c_2 & \ldots & c_n \\ 1\delta_1 & 2\delta_2 & \ldots & n\delta_n \end{pmatrix}.$$

Rearrange the columns in such a way that $\binom{c_i}{i}$ occurs to the left of $\binom{c_j}{j}$, if either $c_i > c_j$, or $c_i = c_j$, $i < j$. The bottom of the new matrix is a *signed* permutation w. After those two transformations the new matrix reads

$$\begin{pmatrix} c' \\ w \end{pmatrix} = \begin{pmatrix} c'_1 & c'_2 & \ldots & c'_n \\ x_1 & x_2 & \ldots & x_n \end{pmatrix}.$$

The sequences ϵ and z are defined as above, as well as the sequence $a := (c' - \epsilon)/2$. As the sequence c' is nonincreasing, the inequality $c'_k - \epsilon_k \geq c'_{k+1} - \epsilon_{k+1}$ holds if $\epsilon_k = 0$ or if $\epsilon_k = \epsilon_{k+1} = 1$. It also holds when $\epsilon_k = 1$ and $\epsilon_{k+1} = 0$, because in such a case c'_k is odd and c'_{k+1} even and then $c'_k \geq 1 + c'_{k+1}$. Hence, a is also nonincreasing.

Finally, define $b := a - z$. Because of (4.7) we have $z_k = z_{k+1} + 1 \Rightarrow c'_k > c'_{k+1}$. If c'_k and c'_{k+1} are of the same parity, then $c'_k > c'_{k+1} \Rightarrow a_k > a_{k+1}$. If $c'_k > c'_{k+1}$ holds and the two terms are of different parity, then c'_k is even and c'_{k+1} odd. Hence, $a_k = c'_k/2 > (c'_{k+1} - \epsilon_{k+1})/2 = a_{k+1}$. Thus $z_k = z_{k+1} + 1 \Rightarrow a_k > a_{k+1}$. As $z_n = 0$, we conclude by a decreasing induction that $b_k = a_k - z_k \geq a_{k+1} - z_{k+1} = b_{k+1}$, so that b is a nonincreasing sequence of nonnegative integers.

There remains to prove (4.3). First, the letter c_j of c is odd, if and only if j occurs with the minus sign in w, so that $\operatorname{neg} w = \operatorname{odd} c$. Next, suppose that x_j is a positive lower record of $w = x_1 x_2 \ldots x_n$. For $j < i$ we have $x_j < |x_i|$. Hence, the following implications hold: $|x_i| < x_j \Rightarrow i < j \Rightarrow c'_i \geq c'_j \Rightarrow c_{|x_j|} \leq c_{|x_i|}$ and c_{x_i} is an even lower record of c. If x_j is a negative lower record of w, then $j < i \Rightarrow -x_j < |x_i|$, so that $|x_i| < -x_j \Rightarrow i < j \Rightarrow c'_i > c'_j \Rightarrow c_{|x_j|} < c_{|x_i|}$ and $c_{|x_i|}$ is an odd lower record of c. $\square$

5. The Insertion Method

Another method for proving identity (4.1) is to make use of the insertion method. Each signed permutation $w' = x'_1 \ldots x'_{n-1}$ of order $(n-1)$ gives rise to $2n$ signed permutations of order n when n or $-n$ is inserted to the left or to the right w', or between two letters of w'. Assuming that w' has a flag descent number equal to $\operatorname{fdes} w' = k$, our duty is then to watch how the statistics "fmaj", "lowerp", "lowern", "neg" are modified after the insertion of n or $-n$ into the possible n slots. Such a method has already been used by Adin et al. [ABR01], Chow and Gessel [ChGe04], Haglund et al. [HLR04], for "fmaj" only. They all have observed that for each $j = 0, 1, \ldots, 2n-1$ there is one and only one signed permutation of order n derived by the insertion of n or $-n$ whose flag-major index is increased by j.

In our case we observe that the number of positive (resp. negative) lower records remains alike, except when n (resp. $-n$) is inserted to the right of w', where it increases by 1. The number of negative letters increases only when $-n$ is inserted. For controlling "fdes" we make a distinction between the signed permutations having an even flag descent number and those having an odd one. For the former ones the first letter is positive. When n (resp. $-n$) is inserted to the left of w', the flag descent number increases by 2 (resp. by 1). For the latter ones the first letter is negative. When n (resp. $-n$) is inserted to the left of w', the flag descent number increases by 1 (resp. remains invariant).

For $n \geq 2$, $0 \leq k \leq 2n-1$ let

$$ {}^{\mathrm{fmaj}}B_{n,k} := \sum_{w \in B_n,\, \mathrm{fdes}\, w = k} q^{\mathrm{fmaj}\, w} X^{\mathrm{lowerp}\, w} Y^{\mathrm{lowern}\, w} Z^{\mathrm{neg}\, w}, $$

and ${}^{\mathrm{fmaj}}B_{0,0} := 1$, ${}^{\mathrm{fmaj}}B_{0,k} := 0$ when $k \neq 0$; ${}^{\mathrm{fmaj}}B_{1,0} = X$, ${}^{\mathrm{fmaj}}B_{1,1} = qYZ$. Making use of the observations above we easily see that the polynomials ${}^{\mathrm{fmaj}}B_{n,k}$ satisfy the same recurrence relation, displayed in (3.4), as the $B_{n,k}(q,Y,Y,Z)$.

Remark. How to compare the MacMahon Verfahren and the insertion method? Recurrence relations may be difficult to be truly verified. The first procedure has the advantage of giving both a new identity on ordinary words (Theorem 2.1) and a closed expression for the factorial generating function for the polynomials ${}^{\mathrm{fmaj}}B_n(t,q,X,Y,Z)$ (formula (1.8)).

6. Specializations

When a variable has been deleted in the following specializations of the polynomials $B_n(t,q,X,Y,Z)$, this means that the variable has been given the value 1. For instance, $B_n(q,X,Y,Z) := B_n(1,q,X,Y,Z)$.

When s tends to infinity, then $H_s(u)$ tends to $H_\infty(u)$ given in (1.1). Hence, when identity (3.1) is multiplied by $(1-t)$ and t is replaced by 1, identity (3.1) specializes into

$$ \sum_{n\geq 0} B_n(q,X,Y,Z)\frac{u^n}{(q^2;q^2)_n} = H_\infty(u) = \frac{\Bigl(uq\Bigl(\dfrac{Z+q}{1-q^2} - ZY\Bigr);q^2\Bigr)_\infty}{\Bigl(u\Bigl(\dfrac{q(Z+q)}{1-q^2}+X\Bigr);q^2\Bigr)_\infty}. \tag{6.1} $$

Now expand $H_\infty(u)$ by means of the q-binomial theorem [GaRa90, chap. 1]. We get:

$$ H_\infty(u) = \sum_{n\geq 0} \left(\frac{q\Bigl(\dfrac{Z+q}{1-q^2} - YZ\Bigr)}{\dfrac{q(Z+q)}{1-q^2}+X};q^2\right)_n \Bigl(u\Bigl(\frac{q(Z+q)}{1-q^2}+X\Bigr)\Bigr)^n \Big/ (q^2;q^2)_n. $$

By identification

$$ B_n(q,X,Y,Z) = \left(\frac{q\Bigl(\dfrac{Z+q}{1-q^2} - YZ\Bigr)}{\dfrac{q(Z+q)}{1-q^2}+X};q^2\right)_n \Bigl(\frac{q(Z+q)}{1-q^2}+X\Bigr)^n, $$

which can also be written as

$$ \begin{aligned} B_n(q,X,Y,Z) &= (X+qZ+q^2+q^3Z+\cdots+q^{2n-2}+YZq^{2n-1}) \\ &\quad \cdots \times (X+qZ+q^2+q^3Z+q^4+q^5YZ)(X+qZ+q^2+q^3YZ)(X+qYZ), \end{aligned} \tag{6.2} $$

or, by induction,

$$ B_n(q,X,Y,Z) = (X+qZ+q^2+q^3Z+\cdots+q^{2n-2}+YZq^{2n-1})B_{n-1}(q,X,Y,Z) \tag{6.3} $$

for $n \geq 2$ and $\quad B_1(q,X,Y,Z) = X + qYZ$.

In particular, the polynomial ${}^{\mathrm{fmaj}}B_n(q,X,Y)$ defined in the introduction is equal to

$$(6.4)\quad B_n(q,X,Y)=(X+q+q^2+q^3+\cdots+q^{2n-2}+Yq^{2n-1})\\ \cdots\times(X+q+q^2+q^3+q^4+q^5Y)(X+q+q^2+q^3Y)(X+qY).$$

Finally, when $X=Y=Z:=1$, identity (6.4) reads

$$(6.5)\qquad \sum_{n\geq 0}(1+t)B_n(t,q)\frac{u^n}{(t^2;q^2)_{n+1}}=\sum_{s\geq 0}t^s\frac{1}{1-u(1+q+q^2+\cdots+q^s)},$$

an identity derived by several authors (Adin et al. [ABR01], Chow & Gessel [ChGe04], Haglund et al. [HLR04]) with *ad hoc* methods. Also notice that for $X=Y=Z:=1$ formula (6.2) yields $B_n(q)=(q^2;q^2)_n/(1-q)^n$.

For $Z=0$ we get

$${}^{\mathrm{fmaj}}B_n(t,q,X,Y,0)=\sum_{w\in\mathfrak{S}_n}t^{\mathrm{fdes}\,w}q^{\mathrm{fmaj}\,w}X^{\mathrm{lowerp}\,w},$$

since the monomials corresponding to signed permutations having negative letters vanish. The summation is then over the *ordinary permutations*. Also notice that ${}^{\mathrm{fmaj}}B_n(t,q,X,Y,0)={}^{\mathrm{fmaj}}B_n(t,q,X,0,0)$. Moreover, for each ordinary permutation w we have: $\mathrm{fdes}\,w=2\,\mathrm{des}\,w$ and $\mathrm{fmaj}\,w=2\,\mathrm{maj}\,w$. When $Y=Z=0$ we also have

$$H_{2+1}(u)=H_{2s}(u)=\frac{\Bigl(\dfrac{uq^2}{1-q^2+uq^{2(s+1)}};q^2\Bigr)_{s+1}}{\Bigl(u\dfrac{X(1-q^2)+q^2}{1-q^2+uq^{2(s+1)}};q^2\Bigr)_{s+1}}$$

and

$$\sum_{n\geq 0}(1+t)\ {}^{\mathrm{fmaj}}B_n(t,q,X,0,0)\frac{u^n}{(t^2;q^2)_{n+1}}=\sum_{s\geq 0}(t^{2s}+t^{2s+1})H_{2s}(u).$$

With $A_n(t,q,X):={}^{\mathrm{fmaj}}B_n(t^{1/2},q^{1/2},X,0,0)=\sum\limits_{w\in\mathfrak{S}_n}t^{\mathrm{des}\,w}q^{\mathrm{maj}\,w}X^{\mathrm{lowerp}\,w}$ we obtain the identity

$$(6.7)\qquad \sum_{n\geq 0}A_n(t,q,X)\frac{u^n}{(t;q)_{n+1}}=\sum_{s\geq 0}t^s\frac{\Bigl(\dfrac{uq}{1-q+uq^{s+1}};q\Bigr)_{s+1}}{\Bigl(u\dfrac{X(1-q)+q}{1-q+uq^{s+1}};q\Bigr)_{s+1}},$$

apparently a new identity for the generating function for the symmetric groups $\mathfrak{S}_n$ by the three-variable statistic $(\mathrm{des},\mathrm{maj},\mathrm{lowerp})$, as well as the following one obtained by multiplying the identity by $(1-t)$ and letting t go to infinity:

$$(6.8)\qquad \sum_{n\geq 0}A_n(q,X)\frac{u^n}{(q;q)_n}=\frac{\Bigl(\dfrac{uq}{1-q};q\Bigr)_\infty}{\Bigl(uX+\dfrac{uq}{1-q};q\Bigr)_\infty}.$$

7. Lower Records and Flag-inversion Number

The purpose of this section is prove Theorem 1.2. We proceed as follows. For each (ordinary) permutation $\sigma = \sigma_1\sigma_2\ldots\sigma_n$ of order n and for each $i = 1, 2, \ldots, n$ let $b_i(\sigma)$ denote the number of letters σ_j *to the left of* σ_i such that $\sigma_j < \sigma_i$. On the other hand, for each *signed* permutation $w = x_1x_2\ldots x_n$ of order n let $\operatorname{abs} w$ denote the (ordinary) permutation $|x_1|\,|x_2|\ldots|x_n|$. It is straightforward to see that another expression for the flag-inversion number (or the length function), $\operatorname{finv} w$, of w is the following

$$\operatorname{finv} w = \operatorname{inv}\operatorname{abs} w + \sum_{1\le i\le n}(2b_i(\operatorname{abs} w)+1)\chi(x_i<0). \tag{7.1}$$

The generating polynomial ${}^{\operatorname{finv}}B_n(q, X, Y)$ can be derived as follows. Let $\operatorname{Lower}\sigma$ denote the set of the (necessarily positive) records of the ordinary permutation σ. By (7.1) we have

$$\begin{aligned}
{}^{\operatorname{finv}}B_n(q,X,Y,Z) &= \sum_{w\in B_n} q^{\operatorname{finv} w}X^{\operatorname{lowerp} w}Y^{\operatorname{lowern} w}Z^{\operatorname{neg} w}\\
&= \sum_{\sigma\in\mathfrak{S}_n}\sum_{\operatorname{abs} w=\sigma} q^{\operatorname{finv} w}X^{\operatorname{lowerp} w}Y^{\operatorname{lowern} w}Z^{\operatorname{neg} w}\\
&= \sum_{\sigma\in\mathfrak{S}_n} q^{\operatorname{inv}\sigma}\prod_{\sigma_i\in\operatorname{Lower}\sigma}(X+YZq^{2b_i(\sigma)+1})\prod_{\sigma_i\notin\operatorname{Lower}\sigma}(1+Zq^{2b_i(\sigma)+1})\\
&:= \sum_{\sigma\in\mathfrak{S}_n} f(\sigma),
\end{aligned}$$

showing that ${}^{\operatorname{finv}}B_n(q, X, Y, Z)$ is simply a generating polynomial for the group $\mathfrak{S}_n$ itself.

But $\mathfrak{S}_n$ can be generated from $\mathfrak{S}_{n-1}$ by inserting the letter n into the possible n slots of each permutation of order $n-1$. Let $\sigma' = \sigma'_1\sigma'_2\ldots\sigma'_{n-1}$ be such a permutation. Then

$$\begin{aligned}
f(\sigma'_1\sigma'_2\ldots\sigma'_{n-2}\sigma'_{n-1}n) &= f(\sigma')(X+YZq^{2n-1});\\
f(\sigma'_1\sigma'_2\ldots\sigma'_{n-2}n\sigma'_{n-1}) &= f(\sigma')q(1+Zq^{2n-3});\\
\cdots\qquad & \qquad\cdots\\
f(\sigma'_1 n\sigma'_2\ldots\sigma'_{n-2}\sigma'_{n-1}) &= f(\sigma')q^{n-2}(1+Zq^3);\\
f(n\sigma'_1\sigma'_2\ldots\sigma'_{n-2}\sigma'_{n-1}) &= f(\sigma')q^{n-1}(1+Zq).
\end{aligned}$$

Hence, ${}^{\operatorname{finv}}B_n(q,X,Y,Z) = {}^{\operatorname{finv}}B_{n-1}(q,X,Y,Z)(X+q+\cdots+q^{n-1}+q^nZ+\cdots+q^{2n-2}Z+q^{2n-1}YZ)$ $(n\ge 2)$. As ${}^{\operatorname{finv}}B_1(q,X,Y,Z) = X+qYZ$, we get the expression displayed in (1.9). $\square$

Comparing (6.2) with (1.9) we see that the two polynomials ${}^{\operatorname{finv}}B_n(q, Z)$ and $B_n(q, Z)$ are different as soon as $n \ge 2$, while ${}^{\operatorname{finv}}B_n(q, X, Y) = B_n(q, X, Y)$ for all n. This means that any bijection Ψ of the group B_n onto itself having the property that

$$\operatorname{fmaj} w = \operatorname{finv}\Psi(w)$$

does not leave the number of negative letters "neg" invariant. It was, in particular, the case for the bijection constructed in our previous paper [FoHa05].

Remark 1. Instead of considering lower records from right to left we can introduce lower records *from left to right*. Let $\mathrm{lowerp}' w$ and $\mathrm{lowern}' w$ denote the numbers of such records, positive and negative, respectively and introduce

$$^{\mathrm{finv}}B_n(q,X,Y,X',Y'){:=}\sum_{w\in B_n} q^{\mathrm{finv}\, w} X^{\mathrm{lowerp}\, w} Y^{\mathrm{lowern}\, w} X'^{\,\mathrm{lowerp}'\, w} Y'^{\,\mathrm{lowern}'\, w}.$$

Using the method developed in the proof of the previous theorem we can calculate this polynomial in the form

$$\begin{aligned} &^{\mathrm{finv}}B_n(q,X,Y,X',Y') \\ &\quad = (X+q+\cdots+q^{n-2}+q^{n-1}X'+q^nY'+q^{n+1}+\cdots+q^{2n-2}+q^{2n-1}Y) \\ &\qquad \cdots \times (X+q+q^2X'+q^3Y'+q^4+q^5Y)(X+qX'+q^2Y'+q^3Y)(XX'+qYY'). \end{aligned}$$

Remark 2. The same method may be used for ordinary permutations. For each permutation σ let $\mathrm{upper}\,\sigma$ denote the number of its *upper* records *from right to left* and define:

$$^{\mathrm{inv}}A_n(q,X,V) := \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{inv}\,\sigma} X^{\mathrm{lowerp}\,\sigma} V^{\mathrm{upper}\,\sigma}.$$

Then

$$^{\mathrm{inv}}A_n(q,X,V) = XV(X+qV)(X+q+q^2V)\cdots(X+q+q^2+\cdots+q^{n-1}V),$$

an identity which can be put into the form

$$\text{(7.2)}\qquad \sum_{n\ge 0} {}^{\mathrm{inv}}A_n(q,X,V)\frac{u^n}{(q;q)_n} = 1-\frac{XV}{X+V-1}+\frac{XV}{X+V-1}\,\frac{\Bigl(\dfrac{u}{1-q}-ux;q\Bigr)_\infty}{\Bigl(uX+\dfrac{uq}{1-q};q\Bigr)_\infty},$$

which specializes into

$$\text{(7.3)}\qquad \sum_{n\ge 0} {}^{\mathrm{inv}}A_n(q,X)\frac{u^n}{(q;q)_n} = \frac{\Bigl(\dfrac{uq}{1-q};q\Bigr)_\infty}{\Bigl(uX+\dfrac{uq}{1-q};q\Bigr)_\infty};$$

$$\text{(7.4)}\qquad \sum_{n\ge 0} {}^{\mathrm{inv}}A_n(q,V)\frac{u^n}{(q;q)_n} = \frac{\Bigl(\dfrac{u}{1-q}-uV;q\Bigr)_\infty}{\Bigl(\dfrac{u}{1-q};q\Bigr)_\infty}.$$

Comparing (7.3) with (6.8) we then see that $A_n(q,X) = {}^{\mathrm{inv}}A_n(q,X)$. In other words, the generating polynomial for $\mathfrak{S}_n$ by $(\mathrm{maj},\mathrm{lowerp})$ is the same as the generating polynomial by $(\mathrm{inv},\mathrm{lowerp})$. A combinatorial proof of this result is due to Björner and Wachs [BjW88], who have made use of the transformation constructed in [Fo68]. Finally, the expression of ${}^{\mathrm{inv}}A_n(q,X,V)$ for $q=1$ was derived by David and Barton ([DaBa62], chap. 10).

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SIGNED WORDS AND PERMUTATIONS, III; THE MACMAHON VERFAHREN

Dominique Foata and Guo-Niu Han

Glory to Viennot,	*No one can beat him.*
Wizard of Bordeaux,	*Only verbatim*
A prince in Physics,	*Can we follow him.*
In Mathematics,	*He has admirers,*
Combinatorics,	*Who have got down here.*
Even in Graphics,	*They all celebrate*
Sure, in Viennotics.	*Such a happy fate.*
He builds bijections,	*Sixty years have gone,*
Top calculations.	*He still is our don.*

Dedicated to Xavier.
Lucelle, April 2005.

Abstract. The MacMahon Verfahren is fully exploited to derive further multivariable generating functions for the hyperoctahedral group B_n and for rearrangements of signed words. Using the properties of the fundamental transformation the generating polynomial for B_n by the flag-major and inverse flag-major indices can be derived, and also by the length function, associated with the flag-descent number.

1. Introduction

To paraphrase Leo Carlitz [Ca56], the present paper could have been entitled "Expansions of certain products," as we want to expand the product

$$K_\infty(u) := \prod_{i\ge 0, j\ge 0} \frac{1}{(1-uZ_{ij}q_1^i q_2^j)}, \tag{1.1}$$

in its infinite version, and

$$K_{r,s}(u) := \prod_{0\le i\le r,\, 0\le j\le s} \frac{1}{(1-uZ_{ij}q_1^i q_2^j)}, \tag{1.2}$$

in its graded version, where

$$Z_{ij} := \begin{cases} Z, & \text{if } i \text{ and } j \text{ are both odd;}\\ 1, & \text{if } i \text{ and } j \text{ are both even;}\\ 0, & \text{if } i \text{ and } j \text{ have different parity.}\end{cases}$$

The second pair under study, which depends on r variables $u_1, \ldots, u_r$, reads

$$(1.3)\qquad L_\infty(u_1,\ldots,u_r) := \prod_{1\le i\le r} \frac{1}{(u_i;q^2)_\infty}\frac{1}{(u_iqZ;q^2)_\infty},$$

$$(1.4)\qquad L_s(u_1,\ldots,u_r) := \prod_{1\le i\le r} \frac{1}{(u_i;q^2)_{\lfloor s/2\rfloor+1}}\frac{1}{(u_iqZ;q^2)_{\lfloor (s+1)/2\rfloor}}.$$

In those expressions we have used the usual notations for the q-*ascending factorial*

$$(1.5)\qquad (a;q)_n := \begin{cases} 1, & \text{if } n=0;\\ (1-a)(1-aq)\ \ldots\ (1-aq^{n-1}), & \text{if } n\ge 1;\end{cases}$$

in its *finite* form and

$$(1.6)\qquad (a;q)_\infty := \lim_n (a;q)_n = \prod_{n\ge 0}(1-aq^n);$$

in its *infinite* form.

All those products will be the basic ingredients for deriving the distributions of various statistics attached to signed permutations and words. By *signed word* we understand a word $w = x_1x_2\ldots x_m$, whose letters are integers, positive or negative. If $\mathbf{m} = (m_1, m_2, \ldots, m_r)$ is a sequence of nonnegative integers such that $m_1+m_2+\cdots+m_r = m$, let $B_\mathbf{m}$ be the set of rearrangements $w = x_1x_2\ldots x_m$ of the sequence $1^{m_1}2^{m_2}\ldots r^{m_r}$, with the convention that some letters i $(1\le i\le r)$ may be replaced by their opposite values $-i$. For typographical reasons we shall use the notation $\overline{i} := -i$ in the sequel. When $m_1 = m_2 = \cdots = m_r = 1$, the class $B_\mathbf{m}$ is simply the hyperoctahedral group B_m (see [Bo68], p. 252-253) of the signed permutations of order m $(m=r)$.

Using the χ-notation that maps each statement A onto the value $\chi(A) = 1$ or 0 depending on whether A is true or not, we recall that the usual *inversion number*, $\operatorname{inv} w$, of each signed word $w = x_1x_2\ldots x_n$ is defined by

$$\operatorname{inv} w := \sum_{1\le j\le n}\sum_{i<j}\chi(x_i > x_j).$$

It also makes sense to introduce

$$\overline{\operatorname{inv}}\, w := \sum_{1\le j\le n}\ \sum_{i<j}\chi(\overline{x}_i > x_j),$$

and define the *flag-inversion number* of w by

$$\operatorname{finv} w := \operatorname{inv} w + \overline{\operatorname{inv}}\, w + \operatorname{neg} w,$$

where $\operatorname{neg} w := \sum_{1\le j\le n}\chi(x_j < 0)$. As noted in our previous paper [FoHa05a], the flag-inversion number coincides with the traditional *length function* ℓ, when applied to each signed permutation.

The *flag-major index* "fmaj" and the *flag descent number* "fdes", which were introduced by Adin and Roichman [AR01] for signed permutations, are also valid for signed words. They read

$$\begin{aligned} \operatorname{fmaj} w &:= 2 \operatorname{maj} w + \operatorname{neg} w; \\ \operatorname{fdes} w &:= 2 \operatorname{des} w + \chi(x_1 < 0); \end{aligned}$$

where $\operatorname{maj} w := \sum_j j\, \chi(x_j > x_{j+1})$ denotes the usual *major index* of w and $\operatorname{des} w$ the *number of descents* $\operatorname{des} w := \sum_j \chi(x_j > x_{j+1})$. Finally, for each signed *permutation* w let w^{-1} denote the inverse of w and define $\operatorname{ifdes} w := \operatorname{fdes} w^{-1}$ and $\operatorname{ifmaj} := \operatorname{fmaj} w^{-1}$.

Notations. In the sequel B_n (resp. $B_{\mathbf{m}}$) designates the hyperoctahedral group of order n (resp. the set of signed words of multiplicity $\mathbf{m} = (m_1, m_2, \ldots, m_r)$), as defined above. Each generating polynomial for B_n (resp. for $B_{\mathbf{m}}$) by some k-variable statistic will be denoted by $B_n(t_1, \ldots, t_k)$ (resp. $B_{\mathbf{m}}(t_1, \ldots, t_k)$). When the variable t_i is missing in the latter expression, it means that the variable t_i is given the value 1.

The main two results of this paper corresponding to the two pairs of products earlier introduced can be stated as follows.

Theorem 1.1. *Let*

$$B_n(t_1, t_2, q_1, q_2, Z) := \sum_{w \in B_n} t_1^{\operatorname{fdes} w} t_2^{\operatorname{ifdes} w} q_1^{\operatorname{fmaj} w} q_2^{\operatorname{ifmaj} w} Z^{\operatorname{neg} w} \tag{1.7}$$

be the generating polynomial for the group B_n by the five-variable statistic (fdes, ifdes, fmaj, ifmaj, neg). *Then,*

$$\sum_{n \ge 0} \frac{u^n}{(t_1^2; q_1^2)_{n+1} (t_2^2; q_2^2)_{n+1}} (1 + t_1)(1 + t_2) B_n(t_1, t_2, q_1, q_2, Z) = \sum_{r \ge 0, s \ge 0} t_1^r t_2^s\, K_{r,s}(u), \tag{1.8}$$

where $K_{r,s}(u)$ is defined in (1.2).

Theorem 1.2. *For each sequence $\mathbf{m} = (m_1, \ldots, m_r)$ let*

$$B_{\mathbf{m}}(t, q, Z) := \sum_{w \in B_{\mathbf{m}}} t^{\operatorname{fdes} w} q^{\operatorname{fmaj} w} Z^{\operatorname{neg} w} \tag{1.9}$$

be the generating polynomial for the class $B_{\mathbf{m}}$ of signed words by the three-variable statistic (fdes, fmaj, neg). *Then*

$$\sum_{\mathbf{m}} (1 + t) B_{\mathbf{m}}(t, q, Z) \frac{u_1^{m_1} \cdots u_r^{m_r}}{(t^2; q^2)_{1 + |\mathbf{m}|}} = \sum_{s \ge 0} t^s L_s(u_1, \ldots, u_r), \tag{1.10}$$

where $|\mathbf{m}| := m_1 + \cdots + m_r$ and $L_s(u_1, \ldots, u_r)$ is defined in (1.4).

It is worth noticing that Reiner [Re93a] has calculated the generating polynomial for B_n by another 5-variable statistic involving each signed permutation and its inverse. The bibasic series he has used are normalized by products of the form $(t_1;q_1)_{n+1}(t_2;q_2)_n$ instead of $(t_1^2;q_1^2)_{n+1}(t_2^2;q_2^2)_{n+1}$.

Using the properties of the fundamental transformation on signed words described in our first paper [FoHa05a] we obtain the following specialization of Theorem 1.1. Let

$$ {}^{\mathrm{finv}}B_n(t,q) := \sum_{w\in B_n} t^{\mathrm{fdes}\, w} q^{\mathrm{finv}\, w} \tag{1.11}$$

be the generating polynomial for the group B_n by the pair $(\mathrm{fdes}, \mathrm{finv})$. Then, the q-factorial generating function for the polynomials ${}^{\mathrm{finv}}B_n(t,q)$ $(n\ge 0)$ has the following form:

$$\begin{aligned}(1.12)\qquad \sum_{n\ge 0} \frac{v^n}{(q^2;q^2)_n} {}^{\mathrm{finv}}B_n(t,q) \\ = \frac{1-t}{-t^2+(v(1-t^2);q)_\infty}\big(t+(v(1-t^2)q;q^2)_\infty\big).\end{aligned}$$

From identity (1.12) we deduce the generating function for the polynomials ${}^{\mathrm{dess}}B_n(t,q) := \sum_{w\in B_n} t^{\mathrm{dess}\, w} q^{\mathrm{finv}\, w}$, where "dess" is the traditional number of descents for signed permutations defined by

$$\mathrm{dess}\, w := \mathrm{des}\, w + \chi(x_1<0) \quad \text{instead of} \quad \mathrm{fdes}\, w := 2\,\mathrm{des}\, w + \chi(x_1<0)$$

and recover Reiner's identity [Re93a]

$$\sum_{n\ge 0} \frac{u^n}{(q^2;q^2)_n} {}^{\mathrm{dess}}B_n(t,q) = \frac{1-t}{1-t\, e_q(u(1-t))} \frac{1}{(v(1-t);q^2)_\infty}, \tag{1.13}$$

where $e_q(v(1-t^2)) = 1/(v(1-t^2);q)_\infty$ is the traditional q-exponential. This is done in section 4.

As in our second paper [FoHa05b] we make use of the *MacMahon Verfahren* technique to prove the two theorems, which consists of transferring the topology of the signed permutations or words measured by the various statistics, "fdes", "fmaj", to a set of pairs of matrices with integral entries in the case of Theorem 1.1 and a set of plain words in the case of Theorem 1.2 for which the calculation of the associated statistic is easy. Each time there is then a combinatorial bijection between signed permutations (resp. words) and pairs of matrices (resp. plain words) that has the adequate properties. This is the content of Theorem 3.1 and Theorem 4.1.

In all our results we have tried to include the variable Z that takes the number "neg" of *negative* letters of each signed permutation or word into account. This allows us to re-obtain the classical results on the symmetric group and sets of words.

In the next section we recall the *MacMahon Verfahren* technique, which was developed in our previous paper [FoHa05b] for signed permutations. Notice that Reiner [Re93a, Re93b, Re93c, Re95a, Re95b], extending Stanley's [St72] (P,ω)-partition approach, has successfully developed a (P,ω)-partition theory for the combinatorial study of the hyperoctahedral group, which could have been used in this paper. Section 3 contains the proof of Theorem 1.1, whose specializations are given in Section 4. We end the paper with the proof of Theorem 1.2 and its specializations. Noticeably, the generating polynomial for the class $B_{\mathbf{m}}$ of signed *words* by the two-variable statistic $(\mathrm{fdes}, \mathrm{fmaj})$ is completely explicit, in the sense that we derive the factorial generating function for those polynomials and also a recurrence relation, while only the generating function given by (1.10) has a simple form when the variable Z is kept.

2. The MacMahon Verfahren

Let $\mathbb{N}^n$ (resp. $\mathrm{NIW}(n)$) be the set of words (resp. nonincreasing words) of length n, whose letters are nonnegative integers. As done in [FoHa05b] the MacMahon Verfahren consists of mapping each pair $(b,w) \in \mathrm{NIW}(n) \times B_n$ onto a word $c \in \mathbb{N}^n$ as follows. Write the signed permutation w as the linear word $w = x_1x_2\ldots x_n$, where x_k is the image of the integer k $(1 \le k \le n)$. For each $k = 1, 2, \ldots, n$ let z_k be the number of descents in the right factor $x_kx_{k+1}\ldots x_n$ and ϵ_k be equal to 0 or 1 depending on whether x_k is positive or negative. Next, form the words $z(w) := z_1z_2\ldots z_n$ and $\epsilon(w) := \epsilon_1\epsilon_2\ldots\epsilon_n$.

Now, take a nonincreasing word $b = b_1b_2\ldots b_n$ and define $a_k := b_k + z_k$, $c'_k := 2a_k + \epsilon_k$ $(1 \le k \le n)$, then $a(b,w) := a_1a_2\ldots a_n$ and $c'(b,w) := c'_1c'_2\ldots c'_n$. Finally, form the two-row matrix $\begin{pmatrix} c'_1 & c'_2 & \ldots & c'_n \\ |x_1| & |x_2| & \ldots & |x_n| \end{pmatrix}$. Its bottom row is a permutation of $1\,2\ldots n$; rearrange the columns in such a way that the bottom row is precisely $1\,2\ldots n$. Then the word $c(b,w) = c_1c_2\ldots c_n$ which corresponds to the pair (b,w) is defined to be the top row in the resulting matrix.

Example. Start with the pair (b,w) below and calculate all the necessary ingredients:

$$\begin{array}{rcl}
b & = & 4\ 2\ 2\ 1\ 1\ 0\ 0 \\
w & = & 3\ \overline{5}\ \overline{1}\ 6\ 7\ \overline{4}\ 2 \\
z(w) & = & 2\ 1\ 1\ 1\ 1\ 0\ 0 \\
\epsilon(w) & = & 0\ 1\ 1\ 0\ 0\ 1\ 0 \\
a(b,w) & = & 6\ 3\ 3\ 2\ 2\ 0\ 0 \\
c'(b,w) & = & 12\ 7\ 7\ 4\ 4\ 1\ 0 \\
c(b,w) & = & 7\ 0\ 12\ 1\ 7\ 4\ 4
\end{array}$$

For each $c = c_1 \dots c_n \in \mathbb{N}^n$ and let $\operatorname{tot} c := c_1 + \dots + c_n$, $\max c := \max\{c_1, \dots, c_n\}$ and let $\operatorname{odd} c$ denote the number of *odd* letters in c . The proof of the following theorem can be found in [FoHa05b, Theorem 4.1].

Theorem 2.1. *For each nonnegative integer s the above mapping is a bijection $(b, w) \mapsto c(b, w)$ of the set of pairs*

$$(b, w) = (b_1 b_2 \dots b_n,\ x_1 x_2 \dots x_n) \in \mathrm{NIW}(n) \times B_n$$

such that $2b_1 + \operatorname{fdes} w = s$ onto the set of words $c = c_1 c_2 \dots c_n \in \mathbb{N}^n$ such that $\max c = s$. Moreover,

$$\text{(2.1)} \quad 2b_1 + \operatorname{fdes} w = \max c(b, w); \quad 2 \operatorname{tot} b + \operatorname{fmaj} w = \operatorname{tot} c(b, w);$$
$$\operatorname{neg} w = \operatorname{odd} c(b, w).$$

We end this section by quoting the following classical identity, where b_1 is the first letter of b.

$$\text{(2.2)} \qquad \frac{1}{(u;q)_{n+1}} = \sum_{s \ge 0} u^s \sum_{b \in \mathrm{NIW}(n),\, b_1 \le s} q^{\operatorname{tot} b}.$$

3. Proof of Theorem 1.1

We apply the MacMahon Verfahren just described to the two pairs (b, w) and (β, w^{-1}), where b and β are two nonincreasing words. The pair (b, w) (resp. (β, w^{-1})) is mapped onto a word $c := c(b, w) = c_1 c_2 \dots c_n$ (resp. $C := c(\beta, w^{-1}) = C_1 C_2 \dots C_n$) of length n, with the properties

$$\text{(3.1)} \qquad 2b_1 + \operatorname{fdes} w = \max c; \quad 2 \operatorname{tot} b + \operatorname{fmaj} w = \operatorname{tot} c;$$

$$\text{(3.2)} \qquad 2\beta_1 + \operatorname{ifdes} w = \max C; \quad 2 \operatorname{tot} \beta + \operatorname{ifmaj} w = \operatorname{tot} C.$$

There remains to investigate the relation between the two words c and C before pursuing the calculation. Form the two-row matrix

$$\begin{pmatrix} c' \\ C \end{pmatrix} = \begin{pmatrix} c'_1 & c'_2 & \dots & c'_n \\ C_1 & C_2 & \dots & C_n \end{pmatrix},$$

where $c' = c'_1 c'_2 \dots c'_n$ designates the nonincreasing rearrangement of the word c. The following two properties can be readily verified:

(i) *for each $i = 1, 2, \dots, n$ the letters c'_i and C_i are of the same parity;*
(ii) *if $c'_i = c'_{i+1}$ is even (resp. is odd), then $C_i \ge C_{i+1}$ (resp. $C_i \le C_{i+1}$).*

Conversely, the following property holds:

(iii) *if $\binom{c'}{C}$ is a two-row matrix of length n having properties* (i) *and* (ii), *there exists one and only one signed permutation w and two nonincreasing words b, β satisfying* (3.1) and (3.2).

Example. We take the same example for w as in the previous section, but we also calculate $C = c(\beta, w^{-1})$.

$$\begin{array}{rl}
b & = 4\ 2\ 2\ 1\ 1\ 0\ 0 \\
w & = 3\ \overline{5}\ \overline{1}\ 6\ 7\ \overline{4}\ 2 \\
z(w) & = 2\ 1\ 1\ 1\ 1\ 0\ 0 \\
\epsilon(w) & = 0\ 1\ 1\ 0\ 0\ 1\ 0 \\
a(b,w) & = 6\ 3\ 3\ 2\ 2\ 0\ 0 \\
c' := c'(b,w) & = 12\ 7\ 7\ 4\ 4\ 1\ 0 \\
c := c(b,w) & = 7\ 0\ 12\ 1\ 7\ 4\ 4
\end{array}
\qquad
\begin{array}{rl}
\beta & = 3\ 2\ 1\ 1\ 0\ 0\ 0 \\
w^{-1} & = \overline{3}\ 7\ 1\ \overline{6}\ \overline{2}\ 4\ 5 \\
z(w^{-1}) & = 2\ 2\ 1\ 0\ 0\ 0\ 0 \\
\epsilon(w^{-1}) & = 1\ 0\ 0\ 1\ 1\ 0\ 0 \\
a(\beta,w^{-1}) & = 5\ 4\ 2\ 1\ 0\ 0\ 0 \\
C' := c'(\beta,w^{-1}) & = 11\ 8\ 4\ 3\ 1\ 0\ 0 \\
C := c(\beta,w^{-1}) & = 4\ 1\ 11\ 0\ 0\ 3\ 8
\end{array}$$

$$\binom{c'}{C} = \begin{pmatrix} 12 & 7 & 7 & 4 & 4 & 1 & 0 \\ 4 & 1 & 11 & 0 & 0 & 3 & 8 \end{pmatrix}.$$

Let $(i_1 < i_1 < \cdots < i_r)$ (resp. $(j_1 < j_2 < \cdots < j_s)$) be the increasing sequence of the integers i (resp. the integers j) such that c'_i is *even* (resp. c'_j is *odd*). Define ("e" for "even" and "o" for "odd")

$$2d^e = \binom{2f^e}{2g^e} := \begin{pmatrix} c'_{i_1} & c'_{i_2} & \ldots & c'_{i_r} \\ C_{i_1} & C_{i_2} & \ldots & C_{i_r} \end{pmatrix};$$
$$2d^o + 1 = \binom{2f^o + 1}{2g^o + 1} := \begin{pmatrix} c'_{j_1} & c'_{j_2} & \ldots & c'_{j_s} \\ C_{j_1} & C_{j_2} & \ldots & C_{j_s} \end{pmatrix};$$

so that the two two-row matrices

$$d^e = \binom{f^e}{g^e} := \begin{pmatrix} c'_{i_1}/2 & c'_{i_2}/2 & \ldots & c'_{i_r}/2 \\ C_{i_1}/2 & C_{i_2}/2 & \ldots & C_{i_r}/2 \end{pmatrix},$$
$$d^o = \binom{f^o}{g^o} := \begin{pmatrix} (c'_{j_1}-1)/2 & (c'_{j_2}-1)/2 & \ldots & (c'_{j_s}-1)/2 \\ (C_{j_1}-1)/2 & (C_{j_2}-1)/2 & \ldots & (C_{j_s}-1)/2 \end{pmatrix},$$

may be regarded as another expression for the two-row matrix $\binom{c'}{C}$.

Define the integers $i_{\max}$ and $j_{\max}$ by:

$$i_{\max} := \begin{cases} (\max c - 1)/2, & \text{if } \max c \text{ is odd;} \\ \max c/2, & \text{if } \max c \text{ is even;} \end{cases}$$
$$j_{\max} := \begin{cases} (\max C - 1)/2, & \text{if } \max C \text{ is odd;} \\ \max C/2, & \text{if } \max C \text{ is even.} \end{cases}$$

Then, to the pair d^e, d^o there corresponds a pair of unique *finite* matrices $D^e = (d^e_{ij})$, $D^o = (d^o_{ij})$ $(0 \le i \le i_{\max}, 0 \le j \le j_{\max})$ (and no other pair of smaller dimensions), where d^e_{ij} (resp. d^o_{ij}) is equal to the number of the two-letters in d^e (resp. in d^o) that are equal to $\binom{i}{j}$.

On the other hand, with $|f|$ designating the length of the word f,

$$\begin{aligned}\operatorname{tot} c &= \operatorname{tot} 2f^e + \operatorname{tot}(2f^o+1) = 2\operatorname{tot} f^e + 2\operatorname{tot} f^o + |f^o|\\ &= 2\sum_{i,j} i\, d^e_{ij} + 2\sum_{i,j} i\, d^o_{ij} + \sum_{i,j} d^o_{ij};\\ \operatorname{tot} C &= \operatorname{tot} 2g^e + \operatorname{tot}(2g^o+1) = 2\operatorname{tot} g^e + 2\operatorname{tot} g^o + |g^o|\\ &= 2\sum_{i,j} j\, d^e_{ij} + 2\sum_{i,j} j\, d^o_{ij} + \sum_{i,j} d^o_{ij}.\end{aligned}$$

The following proposition follows from Theorem 2.1.

Proposition 3.1. *For each pair of nonnegative integers* (r,s) *the map* $(b,\beta,w) \mapsto (D^e, D^o)$ *is a bijection of the triples* (b,β,w) *such that* $2b_1 + \operatorname{fdes} w \le r$, $2\beta_1 + \operatorname{ifdes} w \le s$ *onto the pairs of matrices* $D^e = (d^e_{i,j})$, $D^o = (d^o_{i,j})$ $(0 \le i \le r, 0 \le j \le s)$. *Moreover,*

$$2\operatorname{tot} b + \operatorname{fmaj} w = 2\sum_{i,j} i\, d^e_{ij} + 2\sum_{i,j} i\, d^o_{ij} + \sum_{i,j} d^o_{ij}; \tag{3.3}$$

$$2\operatorname{tot} \beta + \operatorname{ifmaj} w = 2\sum_{i,j} j\, d^e_{ij} + 2\sum_{i,j} j\, d^o_{ij} + \sum_{i,j} d^o_{ij}; \tag{3.4}$$

$$\operatorname{neg} w = \sum_{i,j} d^o_{ij}. \tag{3.5}$$

$$\sum_{i,j} d^e_{ij} + \sum_{i,j} d^o_{ij} = |w|. \tag{3.6}$$

Again work with the same example as above. To the two-row matrix

$$\begin{pmatrix} c' \\ C \end{pmatrix} = \begin{pmatrix} 12 & 7 & 7 & 4 & 4 & 1 & 0 \\ 4 & 1 & 11 & 0 & 0 & 3 & 8 \end{pmatrix}$$

there corresponds the pair

$$d^e = \begin{pmatrix} 6 & 2 & 2 & 0 \\ 2 & 0 & 0 & 4 \end{pmatrix}, \quad d^o = \begin{pmatrix} 3 & 3 & 0 \\ 0 & 5 & 1 \end{pmatrix}.$$

As $\max c = 12$ is even (resp. $\max C = 11$ is odd), we have $i_{\max} = 6$, $j_{\max} = 5$ and

$$D^e = \begin{array}{c} \\ 0\\1\\2\\3\\4\\5\\6 \end{array}\!\!\begin{array}{c} \begin{array}{cccccc} 0&1&2&3&4&5 \end{array} \\ \begin{pmatrix} .&.&.&.&1&.\\ .&.&.&.&.&.\\ 2&.&.&.&.&.\\ .&.&.&.&.&.\\ .&.&.&.&.&.\\ .&.&.&.&.&.\\ .&.&1&.&.&. \end{pmatrix}\end{array}; \quad D^o = \begin{array}{c} \\ 0\\1\\2\\3\\4\\5\\6 \end{array}\!\!\begin{array}{c} \begin{array}{cccccc} 0&1&2&3&4&5 \end{array} \\ \begin{pmatrix} .&1&.&.&&.\\ .&.&.&.&.&.\\ .&.&.&.&.&.\\ 1&.&.&.&.&1\\ .&.&.&.&.&.\\ .&.&.&.&.&.\\ .&.&.&.&.&. \end{pmatrix}\end{array}.$$

Also verify that $2\operatorname{tot} b + \operatorname{fmaj} w = 35 = 2\times(2+2+6) + 2\times(3+3)+3$; $2\operatorname{tot}\beta + \operatorname{ifmaj} w = 27 = 2\times(2+4) + 2\times(1+5) + 3$ and $\operatorname{neg} w = 3$.

In the following summations b and β run over the set of nonincreasing words of length n. By using identity (2.2) we have

$$\begin{aligned}
&\sum_{n\ge 0}\frac{u^n}{(t_1^2;q_1^2)_{n+1}(t_2^2;q_2^2)_{n+1}}(1+t_1)(1+t_2)B_n(t_1,t_2,q_1,q_2,Z)\\
&=\sum_{n\ge 0}u^n(1+t_1)(1+t_2)B_n(t_1,t_2,q_1,q_2,Z)\sum_{\substack{r'\ge 0,\, s'\ge 0\\ b,\beta,b_1\le r',\,\beta_1\le s'}} t_1^{2r'}t_2^{2s'}q_1^{2\,\mathrm{tot}\,b}q_2^{2\,\mathrm{tot}\,\beta}\\
&=\sum_{n,r',s'}u^n(t_1^{2r'}+t_1^{2r'+1})(t_2^{2s'}+t_2^{2s'+1})\\
&\qquad\times\sum_{\substack{w\in B_n\\ b,\beta,b_1\le r',\,\beta_1\le s'}} t_1^{\mathrm{fdes}\,w}t_2^{\mathrm{ifdes}\,w}q_1^{\mathrm{fmaj}\,w+2\,\mathrm{tot}\,b}q_2^{\mathrm{ifmaj}\,w+2\,\mathrm{tot}\,\beta}Z^{\mathrm{neg}\,w}\\
&=\sum_{n,r',s'}u^n t_1^{r'}t_2^{s'}\sum_{\substack{w\in B_n\\ b,\beta,2b_1\le r',\,2\beta_1\le s'}} t_1^{\mathrm{fdes}\,w}t_2^{\mathrm{ifdes}\,w}q_1^{\mathrm{fmaj}\,w+2\,\mathrm{tot}\,b}q_2^{\mathrm{ifmaj}\,w+2\,\mathrm{tot}\,\beta}Z^{\mathrm{neg}\,w}\\
&=\sum_{n,r,s}u^n t_1^{r}t_2^{s}\sum_{\substack{w\in B_n,b,\beta\\ 2b_1+\mathrm{fdes}\,w\le r,\,2\beta_1+\mathrm{ifdes}\,w\le s}} q_1^{\mathrm{fmaj}\,w+2\,\mathrm{tot}\,b}q_2^{\mathrm{ifmaj}\,w+2\,\mathrm{tot}\,\beta}Z^{\mathrm{neg}\,w}.
\end{aligned}$$

We can continue the calculation by using (3.3), (3.4), (3.5) the last summation being over matrices D^e, D^o of dimensions $(r+1)\times(s+1)$, that is,

$$\begin{aligned}
&=\sum_{n,r,s}u^n t_1^r t_2^s\sum_{D^e,D^o} q_1^{2\Sigma\, id^e_{ij}+2\Sigma\, id^o_{ij}+\Sigma\, d^o_{ij}}q_2^{2\Sigma\, jd^e_{ij}+2\Sigma\, jd^o_{ij}+\Sigma\, d^o_{ij}}Z^{\Sigma\, d^o_{ij}}\\
&=\sum_{r,s}t_1^r t_2^s\sum_{D^e}u^{\Sigma\, d^e_{ij}}q_1^{\Sigma\,(2i)d^e_{ij}}q_2^{\Sigma\,(2j)d^e_{ij}}\sum_{D^o}u^{\Sigma\, d^o_{ij}}q_1^{\Sigma\,(2i+1)d^o_{ij}}q_2^{\Sigma\,(2j+1)d^o_{ij}}Z^{\Sigma\, d^o_{ij}}\\
&=\sum_{r,s}t_1^r t_2^s\sum_{D^e}\prod_{ij}(uq_1^{2i}q_2^{2j})^{d^e_{ij}}\sum_{D^o}\prod_{ij}(uZq_1^{2i+1}q_2^{2j+1})^{d^o_{ij}}\quad(0\le i\le r,0\le j\le s)\\
&=\sum_{r,s}t_1^r t_2^s\frac{1}{\prod_{ij}(1-uq_1^{2i}q_2^{2j})}\frac{1}{\prod_{ij}(1-uZq_1^{2i+1}q_2^{2j+1})}\quad(0\le i\le r,0\le j\le s)\\
&=\sum_{r\ge 0,s\ge 0}t_1^r t_2^s\frac{1}{\prod_{0\le i\le r,0\le j\le s}(1-uZ_{ij}q_1^iq_2^j)}.\quad\square
\end{aligned}$$

4. Specializations and Flag-inversion number

First, when $Z:=0$ and t_1, t_2, q_1, q_2 are replaced by their square roots, identity (1.8) becomes

$$\sum_{n\ge 0}\frac{u^n}{(t_1;q_1)_{n+1}(t_2;q_2)_{n+1}}A_n(t_1,t_2,q_1,q_2)=\sum_{r\ge 0,s\ge 0}\frac{t_1^rt_2^s}{(u;q_1,q_2)_{r+1,s+1}},\tag{4.1}$$

where $A_n(t_1,t_2,q_1,q_2)$ is the generating polynomial for the symmetric group $\mathfrak{S}_n$ by the quadruple $(\mathrm{des},\mathrm{ides},\mathrm{maj},\mathrm{imaj})$, an identity first derived by Garsia and Gessel [GaGe78]. Other approaches can be found in [Ra79], [DeFo85].

Remember that when a variable is missing in $B_n(t_1,t_2,q_1,q_2,Z)$ it means that the variable has been given the value 1. Multiply both sides of (1.8) by $(1-t_2)$ and let $t_2=1$. We get:

$$\sum_{n\ge 0}\frac{u^n}{(t_1^2;q_1^2)_{n+1}(q_2^2;q_2^2)_n}(1+t_1)B_n(t_1,q_1,q_2,Z) = \sum_{r\ge 0} t_1^r \frac{1}{\prod\limits_{0\le i\le r,\, j\ge 0}(1-uZ_{ij}q_1^iq_2^j)}. \tag{4.2}$$

Again multiply both sides by $(1-t_1)$ and let $t_1=1$:

$$\sum_{n\ge 0}\frac{u^n}{(q_1^2;q_1^2)_n(q_2^2;q_2^2)_n}B_n(q_1,q_2,Z) = \frac{1}{\prod\limits_{i\ge 0,\, j\ge 0}(1-uZ_{ij}q_1^iq_2^j)}. \tag{4.3}$$

Also the (classical) generating function for the polynomials $A_n(q_1,q_2)$ can be derived from identity (4.3) and reads:

$$\sum_{n\ge 0}\frac{u^n}{(q_1;q_1)_n(q_2;q_2)_n}A_n(q_1,q_2) = \frac{1}{(u;q_1,q_2)_{\infty,\infty}}.$$

With $q_1=1$ the denominator of the fraction on the right side of identity (4.2) becomes

$$\begin{cases} (u;q_2^2)_\infty^{(r/2)+1}(uq_2Z;q_2^2)_\infty^{r/2}, & \text{if } r \text{ is even;}\\ (u;q_2^2)_\infty^{(r+1)/2}(uq_2Z;q_2^2)_\infty^{(r+1)/2}, & \text{if } r \text{ is odd.}\end{cases}$$

Hence,

$$\begin{aligned}
&\sum_{n\ge 0}\frac{u^n}{(1-t_1^2)^{n+1}(q_2^2;q_2^2)_n}(1+t_1)B_n(t_1,q_2,Z)\\
&=\sum_{s\ge 0}\frac{t_1^{2s+1}}{((u;q_2^2)_\infty(uq_2Z;q_2^2)_\infty)^{s+1}}+\sum_{s\ge 0}\frac{t_1^{2s}}{((u;q_2^2)_\infty(uq_2Z;q_2^2)_\infty)^{s}}\frac{1}{(u;q_2^2)_\infty}.\\
&=\frac{1}{-t_1^2+(u;q_2^2)_\infty(uq_2Z;q_2^2)_\infty}\big(t_1+(uq_2Z;q_2^2)_\infty\big).
\end{aligned}$$

Now replace u by $v(1-t_1^2)$. This implies the following result stated as a theorem.

Theorem 4.1. *Let $B_n(t_1,q_2,Z)$ be the generating polynomial for the group B_n by the triple* $(\mathrm{fdes},\mathrm{ifmaj},\mathrm{neg})$. *Then,*

$$\begin{aligned}(4.4)\quad &\sum_{n\ge0}\frac{v^n}{(q_2^2;q_2^2)_n}B_n(t_1,q_2,Z)\\ &=\frac{1-t_1}{-t_1^2+(v(1-t_1^2);q_2^2)_\infty\,(v(1-t_1^2)q_2Z;q_2^2)_\infty}\bigl(t_1+(v(1-t_1^2)q_2Z;q_2^2)_\infty\bigr).\end{aligned}$$

Several consequences are drawn from Theorem 4.1. First, when $Z=0$ and when t_1, q_2 are replaced by their square roots, we get

$$(4.5)\qquad \sum_{n\ge0}\frac{v^n}{(q_2;q_2)_n}A_n(t_1,q_2)=\frac{1-t_1}{-t_1+(v(1-t_1);q_2)_\infty},$$

where $A_n(t_1,q_2)$ is the generating polynomial for the group $\mathfrak{S}_n$ by the pair $(\mathrm{des},\mathrm{imaj})$, but also by the pair $(\mathrm{des},\mathrm{inv})$ (see [St76], [FoHa97]).

Let $\mathbf{i}\,w:=w^{-1}$ denote the inverse of the signed permutation w. At this stage we have to remember that the bijection Ψ of B_n onto itself that we have constructed in our first paper [FoHa05a] preserves the *inverse ligne of route* [FoHa05a, Theorem 1.2], so that the chain

$$\begin{matrix} w & \xrightarrow{\ \mathbf{i}\ } & w_1 & \xrightarrow{\ \Psi\ } & w_2 & \xrightarrow{\ \mathbf{i}\ } & w_3\\ \begin{pmatrix}\mathrm{fdes}\\ \mathrm{ifmaj}\end{pmatrix} & & \begin{pmatrix}\mathrm{ifdes}\\ \mathrm{fmaj}\end{pmatrix} & & \begin{pmatrix}\mathrm{ifdes}\\ \mathrm{finv}\end{pmatrix} & & \begin{pmatrix}\mathrm{fdes}\\ \mathrm{finv}\end{pmatrix}\end{matrix}$$

shows that the four pairs $(\mathrm{fdes},\mathrm{ifmaj})$, $(\mathrm{ifdes},\mathrm{fmaj})$, $(\mathrm{ifdes},\mathrm{finv})$ and $(\mathrm{fdes},\mathrm{finv})$ are all *equidistributed* over B_n. Therefore,

$$\sum_w t^{\mathrm{fdes}\,w}q^{\mathrm{ifmaj}\,w}=\sum_w t^{\mathrm{ifdes}\,w}q^{\mathrm{fmaj}\,w}=\sum_w t^{\mathrm{ifdes}\,w}q^{\mathrm{finv}\,w}=\sum_w t^{\mathrm{fdes}\,w}q^{\mathrm{finv}\,w},$$

where w runs over B_n. The rightmost generating polynomial was designated by ${}^{\mathrm{finv}}B_n(t,q)$ in (1.11). Therefore we can derive a formula for ${}^{\mathrm{finv}}B_n(t,q)$ by using its $(\mathrm{fdes},\mathrm{ifmaj})$ interpretation. We have then ${}^{\mathrm{finv}}B_n(t,q)=B_n(t_1,q_2,Z)$ with $t_1=t$, $q_2=q$ and $Z=1$. Let $Z:=1$ in (4.4); as $(v(1-t_1^2);q_2^2)_\infty\,(v(1-t_1^2)q_2;q_2^2)_\infty=(v(1-t_1^2);q_2)_\infty$, identity (4.4) implies identity (1.12).

To recover Reiner's identity (1.13) we make use of (1.12) by sorting the signed permutations according to the parity of their flag descent numbers: $B_n(t,q)\;=:\;B_n'(t^2,q)+t\,B_n''(t^2,q)$, so that ${}^{\mathrm{dess}}B_n(t^2,q)=B_n'(t^2,q)+t^2\,B_n''(t^2,q)$. Hence

$$\begin{aligned}\sum_{n\ge0}\frac{v^n}{(q^2;q^2)_n}{}^{\mathrm{dess}}B_n(t^2,q)&=\frac{(v(1-t^2)q;q^2)_\infty-t^2}{-t^2+(v(1-t^2);q)_\infty}+t^2\frac{1-(v(1-t^2)q;q^2)_\infty}{-t^2+(v(1-t^2);q)_\infty}\\ &=\frac{1-t^2}{-t^2+(v(1-t^2);q)_\infty}(v(1-t^2)q;q^2)_\infty.\end{aligned}$$

As $1/(v(1-t^2);q)_\infty$ can also be expressed as the q-exponential $e_q(v(1-t^2))$, we then get identity (1.13).

5. The MacMahon Verfahren for signed words

Let $w = x_1x_2\ldots x_m$ be a signed *word* belonging to the class $B_{\mathbf{m}}$, where $\mathbf{m} = (m_1, m_2, \ldots, m_r)$ is a sequence of nonnegative integers such that $m_1 + m_2 + \cdots + m_r = m$. Remember that this means that w is a rearrangement of $1^{m_1}2^{m_2}\ldots r^{m_r}$, with the convention that some letters i $(1 \le i \le r)$ may be replaced by their opposite values $\bar{i}$. Again, let $\epsilon := \epsilon(w) := \epsilon_1\epsilon_2\ldots\epsilon_m$ be the binary word defined by $\epsilon_i = 0$ or 1, depending on whether x_i is positive or negative.

The MacMahon Verfahren bijection for signed words is constructed as follows. First, compute the word $z = z_1z_2\ldots z_m$, where z_k is equal to the number of descents in the right factor $x_kx_{k+1}\ldots x_m$, as well as the word $\epsilon = \epsilon_1\epsilon_2\ldots\epsilon_m$ mentioned above, so that, as in the case of signed permutations,

$$\operatorname{fmaj} w = 2\operatorname{tot} z + \operatorname{tot}\epsilon. \tag{5.1}$$

Next, define $a_k := b_k + z_k$, $c'_k := 2a_k + \epsilon_k$ $(1 \le k \le m)$, then $a := a_1a_2\ldots a_m$ and $c' := c'_1c'_2\ldots c'_m$. Finally, form the two-row matrix $\begin{pmatrix} c' \\ \operatorname{abs} w\end{pmatrix} = \begin{pmatrix} c'_1 & c'_2 & \ldots & c'_m \\ |x_1| & |x_2| & \ldots & |x_m|\end{pmatrix}$. Its bottom row is a rearrangement of the word $1^{m_1}2^{m_2}\ \ldots\ r^{m_r}$.

Make the convention that two biletters $\binom{c'_k}{|x_k|}$ and $\binom{c'_l}{|x_l|}$ commute if and only if $|x_k|$ and $|x_l|$ are different and rearrange the biletters of that biword in such a way that the bottom row is precisely $1^{m_1}2^{m_2}\ \ldots\ r^{m_r}$. The top row in the resulting two-row matrix is then the juxtaposition product of r *nonincreasing* words $b^{(1)} = b_1^{(1)}\ldots b_{m_1}^{(1)}$, $b^{(2)} = b_1^{(2)}\ldots b_{m_2}^{(2)}$, $\ldots$, $b^{(r)} = b_1^{(r)}\ldots b_{m_r}^{(r)}$, of length $m_1, m_2, \ldots, m_r$, respectively. Moreover,

$$\begin{aligned}\operatorname{tot} b^{(1)} + \operatorname{tot} b^{(2)} + \cdots + \operatorname{tot} b^{(r)} &= \operatorname{tot} c' = 2\operatorname{tot} a + \operatorname{tot}\epsilon\\ &= 2\operatorname{tot} b + 2\operatorname{tot} z + \operatorname{tot}\epsilon\\ &= 2\operatorname{tot} b + \operatorname{fmaj} w. \end{aligned}\tag{5.2}$$

On the other hand,

$$\begin{aligned}2b_1 + \operatorname{fdes} w &= 2b_1 + 2z_1 + \epsilon_1 = c'_1\\ &= \max\nolimits_i b_1^{(i)}\ (1 \le i \le r).\end{aligned}\tag{5.3}$$

As in the case of signed permutations, we can easily see that for each nonnegative integer s the map $(b, w) \mapsto (b^{(1)}, b^{(2)}, \ldots, b^{(r)})$ is a bijection of the set of pairs $(b, w) \in \mathrm{NIW}(m) \times R_{\mathbf{m}}$ such that $2b_1 + \operatorname{fdes} w = s$ onto the set of juxtaposition products $b^{(1)}b^{(2)} \ldots b^{(r)}$ such that $\max_i b_1^{(i)} = s$. The reverse bijection is constructed in the same way as in the case of signed permutations.

Example. Start with the pair (b, w), where w belongs to $B_{\mathbf{m}}$ with $\mathbf{m} = (2,2,2,2,2,2)$, $r = 6$, $m = 12$, the negative elements being overlined.

$$\begin{array}{lcrrrrrrrrrrrr}
\mathrm{Id} & = & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
b & = & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 2 & 2 & 1 & 1 & 1 \\
w & = & 1 & \bar{5} & \bar{2} & 2 & \bar{3} & \bar{3} & \bar{\mathbf{1}} & 4 & \bar{6} & \bar{5} & \mathbf{4} & \mathbf{6} \\
z & = & 3 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
\epsilon & = & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
a & = & 7 & 6 & 6 & 6 & 5 & 5 & 5 & 3 & 2 & 1 & 1 & 1 \\
c' & = & 14 & 13 & 13 & 12 & 11 & 11 & \mathbf{11} & 6 & 5 & 3 & \mathbf{2} & \mathbf{2} \\
\operatorname{abs} w & = & 1 & 5 & 2 & 2 & 3 & 3 & \mathbf{1} & 4 & 6 & 5 & \mathbf{4} & \mathbf{6} \\
b^{(1)} \ldots b^{(r)} & = & 14 & \mathbf{11}\,\vert & 13 & 12\,\vert & 11 & 11\,\vert & 6 & \mathbf{2}\,\vert & 13 & 3\,\vert & 5 & \mathbf{2} \\
1^{m_1} \ldots r^{m_r} & = & 1 & \mathbf{1}\,\vert & 2 & 2\,\vert & 3 & 3\,\vert & 4 & \mathbf{4}\,\vert & 5 & 5\,\vert & 6 & \mathbf{6}
\end{array}$$

We verify that $2 \operatorname{tot} b + \operatorname{fmaj} w = 2 \operatorname{tot} b + 2 \operatorname{tot} z + \operatorname{tot} \epsilon = 2 \times 35 + 2 \times 13 + 7 = 103 = \operatorname{tot} b^{(1)} + \cdots + \operatorname{tot} b^{(6)}$ and $2b_1 + \operatorname{fdes} w = 2b_1 + 2z_1 + \epsilon_1 = 2 \times 4 + 2 \times 3 + 0 = 14 = c'_1 = \max_i b_1^{(i)}$.

The combinatorial theorem for signed words that corresponds to Theorem 2.1 is now stated.

Theorem 5.1. *For each nonnegative integer s the above mapping is a bijection of the set of pairs $(b, w) = (b_1 b_2 \ldots b_m,\, x_1 x_2 \ldots x_m) \in \mathrm{NIW}(m) \times B_{\mathbf{m}}$ such that $2b_1 + \operatorname{fdes} w = s$ onto the set of juxtaposition products $b^{(1)} \ldots b^{(r)} \in \mathrm{NIW}(m_1) \times \cdots \times \mathrm{NIW}(m_r)$ such that $\max_i b_1^{(i)} = s$. Moreover,* (5.2) *and* (5.3) *hold, together with*

$$\operatorname{neg} w = \operatorname{odd}(b^{(1)} \ldots b^{(r)}) \tag{5.4}$$

Relation (5.4) is obvious, as the negative letters of w are in bijection with the odd letters of the juxtaposition product. Now consider the generating polynomial $B_{\mathbf{m}}(t, q, Z)$, as defined in (1.9). Making use of (2.2) and of the usual q-identities

$$\frac{1}{(u;q)_N} = \sum_{n \ge 0} \begin{bmatrix} N+n-1 \\ n \end{bmatrix}_q u^n;$$

$$\begin{bmatrix} N+n \\ n \end{bmatrix}_q = \sum_{b \in \mathrm{NIW}(N),\, b_1 \le n} q^{\operatorname{tot} b};$$

we have

$$\begin{aligned}\frac{1+t}{(t^2;q^2)_{m+1}}\,B_{\mathbf{m}}(t,q,Z) &= \sum_{s'\geq 0}(t^{2s'}+t^{2s'+1})\begin{bmatrix} m+s' \\ s'\end{bmatrix}_{q^2} B_{\mathbf{m}}(t,q)\\ &= \sum_{s'\geq 0} t^r \begin{bmatrix} m+\lfloor s'/2\rfloor \\ \lfloor s'/2\rfloor\end{bmatrix}_{q^2} B_{\mathbf{m}}(t,q,Z)\\ &= \sum_{s'\geq 0} t^{s'} \sum_{\substack{b\in \mathrm{NIW}(m),\\ 2b_1\leq s'}} q^{2\,\mathrm{tot}\,b} \sum_{w\in B_{\mathbf{m}}} t^{\mathrm{fdes}\,w} q^{\mathrm{fmaj}\,w} Z^{\mathrm{neg}\,w}\\ &= \sum_{s\geq 0} t^s \sum_{\substack{b\in\mathrm{NIW}(m),\, w\in B_{\mathbf{m}}\\ 2b_1+\mathrm{fdes}\,w\leq s}} q^{2\,\mathrm{tot}\,b+\mathrm{fmaj}\,w} Z^{\mathrm{neg}\,w}.\end{aligned}$$

But using (5.2), (5.3), (5.4) we can write

$$\frac{1+t}{(t^2;q^2)_{m+1}}\,B_{\mathbf{m}}(t,q,Z) = \sum_{s\geq 0} t^s \sum_{\substack{b^{(1)},\ldots,b^{(r)},\\ \max_i b_1^{(i)}\leq s}} q^{\mathrm{tot}\,b^{(1)}+\cdots+\mathrm{tot}\,b^{(r)}} Z^{\mathrm{odd}\,b^{(1)}+\cdots+\mathrm{odd}\,b^{(r)}}$$

and

$$\sum_{\mathbf{m}}(1+t)B_{\mathbf{m}}(t,q,Z)\frac{u_1^{m_1}\cdots u_r^{m_r}}{(t^2;q^2)_{1+|\mathbf{m}|}} = \sum_{s\geq 0} t^s \prod_{1\leq i\leq r}\sum_{m_i\geq 0}\sum_{b^{(i)}} u_i^{m_i} q^{\mathrm{tot}\,b^{(i)}} Z^{\mathrm{odd}\,b^{(i)}},$$

using the notation: $|\mathbf{m}| := m_1+\cdots+m_r$.

There remains to evaluate $\sum_m \sum_b u^m q^{\mathrm{tot}\,b} Z^{\mathrm{odd}\,b}$, where the second sum is over all nonincreasing words $b=b_1\ldots b_m$ of length m such that $b_1\leq s$. Let $1\leq i_1<\cdots<i_k\leq m$ (resp. $1\leq j_1<\cdots<j_l\leq m$) be the sequence of the integers i (resp. j) such that b_i is even (resp. b_j is odd). Then, b is completely characterized by the pair (b^e,b^o), where $b^e := (b_{i_1}/2)\ldots(b_{i_k}/2)$ and $b^o := ((b_{j_1}-1)/2)\ldots((b_{j_l}-1)/2)$. Moreover, $\mathrm{tot}\,b = 2\,\mathrm{tot}\,b^e + 2\,\mathrm{tot}\,b^o + |b^o|$. Hence,

$$\begin{aligned}\sum_{m\geq 0}\sum_{b,|b|=m,b_1\leq s} u^m q^{\mathrm{tot}\,b} Z^{\mathrm{odd}\,b} &= \sum_{b,b_1\leq s} u^{|b|} q^{\mathrm{tot}\,b} Z^{\mathrm{odd}\,b}\\ &= \sum_{b^e,\,2b_1^e\leq s} u^{|b^e|} q^{2\,\mathrm{tot}\,b^e} \sum_{b^o,\,2b_1^o\leq s-1} (uqZ)^{|b^o|} q^{2\,\mathrm{tot}\,b^o}\\ &= \frac{1}{(u;q^2)_{\lfloor s/2\rfloor+1}}\,\frac{1}{(uqZ;q^2)_{\lfloor (s+1)/2\rfloor}}.\end{aligned}$$

This achieves the proof of Theorem 1.3.

6. Specializations

As has been seen in this paper a *graded* form such as (1.10) has an *infinite* version (again obtained by multiplying the formula by $(1-t)$ and letting $t := 1$) given by

$$\begin{aligned}(6.1)\qquad \sum_{\mathbf{m}} B_{\mathbf{m}}(q,Z)\frac{u_1^{m_1}\cdots u_r^{m_r}}{(q^2;q^2)_{|\mathbf{m}|}} &= \prod_{1\le i\le r}\frac{1}{(u_i;q^2)_\infty}\frac{1}{(u_iqZ;q^2)_\infty}\\ &= \prod_{1\le i\le r} e_{q^2}(u_i)\, e_{q^2}(u_iqZ),\end{aligned}$$

where $e_{q^2}(u)$ denotes the usual q-exponential with basis q^2 ([GaRa90], p. 9).
The polynomial $B_{\mathbf{m}}(q,Z)$ is the generating polynomial for the class $B_{\mathbf{m}}$ by the pair $(\mathrm{fmaj}, \mathrm{neg})$, namely

$$(6.2)\qquad B_{\mathbf{m}}(q,Z) = \sum_w q^{\mathrm{fmaj}\, w} Z^{\mathrm{neg}\, w} \quad (w\in B_{\mathbf{m}}).$$

On the other hand, Let

$$(6.3)\qquad {}^{\mathrm{finv}}B_{\mathbf{m}}(q,Z) := \sum_{w\in B_{\mathbf{m}}} q^{\mathrm{finv}\, w} Z^{\mathrm{neg}\, w}$$

be the generating polynomial for the class $B_{\mathbf{m}}$ by the pair $(\mathrm{finv}, \mathrm{neg})$. Using a different approach (the derivation is not reproduced in the paper), we can prove the identity

$$\begin{aligned}(6.4)\qquad {}^{\mathrm{finv}}B_{\mathbf{m}}(q,Z) &= (-Zq;q)_{m_1+\cdots+m_r}\frac{(q;q)_{m_1+\cdots+m_r}}{(q;q)_{m_1}\cdots(q;q)_{m_r}}\\ &= (-Zq;q)_{m_1+\cdots+m_r}\begin{bmatrix} m_1+\cdots+m_r\\ m_1,\ldots,m_r\end{bmatrix}_q,\end{aligned}$$

using the traditional notation for the q-multinomial coefficient. In general, $B_{\mathbf{m}}(q,Z) \neq {}^{\mathrm{finv}}B_{\mathbf{m}}(q,Z)$. This can be shown by means of a combinatorial argument.

Let $Z := 1$ in (6.1) and make use of the q-binomial theorem (see [An76], p. 17, or [GaRa90], chap. 1), on the one hand, and let $Z := 1$ in (6.4), on the other hand. We get the evaluations

$$(6.5)\qquad B_{\mathbf{m}}(q) = (-q;q)_{m_1+\cdots+m_r}\begin{bmatrix} m_1+\cdots+m_r\\ m_1,\ldots,m_r\end{bmatrix}_q = {}^{\mathrm{finv}}B_{\mathbf{m}}(q).$$

This shows that "fmaj" and "finv" are equidistributed over each class $B_{\mathbf{m}}$, a property proved "bijectively" in our first paper [FoHa05a].

Next, let $q := 1$ in (6.4). We obtain

$$ {}^{\mathrm{finv}}B_{\mathbf{m}}(Z) = (1+Z)^{m_1+\cdots+m_r}\binom{m_1+\cdots+m_r}{m_1,\ldots,m_r}\bigl(= B_{\mathbf{m}}(Z)\bigr), \tag{6.6}$$

an identity which is equivalent to

$$ \sum_{\mathbf{m}} B_{\mathbf{m}}(Z)\frac{u_1^{m_1}\cdots u_r^{m_r}}{|\mathbf{m}|!} = \prod_{1\le i\le r} \exp(u_i)\,\exp(u_iZ). \tag{6.7}$$

Thus, the q-analog of (6.6) yields (6.4) with a combinatorial interpretation in terms of the flag-inversion number "finv," while (6.1) may be interpreted as q^2-analog of (6.7) with an interpretation in terms of the flag-maj index number "fmaj."

Finally, for $Z = 0$, formula (1.10) yields the identity

$$\sum_{\mathbf{m}} A_{\mathbf{m}}(t,q)\frac{u_1^{m_1}\cdots u_r^{m_r}}{(t;q)_{1+|\mathbf{m}|}} = \sum_{s\ge 0} t^s \prod_{1\le i\le r}\frac{1}{(u_i;q)_{s+1}},$$

where $A_{\mathbf{m}}(t,q)$ is the generating polynomial for the class of the rearrangements of the word $1^{m_1}2^{m_2}\ldots r^{m_r}$ by $(\mathrm{des},\mathrm{maj})$. As done by Rawlings [Ra79], [Ra80], the polynomials $A_{\mathbf{m}}(t,q)$ can also be defined by a recurrence relation involving either the polynomials themselves, or their coefficients.

7. The Signed-Word-Euler-Mahonian polynomials

We end the paper by showing that the polynomials $B_{\mathbf{m}}(t,q) = B_{\mathbf{m}}(t,q,Z)\mid_{Z=1}$ can be calculated not only by their factorial generating function given by (1.10) for $Z := 1$, but also by a *recurrence formula.*

Definition. A sequence $\Bigl(B_{\mathbf{m}}(t,q) = \sum\limits_{k\ge 0} t^k B_{\mathbf{m},k}(q)\Bigr)$ $(\mathbf{m} = (m_1,\ldots,m_r);$ $m_1\ge 0,\ldots,m_r\ge 0))$ of polynomials in two variables t, q, is said to be *signed-word-Euler-Mahonian*, if one of the following four *equivalent* conditions holds:

(1) The (t^2,q^2)-factorial generating function for the polynomials

$$ C_{\mathbf{m}}(t,q) := (1+t)B_{\mathbf{m}}(t,q) \tag{7.1}$$

is given by identity (1.10) when $Z = 1$, that is,

$$ \sum_{\mathbf{m}} C_{\mathbf{m}}(t,q)\frac{u_1^{m_1}\cdots u_r^{m_r}}{(t^2;q^2)_{1+|\mathbf{m}|}} = \sum_{s\ge 0} t^s \prod_{1\le i\le r}\frac{1}{(u_i;q)_{s+1}}. \tag{7.2}$$

(2) For each multiplicity $\mathbf{m}$ we have:

$$\frac{C_{\mathbf{m}}(t,q)}{(t^2;q^2)_{1+|\mathbf{m}|}} = \sum_{s\ge 0} t^s \begin{bmatrix} m_1+s \\ s \end{bmatrix}_q \cdots \begin{bmatrix} m_r+s \\ s \end{bmatrix}_q. \tag{7.3}$$

Let $\mathbf{m}+1_r := (m_1,\ldots,m_{r-1},m_r+1)$.

(3) The recurrence relation

$$\begin{aligned} &(1-q^{m_r+1})B_{\mathbf{m}+1_r}(t,q) \\ &\qquad = (1-t^2q^{2+2|\mathbf{m}|})B_{\mathbf{m}}(t,q) - q^{m_r+1}(1-t)(1+tq)B_{\mathbf{m}}(tq,q), \end{aligned} \tag{7.4}$$

holds with $B_{(0,\ldots,0)}(t,q)=1$.

(4) The following recurrence relation holds for the coefficients $B_{\mathbf{m},k}(q)$

$$\begin{aligned} &(1+q+\cdots+q^{m_r})B_{\mathbf{m}+1_r,k}(q) = (1+q+\cdots+q^{m_r+k})B_{\mathbf{m},k}(q) \\ &\quad + q^{m_r+k}B_{\mathbf{m},k-1}(q) + (q^{m_r+k}+q^{m_r+k+1}+\cdots+q^{2|\mathbf{m}|+1})B_{\mathbf{m},k-2}(q), \end{aligned} \tag{7.5}$$

while $B_{(0,\ldots,0),0}(q)=1$ and $B_{(0,\ldots,0),k}(q)=0$ for every $k\neq 0$.

Theorem 7.1. *The conditions* (1), (2), (3) *and* (4) *in the previous definition are equivalent.*

Proof. The proofs of the equivalences [(1) $\Leftrightarrow$ (2)] and [(3) $\Leftrightarrow$ (4)] are easy and therefore omitted. For proving the equivalence [(1) $\Leftrightarrow$ (3)] proceed as follows. Let $C(t,q;u_1,\ldots,u_r)$ denote the *right* side of (7.2) and form the *q-difference* $C(t,q;u_1,\ldots,u_r) - C(t,q;u_1,\ldots,u_{r-1},u_rq)$ applied to the sole variable u_r. We get

$$\begin{aligned} &C(t,q;u_1,\ldots,u_r) - C(t,q;u_1,\ldots,u_{r-1},u_rq) \\ &\quad = \sum_{s\ge 0} \frac{t^s}{(u_1;q)_{s+1}\ldots(u_r;q)_{s+1}} - \sum_{s\ge 0} \frac{t^s}{(u_1;q)_{s+1}\ldots(u_rq;q)_{s+1}} \\ &\quad = \sum_{s\ge 0} \frac{t^s}{(u_1;q)_{s+1}\ldots(u_r;q)_{s+1}}\Bigl[1-\frac{1-u_r}{1-u_rq^{s+1}}\Bigr] \\ &\quad = u_r\sum_{s\ge 0} \frac{t^s}{(u_1;q)_{s+1}\ldots(u_r;q)_{s+1}}\Bigl[1-q^{s+1}\frac{1-u_r}{1-u_rq^{s+1}}\Bigr] \\ &\quad = u_r\bigl(C(t,q;u_1,\ldots,u_r) - qC(tq,q;u_1,\ldots,u_{r-1},u_rq)\bigr). \end{aligned} \tag{7.6}$$

Now, let $C(t,q;u_1,\ldots,u_r) := \sum_{\mathbf{m}} C_{\mathbf{m}}(t,q)u_1^{m_1}\cdots u_r^{m_r}/(t^2;q^2)_{1+|\mathbf{m}|}$ and express each term $C(\ldots)$ occurring in identity (7.6) as a factorial series in the u_i's. We obtain

$$\begin{aligned}\sum_{\mathbf{m}}(1-q^{m_r+1})C_{\mathbf{m}+1_r}(t,q)\frac{u_1^{m_1}\cdots u_r^{m_r+1}}{(t^2;q^2)_{2+|\mathbf{m}|}}\\ =\sum_{\mathbf{m}}(1-t^2q^{2+2|\mathbf{m}|})C_{\mathbf{m}}(t,q)\frac{u_1^{m_1}\cdots u^{m_r+1}}{(t^2;q^2)_{2+|\mathbf{m}|}}\\ -\sum_{\mathbf{m}}q^{m_r+1}(1-t^2)C_{\mathbf{m}}(tq,q)\frac{u_1^{m_1}\cdots u^{m_r+1}}{(t^2;q^2)_{2+|\mathbf{m}|}}.\end{aligned}$$

Taking the coefficients of $u_1^{m_1}\ldots u_{r-1}^{m_{r-1}}u_r^{m_r+1}$ yields

$$(1-q^{m_r+1})C_{\mathbf{m}+1_r}(t,q)=(1-t^2q^{2+2|\mathbf{m}|})C_{\mathbf{m}}(t,q)-q^{m_r+1}(1-t^2)C_{\mathbf{m}}(tq,q),$$

which in its turn is equivalent to (7.6) in view of (7.1). All the steps of the argument are reversible. ∎

Remark 1. The fact that $B_{\mathbf{m}}(t,q)$ is the generating polynomial for the class $B_{\mathbf{m}}$ by the pair (fdes, fmaj) can also be proved by the insertion technique using (7.5). The argument has been already developed in [ClFo95a, § 6] for ordinary words. Again let $m:=|\mathbf{m}|=m_1+\cdots+m_r$. With each word from $B_{\mathbf{m}+1^r}$ associate (m_r+1) new words obtained by *marking* one and only one letter equal to r or $\overline{r}$. Let $B^*_{\mathbf{m}+1^r}$ denote the class of all those *marked* signed words. If $w^*=x_1\ldots x_i^*\ldots x_{m+1}$ is such a word, where the i-th letter is marked (accordingly, equal to either r or $\overline{r}$), let mark_i be the number of letters equal to r or $\overline{r}$ in the right factor $x_{i+1}x_{i+2}\ldots x_{m+1}$ and define:

$$\mathrm{fmaj}^*\,w^*:=\mathrm{fmaj}\,w+\mathrm{mark}_i\,w^*.$$

On the other hand, let

$$B_{\mathbf{m},k}(q):=\sum_{w\in B_{\mathbf{m}},\,\mathrm{fdes}\,w=k}q^{\mathrm{fmaj}\,w}.$$

Clearly,

$$\sum_{w^*\in B^*_{\mathbf{m}+1^r},\,\mathrm{fdes}\,w=k}q^{\mathrm{fmaj}^*\,w^*}=(1+q+\cdots+q^{m_r})B_{\mathbf{m}+1_r,k}(q).$$

Now each word w from the class $B_{\mathbf{m}}$ gives rise to $2(m+1)$ *distinct* marked signed words of length $(m+1)$, when the marked letter r or $\overline{r}$ is inserted between letters of w, as well as in the beginning of and at the end of the word. As in the case of the signed permutations, we can verify that for each $j=0,1,\ldots,2m+1$ there is one and only one marked signed word w^* of length $(m+1)$ derived by insertion such that $\mathrm{fmaj}^*\,w^*=\mathrm{fmaj}\,w+j$.

On the other hand, "fdes" is not modified if r is inserted to the right of w, or if r or $\overline{r}$ is inserted into a descent $x_i>x_{i+1}$. Furthermore, "fdes" increases by one, if $x_1>0$ (resp. $x_1<0$) and $\overline{r}$ (resp. r) is inserted to the left of w. For all the other insertions "fdes" increases by 2.

Hence, all the marked signed words w^* from $B^*_{\mathbf{m}+1^r}$, such that $\operatorname{fdes} w^* = k$ are derived by insertion from three sources:

(i) the set $\{w \in B_{\mathbf{m}} : \operatorname{fdes} w = k\}$ and the contribution is: $(1 + q + \cdots + q^{m_r+k})B_{\mathbf{m},k}(q)$;

(ii) the set $\{w \in B_{\mathbf{m}} : \operatorname{fdes} w = k-1\}$ and the contribution is: $q^{m_r+k}B_{\mathbf{m},k-1}(q)$;

(iii) the set $\{w \in B_{\mathbf{m}} : \operatorname{fdes} w = k-2\}$ and the contribution is: $(q^{m_r+k} + \cdots + q^{2|\mathbf{m}|+1})B_{\mathbf{m},k-2}(q)$. $\square$

Remark 2. Let $\mathbf{m} := 1^n$ and $B_n(t,q) = B_{\mathbf{m}}(t,q)$, so that $B_n(t,q)$ is now the generating polynomial for the set of signed *permutations* of order n. Then (7.3), (7.4) and (7.5) become

$$\frac{(1+t)B_n(t,q)}{(t^2;q^2)_{n+1}} = \sum_{s\geq 0} t^s(1+q+\cdots+q^s)^n \tag{7.7}$$

$$(1-q)B_n(t,q) = (1-t^2q^{2n})B_{n-1}(t,q) - q(1-t)(1+tq)B_{n-1}(tq,q). \tag{7.8}$$

$$\begin{aligned} B_{n,k}(q) &= (1+q+\cdots+q^k)B_{n-1,k}(q) \\ &\quad + q^kB_{n-1,k-1}(q) + (q^k+q^{k+1}+\cdots+q^{2n-1})B_{n-1,k-2}(q). \end{aligned} \tag{7.9}$$

The last three relations have been derived by Brenti et al. [ABR01], Chow and Gessel [ChGe04], Haglund et al. [HLR04].

Concluding remarks. The statistical study of the hyperoctahedral group B_n was initiated by Reiner ([Re93a], [Re93b], [Re93c], [Re95a], [Re95b]). It had been rejuvenated by Adin and Roichman [AR01] with their introduction of the flag-major index, which was shown [ABR01] to be equidistributed with the length function. See also their recent papers on the subject [ABR05], [ReRo05]. Another approach to Theorems 1.1 and 1.2 would be to make use of the Cauchy identity for the Schur functions, as was done in [ClFo95b].

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SIGNED WORDS AND PERMUTATIONS, IV; FIXED AND PIXED POINTS

Dominique Foata and Guo-Niu Han

Von Jacobs hat er die Statur,
Des Rechnens ernstes Führen,
Von Lottärchen die Frohnatur
und Lust zu diskretieren.

To Volker Strehl, a dedication à la Goethe,
on the occasion of his sixtieth birthday.

Abstract

The flag-major index "fmaj" and the classical length function "ℓ" are used to construct two q-analogs of the generating polynomial for the hyperoctahedral group B_n by number of positive and negative fixed points (resp. pixed points). Specializations of those q-analogs are also derived dealing with signed derangements and desarrangements, as well as several classical results that were previously proved for the symmetric group.

1. Introduction

The statistical study of the hyperoctahedral group B_n, initiated by Reiner ([Re93a], [Re93b], [Re93c], [Re95a], [Re95b]), has been rejuvenated by Adin and Roichman [AR01] with their introduction of the *flag-major index*, which was shown [ABR01] to be equidistributed with the *length function*. See also their recent papers on the subject [ABR05], [ReRo05]. It then appeared natural to extend the numerous results obtained for the symmetric group $\mathfrak{S}_n$ to the groug B_n. Furthermore, flag-major index and length function become the true q-analog makers needed for calculating various multivariable distributions on B_n.

In the present paper we start with a generating polynomial for B_n by a three-variable statistic involving the number of fixed points (see formula (1.3)) and show that there are two ways of q-analogizing it, by using the flag-major index on the one hand, and the length function on the other hand. As will be indicated, the introduction of an extra variable Z makes it possible to specialize all our results to the symmetric group. Let us first give the necessary notations.

Let B_n be the hyperoctahedral group of all *signed permutations* of order n. The elements of B_n may be viewed as words $w = x_1x_2\cdots x_n$,

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where each x_i belongs to $\{-n,\ldots,-1,1,\ldots,n\}$ and $|x_1||x_2|\cdots|x_n|$ is a permutation of $12\ldots n$. The *set* (resp. the *number*) of *negative* letters among the x_i's is denoted by $\operatorname{Neg} w$ (resp. $\operatorname{neg} w$). A *positive fixed point* of the signed permutation $w=x_1x_2\cdots x_n$ is a (positive) integer i such that $x_i=i$. It is convenient to write $\bar{i}:=-i$ for each integer i. Also, when A is a set of integers, let $\overline{A}:=\{\bar{i}:i\in A\}$. If $x_i=\bar{i}$ with i positive, we say that $\bar{i}$ is a *negative fixed point* of w. The set of all positive (resp. negative) fixed points of w is denoted by $\operatorname{Fix}^+ w$ (resp. $\operatorname{Fix}^- w$). Notice that $\operatorname{Fix}^- w\subset \operatorname{Neg} w$. Also let

$$\text{(1.1)}\qquad \operatorname{fix}^+ w:=\#\operatorname{Fix}^+ w;\quad \operatorname{fix}^- w:=\#\operatorname{Fix}^- w.$$

There are $2^n n!$ signed permutations of order n. The symmetric group $\mathfrak{S}_n$ may be considered as the subset of all w from B_n such that $\operatorname{Neg} w=\emptyset$.

The purpose of this paper is to provide *two q-analogs* for the polynomials $B_n(Y_0,Y_1,Z)$ defined by the identity

$$\text{(1.2)}\qquad \sum_{n\ge 0}\frac{u^n}{n!}B_n(Y_0,Y_1,Z)=\bigl(1-u(1+Z)\bigr)^{-1}\times\frac{\exp(u(Y_0+Y_1Z))}{\exp(u(1+Z))}.$$

When $Z=0$, the right-hand side becomes $(1-u)^{-1}\exp(uY_0)/\exp(u)$, which is the exponential generating function for the generating polynomials for the groups $\mathfrak{S}_n$ by number of fixed points (see [Ri58], chap. 4). Also, by identification, $B_n(1,1,1)=2^n n!$ and it is easy to show (see Theorem 1.1) that $B_n(Y_0,Y_1,Z)$ is in fact the generating polynomial for the group B_n by the three-variable statistic $(\operatorname{fix}^+,\operatorname{fix}^-,\operatorname{neg})$, that is,

$$\text{(1.3)}\qquad B_n(Y_0,Y_1,Z)=\sum_{w\in B_n}Y_0^{\operatorname{fix}^+ w}\,Y_1^{\operatorname{fix}^- w}\,Z^{\operatorname{neg} w}.$$

Recall the traditional notations for the q-ascending factorials

$$\text{(1.4)}\qquad (a;q)_n:=\begin{cases}1, & \text{if } n=0;\\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & \text{if } n\ge 1;\end{cases}$$

$$(a;q)_\infty:=\prod_{n\ge 1}(1-aq^{n-1});$$

for the q-multinomial coefficients

$$\text{(1.5)}\qquad \begin{bmatrix} n\\ m_1,\ldots,m_k\end{bmatrix}_q:=\frac{(q;q)_n}{(q;q)_{m_1}\cdots(q;q)_{m_k}}\qquad (m_1+\cdots+m_k=n);$$

and for the two q-exponentials (see [GaRa90, chap. 1])

$$\text{(1.6)}\qquad e_q(u)=\sum_{n\ge 0}\frac{u^n}{(q;q)_n}=\frac{1}{(u;q)_\infty};\quad E_q(u)=\sum_{n\ge 0}\frac{q^{\binom{n}{2}}u^n}{(q;q)_n}=(-u;q)_\infty.$$

Our two q-analogs, denoted by ${}^{\ell}B_n(q,Y_0,Y_1,Z)$ and $B_n(q,Y_0,Y_1,Z)$, are respectively defined by the identities:

$$(1.7)\quad \sum_{n\ge 0}\frac{u^n}{(-Zq;q)_n\,(q;q)_n}\,{}^{\ell}B_n(q,Y_0,Y_1,Z) = \Bigl(1-\frac{u}{1-q}\Bigr)^{-1}\times (u;q)_\infty\Bigl(\sum_{n\ge 0}\frac{(-qY_0^{-1}Y_1Z;q)_n\,(uY_0)^n}{(-Zq;q)_n\,(q;q)_n}\Bigr);$$

$$(1.8)\quad \sum_{n\ge 0}\frac{u^n}{(q^2;q^2)_n}B_n(q,Y_0,Y_1,Z) = \Bigl(1-u\frac{1+qZ}{1-q^2}\Bigr)^{-1}\times\frac{(u;q^2)_\infty}{(uY_0;q^2)_\infty}\,\frac{(-uqY_1Z;q^2)_\infty}{(-uqZ;q^2)_\infty}.$$

Those two identities can be shown to yield (1.2) when $q=1$.

There is also a graded form of (1.8) in the sense that an extra variable t can be added to form a new polynomial $B_n(t,q,Y_0,Y_1,Z)$ with nonnegative integral coefficients that specializes into $B_n(q,Y_0,Y_1,Z)$ for $t=1$. Those polynomials are defined by the identity

$$(1.9)\quad \sum_{n\ge 0}(1+t)B_n(t,q,Y_0,Y_1,Z)\frac{u^n}{(t^2;q^2)_{n+1}} = \sum_{s\ge 0} t^s\Bigl(1-u\sum_{i=0}^{s}q^iZ^{\chi(i\ \text{odd})}\Bigr)^{-1}\times\frac{(u;q^2)_{\lfloor s/2\rfloor+1}}{(uY_0;q^2)_{\lfloor s/2\rfloor+1}}\,\frac{(-uqY_1Z;q^2)_{\lfloor (s+1)/2\rfloor}}{(-uqZ;q^2)_{\lfloor (s+1)/2\rfloor}},$$

where for each statement A we let $\chi(A)=1$ or 0 depending on whether A is true or not. The importance of identity (1.9) lies in its numerous specializations, as can be seen in Fig. 1.

The two q-extensions ${}^{\ell}B_n(q,Y_0,Y_1,Z)$ and $B_n(t,q,Y_0,Y_1,Z)$ being now defined, the program is to derive appropriate combinatorial interpretations for them. Before doing so we need have a second combinatorial interpretation for the polynomial $B_n(Y_0,Y_1,Z)$ besides the one mentioned in (1.3). Let $w=x_1x_2\cdots x_n$ be a word, all letters of which are integers without any repetitions. Say that w is a *desarrangement* if $x_1>x_2>\cdots>x_{2k}$ and $x_{2k}<x_{2k+1}$ for some $k\ge 1$. By convention, $x_{n+1}=\infty$. We could also say that the *leftmost trough* of w occurs at an *even* position. This notion was introduced by Désarménien [De84] and elegantly used in a subsequent paper [DeWa88]. Notice that there is no one-letter desarrangement. By convention, the empty word e is also a desarrangement.

Now let $w=x_1x_2\cdots x_n$ be a signed permutation. Unless w is increasing, there is always a nonempty right factor of w which is a desarrangement. It then makes sense to define w^d as the *longest* such a right factor. Hence, w admits a unique factorization $w=w^-w^+w^d$, called its *pixed*[(1)]

[(1)] "Pix," of course, must not be taken here for the abbreviated form of "pictures."

factorization, where w^- and w^+ are both *increasing*, the letters of w^- being *negative*, those of w^+ *positive* and where w^d is the longest right factor of w which is a desarrangement.

For example, the pixed factorizations of the following signed permutations are materialized by vertical bars: $w = \overline{5}\,\overline{2} \mid e \mid \overline{3}\,\overline{4}\,1$; $w = \overline{5} \mid e \mid \overline{2}\,\overline{3}\,1\,\overline{4}$; $w = \overline{5}\,\overline{3}\,\overline{2} \mid 1\,4 \mid e$; $\quad w = \overline{5}\,\overline{3} \mid 1 \mid 4\,2$; $\quad w = \overline{5}\,\overline{3} \mid e \mid 4\,1\,2$.

Let $w = w^- w^+ w^d$ be the pixed factorization of $w = x_1 x_2 \cdots x_n$. If $w^- = x_1 \cdots x_k$, $w^+ = x_{k+1} \cdots x_{k+l}$, define $\mathrm{Pix}^- w := \{x_1, \ldots, x_k\}$, $\mathrm{Pix}^+ w := \{x_{k+1}, \ldots, x_{k+l}\}$, $\mathrm{pix}^- w := \#\,\mathrm{Pix}^- w$, $\mathrm{pix}^+ w := \#\,\mathrm{Pix}^+ w$.

Theorem 1.1. *The polynomial $B_n(Y_0, Y_1, Z)$ defined by* (1.2) *admits the following two combinatorial interpretations:*

$$B_n(Y_0, Y_1, Z) = \sum_{w \in B_n} Y_0^{\mathrm{fix}^+ w}\, Y_1^{\mathrm{fix}^- w}\, Z^{\mathrm{neg}\, w} = \sum_{w \in B_n} Y_0^{\mathrm{pix}^+ w}\, Y_1^{\mathrm{pix}^- w}\, Z^{\mathrm{neg}\, w}.$$

Theorem 1.1 is proved in section 2. A bijection ϕ of B_n onto itself will be constructed that satisfies $(\mathrm{Fix}^-, \mathrm{Fix}^+, \mathrm{Neg})\, w = (\mathrm{Pix}^-, \mathrm{Pix}^+, \mathrm{Neg})\, \phi(w)$.

Let "ℓ" be the length function of B_n (see [Bo68, p. 7], [Hu90, p. 12] or the working definition given in (3.1)). As seen in Theorem 1.2, "ℓ" is to be added to the three-variable statistic $(\mathrm{pix}^+, \mathrm{pix}^-, \mathrm{neg})$ (and not to $(\mathrm{fix}^+, \mathrm{fix}^-, \mathrm{neg})$) for deriving the combinatorial interpretation of ${}^{\ell}B_n(q, Y_0, Y_1, Z)$. This theorem is proved in section 3.

Theorem 1.2. *For each $n \geq 0$ let ${}^{\ell}B_n(q, Y_0, Y_1, Z)$ be the polynomial defined in* (1.7)*. Then*

$$\tag{1.10} {}^{\ell}B_n(q, Y_0, Y_1, Z) = \sum_{w \in B_n} q^{\ell(w)}\, Y_0^{\mathrm{pix}^+ w}\, Y_1^{\mathrm{pix}^- w}\, Z^{\mathrm{neg}\, w}.$$

The variables t and q which are added to interpret our second extension $B_n(t, q, Y_0, Y_1, Z)$ will carry the flag-descent number "fdes" and the flag-major index "fmaj." For each signed permutation $w = x_1 x_2 \cdots x_n$ the usual *number of descents* "des" is defined by $\mathrm{des}\, w := \sum_{i=1}^{n-1} \chi(x_i > x_{i+1})$, the *major index* "maj" by $\mathrm{maj}\, w := \sum_{i=1}^{n-1} i\, \chi(x_i > x_{i+1})$, the *flag descent number* "fdes" and the *flag-major index* "fmaj" by

$$\tag{1.11} \mathrm{fdes}\, w := 2\,\mathrm{des}\, w + \chi(x_1 < 0); \quad \mathrm{fmaj}\, w := 2\,\mathrm{maj}\, w + \mathrm{neg}\, w.$$

Theorem 1.3. *For each $n \geq 0$ let $B_n(t, q, Y_0, Y_1, Z)$ be the polynomial defined in* (1.9)*. Then*

$$\tag{1.12} B_n(t, q, Y_0, Y_1, Z) = \sum_{w \in B_n} t^{\mathrm{fdes}\, w}\, q^{\mathrm{fmaj}\, w}\, Y_0^{\mathrm{fix}^+ w}\, Y_1^{\mathrm{fix}^- w}\, Z^{\mathrm{neg}\, w}.$$

Theorem 1.3 is proved in Section 5 after discussing the combinatorics of the so-called *weighted signed permutations* in Section 4. Section 6 deals with numerous specializations of Theorem 1.2 and 1.3 obtained by taking numerical values, essentially 0 or 1, for certain variables. Those specializations are illustrated by the following diagram (Fig. 1). When $Z = 0$, the statistic "neg" plays no role and the signed permutations become plain permutations; the second column of the diagram is then mapped on the third one that only involves generating polynomials for $\mathfrak{S}_n$ or subsets of that group.

$$
\begin{tikzcd}
D_n^B \arrow[r, "(\mathrm{fmaj},\mathrm{fix}^-,\mathrm{neg})"] & D_n^B(q,Y_1,Z) \arrow[d, "\mathrm{fdes}"] & D_n(q) \arrow[l, "\mathrm{neg}"'] \arrow[d, "\mathrm{des}"] & D_n \arrow[l, "\mathrm{maj}"'] \\
 & D_n^B(t,q,Y_1,Z) \arrow[d, "\mathrm{fix}^+"] & D_n(t,q) \arrow[l, "\mathrm{neg}"'] \arrow[d, "\mathrm{fix}^+"] & \\
 & \mathbf{B_n(t,q,Y_0,Y_1,Z)} & A_n(t,q,Y_0) \arrow[l, "\mathrm{neg}"'] & \\
 & B_n(q,Y_0,Y_1,Z) \arrow[u, "\mathrm{fdes}"] & A_n(q,Y_0) \arrow[l, "\mathrm{neg}"'] \arrow[u, "\mathrm{des}"] & \\
B_n \arrow[r, "(\mathrm{fix}^+,\mathrm{fix}^-,\mathrm{neg})", "(\mathrm{pix}^+,\mathrm{pix}^-,\mathrm{neg})"'] & \mathbf{B_n(Y_0,Y_1,Z)} \arrow[u, "\mathrm{fmaj}"] \arrow[d, "\ell"] & A_n(Y_0) \arrow[l, "\mathrm{neg}"'] \arrow[u, "\mathrm{maj}"] \arrow[d, "\mathrm{inv}"] & \mathfrak{S}_n \arrow[l, "\mathrm{fix}^+"', "\mathrm{pix}^+"] \\
 & {}^{\ell}\mathbf{B_n(q,Y_0,Y_1,Z)} & {}^{\mathrm{inv}}A_n(q,Y_0) \arrow[l, "\mathrm{neg}"'] & \\
K_n^B \arrow[r, "(\ell,\mathrm{pix}^-,\mathrm{neg})"'] & {}^{\ell}K_n^B(q,Y_1,Z) \arrow[u, "\mathrm{pix}^+"] & {}^{\mathrm{inv}}K_n(q) \arrow[l, "\mathrm{neg}"'] \arrow[u, "\mathrm{pix}^+"] & K_n \arrow[l, "\mathrm{inv}"', "\mathrm{imaj}"]
\end{tikzcd}
$$

Fig. 1

The first (resp. fourth) column refers to specific subsets of B_n (resp. of $\mathfrak{S}_n$):

$$
\begin{aligned}
D_n &:= \{w \in B_n : \mathrm{Fix}^+ w = \mathrm{Neg}\, w = \emptyset\};\\
K_n &:= \{w \in B_n : \mathrm{Pix}^+ w = \mathrm{Neg}\, w = \emptyset\};\\
D_n^B &:= \{w \in B_n : \mathrm{Fix}^+ w = \emptyset\};\\
K_n^B &:= \{w \in B_n : \mathrm{Pix}^+ w = \emptyset\}.
\end{aligned}
\tag{1.13}
$$

The elements of D_n are the classical *derangements* and provide the most natural combinatorial interpretations of the derangement numbers $d_n = \#D_n$ (see [Co70], p. 9–12). By analogy, the elements of D_n^B are called *signed derangements*. They have been studied by Chow [Ch06] in a recent note. The elements of K_n (resp. of K_n^B) are called *desarrangements* (resp.

signed desarrangements) of order n. When $Y_0 = 0$, the statistic fix^+ (resp. pix^+) plays no role. We can then calculate generating functions for signed and for plain derangements (resp. desarrangements), as shown in the first two rows (resp. last row). The initial polynomial, together with its two q-analogs are reproduced in boldface.

2. Proof of Theorem 1.1

As can be found in ([Co70], p. 9–12)), the generating function for the derangement numbers d_n $(n \geq 0)$ is given by

$$\sum_{n\geq 0} d_n \frac{u^n}{n!} = (1-u)^{-1} e^{-u}. \tag{2.1}$$

An easy calculation then shows that the polynomials $B_n(Y_0, Y_1, Z)$, introduced in (1.2), can also be defined by the identity

$$B_n(Y_0, Y_1, Z) = \sum_{i+j+k+l=n} \binom{n}{i, j, k, l} Y_0^i\, Y_1^j\, Z^{j+k}\, d_{k+l} \quad (n \geq 0). \tag{2.2}$$

For each signed permutation $w = x_1 x_2 \cdots x_n$ let $A := \mathrm{Fix}^+ w$, $B := \mathrm{Fix}^- w$, $C := \mathrm{Neg}\, w \setminus \mathrm{Fix}^- w$, $D := [n] \setminus (A \cup \overline{B} \cup \overline{C})$. Then $(A, \overline{B}, \overline{C}, D)$ is a sequence of disjoint subsets of integers, whose union is the interval $[n] := \{1, 2, \ldots, n\}$. Also the mapping τ defined by $\tau(\overline{j}) = x_j$ if $\overline{j} \in C$ and $\tau(j) = x_j$ if $j \in D$ is a *derangement* of the set $C + D$. Hence, w is completely characterized by the sequence (A, B, C, D, τ). The generating polynomial for B_n by the statistic $(\mathrm{fix}^+, \mathrm{fix}^-, \mathrm{neg})$ is then equal to the right-hand side of (2.2). This proves the first identity of Theorem 1.1.

Each signed permutation $w = x_1 x_2 \cdots x_n$ can be characterized, either by the four-term sequence $(\mathrm{Fix}^+ w, \mathrm{Fix}^- w, \mathrm{Neg}\, w, \tau)$, as just described, or by $(\mathrm{Pix}^+ w, \mathrm{Pix}^- w, \mathrm{Neg}\, w, w^d)$, where w^d is the desarrangement occurring as the third factor in its pixed factorization. To construct a bijection ϕ of B_n onto B_n such that $(\mathrm{fix}^-, \mathrm{fix}^+, \mathrm{neg})\, w = (\mathrm{pix}^-, \mathrm{pix}^+, \mathrm{neg})\, \phi(w)$ and accordingly prove the second identity of Theorem 1.1, we only need a bijection $\tau \mapsto f(\tau)$, that maps each derangement τ onto a desarrangement $f(\tau)$ by rearranging the letters of τ. But such a bijection already exists. It is due to Désarménien (*op. cit.*). We describe it by means of an example.

Start with a derangement $\tau = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8\,9 \\ 9\,7\,4\,3\,8\,2\,6\,5\,1 \end{pmatrix}$ and express it as a product of its disjoint cycles: $\tau = (1\,9)(2\,7\,6)(3\,4)(5\,8)$. In each cycle, write the minimum in the *second* position: $\tau = (9\,1)(6\,2\,7)(4\,3)(8\,5)$. Then, reorder the cycles in such a way that the sequence of those minima, when reading from left to right, is *decreasing*: $\tau = (8\,5)(4\,3)(6\,2\,7)(9\,1)$. The desarrangement $f(\tau)$ is derived from the latter expression by removing the parentheses: $f(\tau) = 8\,5\,4\,3\,6\,2\,7\,9\,1$.

Let $(\mathrm{Fix}^+ w, \mathrm{Fix}^- w, \mathrm{Neg}\, w, \tau)$ be the sequence associated with the signed permutation w and let v^- (resp. v^+) be the *increasing* sequence of the elements of $\mathrm{Fix}^- w$ (resp. of $\mathrm{Fix}^+ w$). Then, $v^- \mid v^+ \mid f(\tau)$ is the pixed factorization of $v^- v^+ f(\tau)$ and we may define $\phi(w)$ by

$$\phi(w) := v^- v^+ f(\tau). \tag{2.3}$$

This defines a bijection of B_n onto itself, which has the further property:

$$(\mathrm{Fix}^-, \mathrm{Fix}^+, \mathrm{Neg})\, w = (\mathrm{Pix}^-, \mathrm{Pix}^+, \mathrm{Neg})\, \phi(w). \tag{2.4}$$

For instance, with $w = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8\,9 \\ 3\,\overline{2}\,8\,4\,5\,\overline{1}\,9\,\overline{6}\,7 \end{pmatrix}$ we have $v^+ = 4\,5$, $v^- = \overline{2}$, $\tau = \begin{pmatrix} \overline{1}\,3\,\overline{6}\,7\,8\,9 \\ 3\,8\,\overline{1}\,9\,\overline{6}\,7 \end{pmatrix} = (9\,7)(8\,\overline{6}\,\overline{1}\,3)$ and $f(\tau) = 9\,7\,8\,\overline{6}\,\overline{1}\,3$. Hence, the pixed factorization of $\phi(w)$ reads $\overline{2} \mid 4\,5 \mid 9\,7\,8\,\overline{6}\,\overline{1}\,3$ and $\phi(w) = \overline{2}\,4\,5\,9\,7\,8\,\overline{6}\,\overline{1}\,3$.

3. Proof of Theorem 1.2

The length function "ℓ" for B_n is expressed in many ways. We shall use the following expression derived by Brenti [Br94]. Let $w = x_1 x_2 \cdots x_n$ be a signed permutation; its *length* $\ell(w)$ is defined by

$$\ell(w) := \mathrm{inv}\, w + \sum_i |x_i|\, \chi(\, x_i < 0), \tag{3.1}$$

where "inv" designates the usual *number of inversions* for words:

$$\mathrm{inv}\, w := \sum_{1 \le i < j \le n} \chi(x_i > x_j).$$

The generating polynomial for K_n (as defined in (1.13)) by "inv" (resp. for D_n by "maj") is denoted by $K_n(q) := {}^{\mathrm{inv}}K_n(q)$ (resp. $D_n(q)$). As was proved in [DeWa93] we have:

$$K_n(q) = D_n(q). \tag{3.2}$$

Also

$$\sum_{n \ge 0} \frac{u^n}{(q;q)_n} D_n(q) = \Bigl(1 - \frac{u}{1-q}\Bigr)^{-1} \times (u;q)_\infty, \tag{3.3}$$

as shown by Wachs [Wa90] in an equivalent form. Another expression for $D_n(q)$ will be derived in section 6, Proposition 6.2.

If A is a finite set of positive integers, let $\mathrm{tot}\, A$ denote the sum $\sum a$ $(a \in A)$. For the proof of Theorem 1.2 we make use of the following classical result, namely that $q^{N(N+1)/2} \begin{bmatrix} n \\ N \end{bmatrix}_q$ is equal to the sum $\sum q^{\mathrm{tot}\, A}$, where the sum is over all subsets A of cardinality N of the set $[n]$.

Remember that each signed permutation $w = x_1x_2\ldots x_n$ is characterized by a sequence (A, B, C, D, τ), where $A = \mathrm{Pix}^+ w$, $B = \mathrm{Pix}^- w$, $C = \mathrm{Neg}\, w \setminus B$, $D = [n] \setminus (A \cup \overline{B} \cup \overline{C})$ and τ is a *desarrangement* of the set $C + D$. Let $\mathrm{inv}(B, C)$ be the number of pairs of integers (i, j) such that $i \in B$, $j \in C$ and $i > j$. As $\mathrm{inv}(B, C) = \mathrm{inv}(\overline{C}, \overline{B})$, we have $\mathrm{inv}\, w = \mathrm{inv}(\overline{B}, \overline{C}) + \mathrm{inv}(A, D) + \#A \times \#C + \mathrm{inv}\, \tau$. From (3.1) it follows that

$$\begin{aligned}\ell(w) &= \mathrm{inv}\, w + \sum_{x_i<0} |x_i| = \mathrm{inv}\, w + \mathrm{tot}\, \overline{B} + \mathrm{tot}\, \overline{C}\\ &= \mathrm{tot}\, \overline{B} + \mathrm{tot}\, \overline{C} + \mathrm{inv}(\overline{C}, \overline{B}) + \mathrm{inv}(A, D) + \#A \times \#C + \mathrm{inv}\, \tau.\end{aligned}$$

Denote the right-hand side of (1.10) by $G_n := G_n(q, Y_0, Y_1, Z)$. We will calculate $G_n(q, Y_0, Y_1, Z)$ by first summing over all sequences (A, B, C, D, τ) such that $\#A = i$, $\#B = j$, $\#C = k$, $\#D = l$. Accordingly, τ is a desarrangement of a set of cardinality $k + l$. We may write:

$$\begin{aligned}G_n &= \sum_{i+j+k+l=n} \sum_{(A,B,C,D)} q^{\mathrm{tot}\, \overline{B}+\mathrm{tot}\, \overline{C}+\mathrm{inv}(\overline{C},\overline{B})+\mathrm{inv}(A,D)+i\cdot k}\\ &\qquad\qquad \times Y_0^i Y_1^j Z^{j+k} \sum_{\tau \in K_{k+l}} q^{\mathrm{inv}\, \tau}\\ &= \sum_{m+p=n} \sum_{\substack{j+k=m,\\ i+l=p}} \sum_{\substack{\#E=m,\\ F=[n]\setminus E}} \sum_{\substack{\overline{C}+\overline{B}=E,\\ A+D=F}} q^{\mathrm{tot}\, E+\mathrm{inv}(\overline{C},\overline{B})+\mathrm{inv}(A,D)+i\cdot k}\\ &\qquad\qquad \times Y_0^i (Y_1 Z)^j Z^k D_{k+l}(q)\\ &= \sum_{m+p=n} \sum_{\substack{j+k=m,\\ i+l=p}} Y_0^i (Y_1 Z)^j (Z q^i)^k D_{k+l}(q)\\ &\qquad\qquad \times \sum_{\substack{\#E=m,\\ F=[n]\setminus E}} q^{\mathrm{tot}\, E} \sum_{\substack{\overline{C}+\overline{B}=E,\\ A+D=F}} q^{\mathrm{inv}(\overline{C},\overline{B})+\mathrm{inv}(A,D)}\\ &= \sum_{m+p=n} \sum_{\substack{j+k=m,\\ i+l=p}} Y_0^i (Y_1 Z)^j (Z q^i)^k D_{k+l}(q)\\ &\qquad\qquad \times q^{m(m+1)/2} \begin{bmatrix} n \\ m \end{bmatrix}_q \begin{bmatrix} m \\ j, k \end{bmatrix}_q \begin{bmatrix} p \\ i, l \end{bmatrix}_q.\end{aligned}$$

Thus

$$G_n = \sum_{i+j+k+l=n} \begin{bmatrix} n \\ i, j, k, l \end{bmatrix}_q q^{\binom{j+k+1}{2}} Y_0^i (Y_1 Z)^j (Z q^i)^k D_{k+l}(q). \tag{3.4}$$

Now form the factorial generating function

$$G(q, Y_0, Y_1, Z; u) := \sum_{n\geq 0} \frac{u^n}{(-Zq; q)_n\, (q; q)_n} G_n(q, Y_0, Y_1, Z).$$

It follows from (3.4) that

$$G(q,Y_0,Y_1,Z;u)=\sum_{n\ge 0}\frac{1}{(-Zq;q)_n}\sum_{i+j+k+l=n} q^{\binom{j+k+1}{2}}\frac{(uY_0)^i}{(q;q)_i}\frac{(uY_1Z)^j}{(q;q)_j} \times u^{n-i-j}\frac{D_{k+l}(q)(Zq^i)^k}{(q;q)_k(q;q)_l}.$$

But $\binom{j+k+1}{2}=\binom{j+1}{2}+(j+1)k+\binom{k}{2}$. Hence

$$G(q,Y_0,Y_1,Z;u)=\sum_{n\ge 0}\frac{1}{(-Zq;q)_n}\sum_{m=0}^{n}\sum_{i+j=m} q^{\binom{j+1}{2}}\frac{(uY_0)^i}{(q;q)_i}\frac{(uY_1Z)^j}{(q;q)_j} \times\frac{u^{n-m}}{(q;q)_{n-m}}D_{n-m}(q)\sum_{k+l=n-m}\begin{bmatrix}n-m\\k,l\end{bmatrix}_q (Zq^{m+1})^k q^{\binom{k}{2}}.$$

Now

$$(-Zq^{m+1};q)_{n-m}=\sum_{k+l=n-m}\begin{bmatrix}n-m\\k,l\end{bmatrix}_q (Zq^{m+1})^k q^{\binom{k}{2}};$$

and

$$(-Zq;q)_n=(-Zq;q)_m\,(-Zq^{m+1};q)_{n-m}.$$

Hence

$$\begin{aligned}G(q,Y_0,Y_1,Z;u)&=\sum_{n\ge 0}\sum_{m=0}^{n}\frac{1}{(-Zq;q)_m}\sum_{i+j=m}\frac{(uY_0)^i}{(q;q)_i}q^{\binom{j+1}{2}}\frac{(uY_1Z)^j}{(q;q)_j} \times\frac{u^{n-m}}{(q;q)_{n-m}}D_{n-m}(q)\\ &=\Bigl(\sum_{n\ge 0}\frac{a_n\,u^n}{(-Zq;q)_n\,(q;q)_n}\Bigr)\Bigl(\sum_{n\ge 0}\frac{u^n}{(q;q)_n}D_n(q)\Bigr),\end{aligned}$$

with

$$\begin{aligned}a_n&=\sum_{i+j=n}\begin{bmatrix}n\\i,j\end{bmatrix}_q Y_0^i q^{\binom{j}{2}}(qY_1Z)^j\\ &=Y_0^n\sum_{i+j=n}\begin{bmatrix}n\\i,j\end{bmatrix}_q (qY_0^{-1}Y_1Z)^j q^{\binom{j}{2}}\\ &=Y_0^n(-qY_0^{-1}Y_1Z;q)_n.\end{aligned}$$

By taking (3.3) into account this shows that $G(q,Y_0,Y_1,Z;u)$ is equal to the right-hand side of (1.7) and then $G_n(q,Y_0,Y_1,Z)={}^{\ell}B_n(q,Y_0,Y_1,Z)$ holds for every $n\ge 0$. The proof of Theorem 1.2 is completed. By (3.4) we also conclude that the identity

$$(3.5)\quad {}^{\ell}B_n(q,Y_0,Y_1,Z)=\sum_{i+j+k+l=n}\begin{bmatrix}n\\i,j,k,l\end{bmatrix}_q q^{\binom{j+k+1}{2}}Y_0^i(Y_1Z)^j(Zq^i)^k D_{k+l}(q)$$

is equivalent to (1.7). As its right-hand side tends to the right-hand side of (2.2) when $q \to 1$, we can then assert that (1.7) specializes into (1.2) for $q = 1$.

4. Weighted signed permutations

We use the following notations: if $c = c_1 c_2 \cdots c_n$ is a word, whose letters are nonnegative integers, let $\lambda(c) := n$ be the *length* of c, $\operatorname{tot} c := c_1 + c_2 + \cdots + c_n$ the *sum* of its letters and $\operatorname{odd} c$ the number of its *odd* letters. Furthermore, NIW_n (resp. $\mathrm{NIW}_n(s)$) designates the set of all *nonincreasing* words of length n, whose letters are nonnegative integers (resp. nonnegative integers at most equal to s). Also let $\mathrm{NIW}_n^e(s)$ (resp. $\mathrm{DW}_n^o(s)$) be the subset of $\mathrm{NIW}_n(s)$ of the nonincreasing (resp. strictly decreasing) words all letters of which are *even* (resp. *odd*).

Next, each pair $\binom{c}{w}$ is called a *weighted signed permutation* of order n if the four properties (*wsp1*)–(*wsp4*) hold:

(*wsp1*) c is a word $c_1 c_2 \cdots c_n$ from NIW_n;

(*wsp2*) w is a signed permutation $x_1 x_2 \cdots x_n$ from B_n;

(*wsp3*) $c_k = c_{k+1} \Rightarrow x_k < x_{k+1}$ for all $k = 1, 2, \ldots, n-1$;

(*wsp4*) x_k is positive (resp. negative) whenever c_k is even (resp. odd).

When w has no fixed points, either negative or positive, we say that $\binom{c}{w}$ is a *weighted signed derangement*. The set of weighted signed permutations (resp. derangements) $\binom{c}{w} = \binom{c_1 c_2 \cdots c_n}{x_1 x_2 \cdots x_n}$ of order n is denoted by WSP_n (resp. by WSD_n). The subset of all those weighted signed permutations (resp. derangements) such that $c_1 \le s$ is denoted by $\mathrm{WSP}_n(s)$ (resp. by $\mathrm{WSD}_n(s)$).

For example, the following pair

$$\binom{c}{w} = \left(\begin{array}{cc|c|ccc|ccc|c|cc|c} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ 10 & 10 & 9 & 7 & 7 & 7 & 4 & 4 & 4 & 3 & 2 & 2 & 1 \\ 1 & 2 & \overline{7} & \overline{6} & \overline{5} & \overline{4} & 3 & 8 & 9 & \overline{10} & 12 & 13 & \overline{11} \end{array}\right)$$

is a weighted signed permutation of order 13. It has four positive fixed points (1, 2, 8, 9) and two negative fixed points ($\overline{5}$, $\overline{10}$).

Proposition 4.1. *With each weighted signed permutation $\binom{c}{w}$ from the set $\mathrm{WSP}_n(s)$ can be associated a unique sequence $(i, j, k, \binom{c'}{w'}, v^e, v^o)$ such that*

(1) *i, j, k are nonnegative integers of sum n;*

(2) *$\binom{c'}{w'}$ is a weighted signed derangement from the set $\mathrm{WSD}_i(s)$;*

(3) *v^e is a nonincreasing word with even letters from the set $\mathrm{NIW}_j^e(s)$;*

(4) *v^o is a decreasing word with odd letters from the set $\mathrm{DW}_k^o(s)$;*

having the following properties:

$$\begin{gathered} \operatorname{tot} c = \operatorname{tot} c' + \operatorname{tot} v^e + \operatorname{tot} v^o; \quad \operatorname{neg} w = \operatorname{neg} w' + \lambda(v^o); \\ \operatorname{fix}^+ w = \lambda(v^e); \quad \operatorname{fix}^- w = \lambda(v^o). \end{gathered} \tag{4.1}$$

The bijection $\binom{c}{w} \mapsto (\binom{c'}{w'}, v^e, v^o)$ is quite natural to define. Only its reverse requires some attention. To get the latter three-term sequence from $\binom{c}{w}$ proceed as follows:

(a) let $l_1, \ldots, l_\alpha$ (resp. $m_1, \ldots, m_\beta$) be the increasing sequence of the integers l_i (resp. m_i) such that x_{l_i} (resp. x_{m_i}) is a positive (resp. negative) fixed point of w;

(b) define: $v^e := c_{l_1} \cdots c_{l_\alpha}$ and $v^o := c_{m_1} \cdots c_{m_\beta}$;

(c) remove all the columns $\binom{c_{l_1}}{x_{l_1}}, \ldots, \binom{c_{l_\alpha}}{x_{l_\alpha}}$, $\binom{c_{m_1}}{x_{m_1}}, \ldots, \binom{c_{m_\beta}}{x_{m_\beta}}$ from $\binom{c}{w}$ and let c' be the nonincreasing word derived from c after the removal;

(d) once the letters $x_{l_1}, \ldots, x_{l_\alpha}, x_{m_1}, \ldots, x_{m_\beta}$ have been removed from the signed permutation w the remaining ones form a signed permutation of a subset A of $[n]$, of cardinality $n - \alpha - \beta$. Using the unique increasing bijection ϕ of A onto the interval $[n - \alpha - \beta]$ replace each remaining letter x_i by $\phi(x_i)$ if $x_i > 0$ or by $-\phi(-x_i)$ if $x_i < 0$. Let w' be the signed derangement of order $n - \alpha - \beta$ thereby obtained.

For instance, with the above weighted signed permutation we have: $v^e = 10, 10, 4, 4$ and $v^o = 7, 3$. After removing the fixed point columns we obtain:

$$\left(\begin{array}{c|cc|c|cc|c} 3 & 4 & 6 & 7 & 11 & 12 & 13 \\ 9 & 7 & 7 & 4 & 2 & 2 & 1 \\ \bar{7} & \bar{6} & \bar{4} & 3 & 12 & 13 & \overline{11} \end{array}\right) \quad \text{and then} \quad \begin{pmatrix} c' \\ w' \end{pmatrix} = \left(\begin{array}{c|cc|c|cc|c} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 9 & 7 & 7 & 4 & 2 & 2 & 1 \\ \bar{4} & \bar{3} & \bar{2} & 1 & 6 & 7 & \bar{5} \end{array}\right).$$

There is no difficulty verifying that the properties listed in (4.1) hold. For reconstructing $\binom{c}{w}$ from the sequence $(\binom{c'}{w'}, v^e, v^o)$ consider the nonincreasing rearrangement of the juxtaposition product $v^e v^o$ in the form $b_1^{h_1} \cdots b_m^{h_m}$, where $b_1 > \cdots > b_m$ and $h_i \geq 1$ (resp. $h_i = 1$) if b_i is even (resp. odd). The pair $\binom{c'}{w'}$ being decomposed into matrix blocks, as shown in the example, each letter b_i indicates where the h_i fixed point columns are to be inserted. We do not give more details and simply illustrate the construction with the running example.

With the previous example $b_1^{h_1} \cdots b_m^{h_m} = 10^2\, 7\, 4^2\, 3$. First, implement 10^2:

$$\left(\begin{array}{cc|c|cc|c|cc|c} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \mathbf{10} & \mathbf{10} & 9 & 7 & 7 & 4 & 2 & 2 & 1 \\ 1 & 2 & \bar{6} & \bar{5} & \bar{4} & 3 & 8 & 9 & \bar{7} \end{array}\right);$$

then 7:

$$\left(\begin{array}{cc|c|ccc|c|cc|c} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 10 & 10 & 9 & 7 & \mathbf{7} & 7 & 4 & 2 & 2 & 1 \\ 1 & 2 & \bar{7} & \bar{6} & \bar{5} & \bar{4} & 3 & 9 & 10 & \bar{8} \end{array}\right);$$

notice that because of condition (*wsp3*) the letter **7** is to be inserted in *second* position in the third block;

then insert 4^2:

$$\begin{pmatrix} 1 & 2 & | & 3 & | & 4\ 5\ 6 & | & 7\ 8\ 9 & | & 10\ 11 & | & 12 \\ 10 & 10 & | & 9 & | & 7\ 7\ 7 & | & 4\ \mathbf{4}\ \mathbf{4} & | & 2\ \ 2 & | & 1 \\ 1 & 2 & | & \overline{7} & | & \overline{6}\ \overline{5}\ \overline{4} & | & 3\ 8\ 9 & | & 11\ 12 & | & \overline{10} \end{pmatrix}.$$

The implementation of 3 gives back the original weighted signed permutation $\binom{c}{w}$.

5. Proof of Theorem 1.3

It is q-routine (see, e.g., [An76, chap. 3]) to prove the following identities, where v_1 is the first letter of v:

$$\frac{1}{(u;q)_N} = \sum_{n\geq 0} \begin{bmatrix} N+n-1 \\ n \end{bmatrix}_q u^n; \qquad \begin{bmatrix} N+n \\ n \end{bmatrix}_q = \sum_{v\in \mathrm{NIW}_n(N)} q^{\mathrm{tot}\, v};$$

$$\frac{1}{(u;q)_{N+1}} = \sum_{n\geq 0} u^n \sum_{v\in \mathrm{NIW}_n(N)} q^{\mathrm{tot}\, v} = \frac{1}{1-u} \sum_{v\in \mathrm{NIW}_n} q^{\mathrm{tot}\, v} u^{v_1};$$

$$\frac{1}{(u;q^2)_{\lfloor s/2\rfloor +1}} = \sum_{n\geq 0} u^n \sum_{v^e\in \mathrm{NIW}^e_n(s)} q^{\mathrm{tot}\, v^e}; \tag{5.1}$$

$$(-uq;q^2)_{\lfloor (s+1)/2\rfloor} = \sum_{n\geq 0} u^n \sum_{v^o\in \mathrm{DW}^o_n(s)} q^{\mathrm{tot}\, v^o}. \tag{5.2}$$

The last two formulas and Proposition 4.1 are now used to calculate the generating function for the weighted signed permutations. The symbols $\mathrm{NIW}^e(s)$, $\mathrm{DW}^o(s)$, $\mathrm{WSP}(s)$, $\mathrm{WSD}(s)$ designate the unions for $n \geq 0$ of the corresponding symbols with an n-subscript.

Proposition 5.2. *The following identity holds:*

$$\begin{aligned} &\sum_{n\geq 0} u^n \sum_{\binom{c}{w}\in \mathrm{WSP}_n(s)} q^{\mathrm{tot}\, c} Y_0^{\mathrm{fix}^+ w} Y_1^{\mathrm{fix}^- w} Z^{\mathrm{neg}\, w} \\ &\quad = \frac{(u;q^2)_{\lfloor s/2\rfloor +1}}{(uY_0;q^2)_{\lfloor s/2\rfloor +1}} \frac{(-uqY_1Z;q^2)_{\lfloor (s+1)/2\rfloor}}{(-uqZ;q^2)_{\lfloor (s+1)/2\rfloor}} \times \sum_{n\geq 0} u^n \sum_{\binom{c}{w}\in \mathrm{WSP}_n(s)} q^{\mathrm{tot}\, c} Z^{\mathrm{neg}\, w}. \end{aligned} \tag{5.3}$$

Proof. First, summing over $(w^e, w^o, \binom{c}{w}) \in \mathrm{NIW}^e(s) \times \mathrm{DW}^o(s) \times \mathrm{WSP}(s)$, we have

$$\begin{aligned} &\sum_{w^e,w^o,\binom{c}{w}} u^{\lambda(w^e)} q^{\mathrm{tot}\, w^e} \times (uZ)^{\lambda(w^o)} q^{\mathrm{tot}\, w^o} \times u^{\lambda(c)} q^{\mathrm{tot}\, c} Y_0^{\mathrm{fix}^+ w} Y_1^{\mathrm{fix}^- w} Z^{\mathrm{neg}\, w} \\ &\qquad = \frac{(-uqZ;q^2)_{\lfloor (s+1)/2\rfloor}}{(u;q^2)_{\lfloor s/2\rfloor +1}} \times \sum_{\binom{c}{w}} u^{\lambda(c)} q^{\mathrm{tot}\, c} Y_0^{\mathrm{fix}^+ w} Y_1^{\mathrm{fix}^- w} Z^{\mathrm{neg}\, w} \end{aligned} \tag{5.4}$$

by (5.1) and (5.2). Now, Proposition 4.1 implies that the initial expression can also be summed over five-term sequences $\left(\binom{c'}{w'}, v^e, v^o, w^e, w^o\right)$ from $\mathrm{WSD}(s) \times \mathrm{NIW}^e(s) \times \mathrm{DW}^o(s) \times \mathrm{NIW}^e(s) \times \mathrm{DW}^o(s)$ in the form

$$\begin{aligned}
&\sum_{\binom{c'}{w'}, v^e, v^o, w^e, w^o} u^{\lambda(c')} q^{\operatorname{tot} c'} Z^{\operatorname{neg} w'} \times (uY_0)^{\lambda(v^e)} q^{\operatorname{tot} v^e} \times (uY_1Z)^{\lambda(v^o)} q^{\operatorname{tot} v^o} \\
&\qquad\qquad\qquad\qquad \times u^{\lambda(w^e)} q^{\operatorname{tot} w^e} \times (uZ)^{\lambda(w^o)} q^{\operatorname{tot} w^o} \\
&\quad = \sum_{v^e, v^o} (uY_0)^{\lambda(v^e)} q^{\operatorname{tot} v^e} \times (uY_1Z)^{\lambda(v^o)} q^{\operatorname{tot} v^o} \\
&\qquad \times \sum_{\binom{c'}{w'}, w^e, w^o} u^{\lambda(c')} q^{\operatorname{tot} c'} Z^{\operatorname{neg} w'} \times u^{\lambda(w^e)} q^{\operatorname{tot} w^e} \times (uZ)^{\lambda(w^o)} q^{\operatorname{tot} w^o}.
\end{aligned}$$

The first summation can be evaluated by (5.1) and (5.2), while by Proposition 4.1 again the second sum can be expressed as a sum over weighted signed permutations $\binom{c}{w} \in \mathrm{WSP}(s)$. Therefore, the initial sum is also equal to

$$\frac{(-uqY_1Z;q^2)_{\lfloor (s+1)/2 \rfloor}}{(uY_0;q^2)_{\lfloor s/2 \rfloor + 1}} \times \sum_{\binom{c}{w} \in \mathrm{WSP}(s)} u^{\lambda(c)} q^{\operatorname{tot} c} Z^{\operatorname{neg} w}. \tag{5.5}$$

Identity (5.3) follows by equating (5.4) with (5.5). ∎

Proposition 5.3. *The following identity holds:*

$$\sum_{n \geq 0} u^n \sum_{\binom{c}{w} \in \mathrm{WSP}_n(s)} q^{\operatorname{tot} c} Z^{\operatorname{neg} w} = \Bigl(1 - u \sum_{i=0}^{s} q^i Z^{\chi(i \text{ odd})}\Bigr)^{-1}. \tag{5.6}$$

Proof. For proving the equivalent identity

$$\sum_{\binom{c}{w} \in \mathrm{WSP}_n(s)} q^{\operatorname{tot} c} Z^{\operatorname{neg} w} = \Bigl(\sum_{i=0}^{s} q^i Z^{\chi(i \text{ odd})}\Bigr)^n \quad (n \geq 0) \tag{5.7}$$

it suffices to construct a bijection $\binom{c}{w} \mapsto d$ of $\mathrm{WSP}_n(s)$ onto $\{0, 1, \ldots, s\}^n$ such that $\operatorname{tot} c = \operatorname{tot} d$ and $\operatorname{neg} w = \operatorname{odd} d$. This bijection is one of the main ingredients of the *MacMahon Verfahren* for signed permutations that has been fully described in [FoHa05, § 4]. We simply recall the construction of the bijection by means of an example.

Start with $\begin{pmatrix} c \\ w \end{pmatrix} = \begin{pmatrix} 10\,9\,7\,4\,4\,2\,2\,1\,1 \\ 1\,\overline{4}\,\overline{3}\,2\,5\,6\,8\,\overline{9}\,\overline{7} \end{pmatrix}$. Then, form the two-matrix $\begin{pmatrix} 10\,9\,7\,4\,4\,2\,2\,1\,1 \\ 1\,4\,3\,2\,5\,6\,8\,9\,7 \end{pmatrix}$, where the *negative* integers on the bottom row have been replaced by their opposite values. Next, rearrange its columns in such a way that the bottom row is precisely $1\,2\,\ldots\,n$. The word d is defined to be the top row in the resulting matrix. Here $\begin{pmatrix} d \\ \mathrm{Id}\end{pmatrix} = \begin{pmatrix} 10\,4\,7\,9\,4\,2\,1\,2\,1 \\ 1\,2\,3\,4\,5\,6\,7\,8\,9\end{pmatrix}$. As d is a rearrangement of c, we have $\operatorname{tot} c = \operatorname{tot} d$ and $\operatorname{neg} w = \operatorname{odd} d$. For reconstructing the pair $\binom{c}{w}$ from $d = d_1 d_2 \cdots d_n$ simply make a full use of condition (*wsp3*).

Using the properties of this bijection we have:

$$\sum_{\binom{c}{w}\in \mathrm{WSP}_n(s)} q^{\operatorname{tot} c} Z^{\operatorname{neg} w} = \sum_{d\in\{0,1,\ldots,s\}^n} q^{\operatorname{tot} d} Z^{\operatorname{odd} d} = \sum_{d\in\{0,1,\ldots,s\}^n} \prod_{i=1}^{n} q^{d_i} Z^{\chi(d_i \text{ odd})}$$
$$= \prod_{i=1}^{n} \sum_{d_i\in\{0,1,\ldots,s\}} q^{d_i} Z^{\chi(d_i \text{ odd})} = \Bigl(\sum_{i=0}^{s} q^i Z^{\chi(i \text{ odd})}\Bigr)^n. \quad \square$$

Proposition 5.4. *Let* $G_n := G_n(t, q, Y_0, Y_1, Z)$ *denote the right-hand side of* (1.12) *in the statement of Theorem* 1.3. *Then*

$$\frac{1+t}{(t^2;q^2)_{n+1}}\, G_n = \sum_{s\ge 0} t^s \sum_{\binom{c}{w}\in \mathrm{WSP}_n(s)} q^{\operatorname{tot} c} Y_0^{\operatorname{fix}^+ w} Y_1^{\operatorname{fix}^- w} Z^{\operatorname{neg} w}. \tag{5.8}$$

Proof. A very similar calculation has been made in the proof of Theorem 4.1 in [FoHa05]. We also make use of the identities on the q-ascending factorials that were recalled in the beginning of this section. First,

$$\frac{1+t}{(t^2;q^2)_{n+1}} = \sum_{r'\ge 0} (t^{2r'} + t^{2r'+1}) \begin{bmatrix} n+r' \\ r' \end{bmatrix}_{q^2}$$
$$= \sum_{r\ge 0} t^r \begin{bmatrix} n+\lfloor r/2\rfloor \\ \lfloor r/2\rfloor \end{bmatrix}_{q^2} = \sum_{r\ge 0} t^r \sum_{b\in \mathrm{NIW}_n(\lfloor r/2\rfloor)} q^{2\operatorname{tot} b}.$$

Then,

$$\frac{1+t}{(t^2;q^2)_{n+1}}\, G_n = \sum_{r\ge 0} t^r \sum_{\substack{b\in \mathrm{NIW}_n,\\ 2b_1\le r}} q^{2\operatorname{tot} b} \sum_{w\in B_n} t^{\operatorname{fdes} w} q^{\operatorname{fmaj} w} Y_0^{\operatorname{fix}^+ w} Y_1^{\operatorname{fix}^- w} Z^{\operatorname{neg} w}$$
$$= \sum_{s\ge 0} t^s \sum_{\substack{b\in \mathrm{NIW}_n,\, w\in B_n\\ 2b_1+\operatorname{fdes} w\le s}} q^{2\operatorname{tot} b+\operatorname{fmaj} w} Y_0^{\operatorname{fix}^+ w} Y_1^{\operatorname{fix}^- w} Z^{\operatorname{neg} w}.$$

As proved in [FoHa05, § 4] to each $\binom{c}{w} = \binom{c_1\cdots c_n}{x_1\cdots x_n} \in \mathrm{WSP}_n(s)$ there corresponds a unique $b = b_1\cdots b_n \in \mathrm{NIW}_n$ such that $2b_1 + \operatorname{fdes} w = c_1$ and $2\operatorname{tot} b + \operatorname{fmaj} w = \operatorname{tot} c$. Moreover, the mapping $\binom{c}{w} \mapsto (b, w)$ is a bijection of $\mathrm{WSP}_n(s)$ onto the set of all pairs (b, w) such that $b = b_1 \cdots b_n \in \mathrm{NIW}_n$ and $w \in B_n$ with the property that $2b_1 + \operatorname{fdes} w \le s$.

The word b is determined as follows: write the signed permutation w as a linear word $w = x_1 x_2 \ldots x_n$ and for each $k = 1, 2, \ldots, n$ let z_k be the number of descents ($x_i > x_{i+1}$) in the right factor $x_k x_{k+1} \cdots x_n$ and let ϵ_k be equal to 0 or 1 depending on whether x_k is positive or negative. Also for each $k = 1, 2, \ldots, n$ define $a_k := (c_k - \epsilon_k)/2$, $b_k := (a_k - z_k)$ and form the word $b = b_1 \cdots b_n$.

For example,

$$\begin{array}{rl}
\mathrm{Id} &= 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ \ 9\ \ 10 \\
c &= 9\ 7\ 7\ 4\ 4\ 4\ 2\ 2\ \ 1\ \ \ 1 \\
w &= \overline{4}\ \overline{3}\ \overline{2}\ 1\ 5\ 6\ 8\ 9\ \overline{10}\ \ \overline{7} \\
z &= 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ \ 0\ \ \ 0 \\
\epsilon &= 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ \ 1\ \ \ 1 \\
a &= 4\ 3\ 3\ 2\ 2\ 2\ 1\ 1\ \ 0\ \ \ 0 \\
b &= 3\ 2\ 2\ 1\ 1\ 1\ 0\ 0\ \ 0\ \ \ 0
\end{array}$$

Pursuing the above calculation we get (5.8). $\square$

We can complete the proof of Theorem 1.3:

$$\begin{aligned}
&\sum_{n\ge 0}(1+t)G_n(t,q,Y_0,Y_1,Z)\frac{u^n}{(t^2;q^2)_{n+1}} \\
&\quad = \sum_{s\ge 0} t^s \sum_{n\ge 0} u^n \sum_{\binom{c}{w}\in \mathrm{WSP}_n(s)} q^{\operatorname{tot} c} Y_0^{\operatorname{fix}^+ w} Y_1^{\operatorname{fix}^- w} Z^{\operatorname{neg} w} && \text{[by (5.8)]} \\
&\quad = \sum_{s\ge 0} t^s \frac{(u;q^2)_{\lfloor s/2\rfloor+1}}{(uY_0;q^2)_{\lfloor s/2\rfloor+1}} \frac{(-uqY_1Z;q^2)_{\lfloor (s+1)/2\rfloor}}{(-uqZ;q^2)_{\lfloor (s+1)/2\rfloor}} \\
&\qquad\qquad \times \sum_{n\ge 0} u^n \sum_{\binom{c}{w}\in \mathrm{WSP}_n(s)} q^{\operatorname{tot} c} Z^{\operatorname{neg} w} && \text{[by (5.3)]} \\
&\quad = \sum_{s\ge 0} t^s \Bigl(1 - u\sum_{i=0}^{s} q^i Z^{\chi(i \text{ odd})}\Bigr)^{-1} \\
&\qquad\qquad \times \frac{(u;q^2)_{\lfloor s/2\rfloor+1}}{(uY_0;q^2)_{\lfloor s/2\rfloor+1}} \frac{(-uqY_1Z;q^2)_{\lfloor (s+1)/2\rfloor}}{(-uqZ;q^2)_{\lfloor (s+1)/2\rfloor}} && \text{[by (5.6)].}
\end{aligned}$$

Hence, $G_n(t, q, Y_0, Y_1, Z) = B_n(t, q, Y_0, Y_1, Z)$ for all $n \ge 0$. $\square$

6. Specializations

For deriving the specializations of the polynomials ${}^{\ell}B_n(q, Y_0, Y_1, Z)$ and $B_n(t, q, Y_0, Y_1, Z)$ with their combinatorial interpretations we refer to the diagram displayed in Fig. 1. Those two polynomials are now regarded as generating polynomials for B_n by the multivariable statistics

$(\ell, \mathrm{pix}^+, \mathrm{pix}^-, \mathrm{neg})$ and $(\mathrm{fdes}, \mathrm{fmaj}, \mathrm{fix}^+, \mathrm{fix}^-, \mathrm{neg})$, their factorial generating functions being given by (1.7) and (1.9), respectively.

First, identity (1.8) is deduced from (1.9) by the traditional token that consists of multiplying (1.9) by $(1-t)$ and making $t=1$. Accordingly, $B_n(q, Y_0, Y_1, Z)$ occurring in (1.8) is the generating polynomial for the group B_n by the statistic $(\mathrm{fmaj}, \mathrm{fix}^+, \mathrm{fix}^-, \mathrm{neg})$.

Now, let

$$B(q, Y_0, Y_1, Z; u) := \sum_{n\ge 0} \frac{u^n}{(q^2;q^2)_n} B_n(q, Y_0, Y_1, Z). \tag{6.1}$$

The *involution* of B_n defined by $w = x_1x_2\cdots x_n \mapsto \overline{w} := \overline{x}_1\overline{x}_2\cdots\overline{x}_n$ has the following properties:

$$\mathrm{fmaj}\, w + \mathrm{fmaj}\, \overline{w} = n^2; \quad \mathrm{neg}\, w + \mathrm{neg}\, \overline{w} = n; \tag{6.2}$$

$$\mathrm{fix}^+\, w = \mathrm{fix}^-\, \overline{w}; \quad \mathrm{fix}^-\, w = \mathrm{fix}^+\, \overline{w}. \tag{6.3}$$

Consequently, the duality between positive and negative fixed points must be reflected in the expression of $B(q, Y_0, Y_1, Z; u)$ itself, as shown next.

Proposition 6.1. *We have:*

$$B(q, Y_0, Y_1, Z; u) = B(q^{-1}, Y_1, Y_0, Z^{-1}; -uq^{-1}Z). \tag{6.4}$$

Proof. The combinatorial proof consists of using the relations written in (6.2), (6.3) and easily derive the identity

$$B_n(q, Y_0, Y_1, Z) = q^{n^2} Z^n\, B_n(q^{-1}, Y_1, Y_0, Z^{-1}). \tag{6.5}$$

With this new expression for the generating polynomial identity (6.1) becomes

$$B(q, Y_0, Y_1, Z; u) = \sum_{n\ge 0} \frac{(-uq^{-1}Z)^n}{(q^{-2};q^{-2})_n} B_n(q^{-1}, Y_1, Y_0, Z^{-1}),$$

which implies (6.4).

The analytical proof consists of showing that the right-hand side of identity (1.8) is invariant under the transformation

$$(q, Y_0, Y_1, Z, u) \mapsto (q^{-1}, Y_1, Y_0, Z^{-1}, -uq^{-1}Z).$$

The factor $1 - u(1+qZ)/(1-q^2)$ is clearly invariant. As for the other two factors it suffices to expand them by means of the q-binomial theorem ([GaRa90], p. 7) and observe that they are simply permuted when the transformation is applied. $\square$

The polynomial $D_n^B(t,q,Y_1,Z) := B_n(t,q,0,Y_1,Z)$ (resp. $D_n^B(q,Y_1,Z) := B_n(q,0,Y_1,Z)$) is the generating polynomial for the set D_n^B of the *signed derangements* by the statistic $(\mathrm{fdes},\mathrm{fmaj},\mathrm{fix}^-,\mathrm{neg})$ (resp. $(\mathrm{fmaj},\mathrm{fix}^-,\mathrm{neg})$). Their factorial generating functions are obtained by letting $Y_0=0$ in (1.9) and (1.8), respectively.

Let $Y_0=0$, $Y_1=1$ in (1.8). We then obtain the factorial generating function for the polynomials $D_n^B(q,Z) := \sum q^{\mathrm{fmaj}\,w} Z^{\mathrm{neg}\,w}$ $(w\in D_n^B)$ in the form

$$\text{(6.6)}\qquad \sum_{n\ge 0}\frac{u^n}{(q^2;q^2)_n}D_n^B(q,Z) = \Bigl(1-u\frac{1+qZ}{1-q^2}\Bigr)^{-1}\times (u;q^2)_\infty.$$

It is worth writing the equivalent forms of that identity:

$$\text{(6.7)}\quad \frac{(q^2;q^2)_n}{(1-q^2)^n}(1+qZ)^n = \sum_{k=0}^n \begin{bmatrix} n\\ k\end{bmatrix}_{q^2} D_k^B(q,Z)\quad (n\ge 0);$$

$$\text{(6.8)}\quad D_n^B(q,Z)=\sum_{k=0}^n \begin{bmatrix} n\\ k\end{bmatrix}_{q^2}(-1)^k q^{k(k-1)}\frac{(q^2;q^2)_{n-k}}{(1-q^2)^{n-k}}(1+qZ)^{n-k}\quad (n\ge 0);$$

$$\text{(6.9)}\quad D_0^B(q,Z)=1,\quad \text{and for } n\ge 0$$
$$D_{n+1}^B(q,Z) = (1+qZ)\frac{1-q^{2n+2}}{1-q^2}D_n^B(q,Z) + (-1)^{n+1}q^{n(n+1)}.$$

$$\text{(6.10)}\ D_0^B(q,Z)=1,\quad D_1^B(q,Z)=Zq,\quad \text{and for } n\ge 1$$
$$D_{n+1}^B(q,Z) = \Bigl(\frac{1-q^{2n}}{1-q^2}+qZ\frac{1-q^{2n+2}}{1-q^2}\Bigr)D_n^B(q,Z) + (1+qZ)q^{2n}\frac{1-q^{2n}}{1-q^2}D_{n-1}^B(q,Z).$$

Note that (6.8) is derived from (6.6) by taking the coefficients of u^n on both sides. Next, multiply both sides of (6.6) by the second q^2-exponential $E_{q^2}(-u)$ and look for the coefficients of u^n on both sides. This yields (6.7). Now, write (6.6) in the form

$$\text{(6.11)}\qquad E_{q^2}(-u) = \Bigl(1-u\frac{1+qZ}{1-q^2}\Bigr)\sum_{n\ge 0}\frac{u^n}{(q^2;q^2)_n}D_n^B(q,Z)$$

and take the coefficients of u^n on both sides. This yields (6.9). Finally, (6.10) is a simple consequence of (6.9).

When $Z=1$, formulas (6.6), (6.8), (6.9) have been proved by Chow [Ch06] with $D_n^B(q)=\sum_w q^{\mathrm{fmaj}\,w}$ $(w\in D_n^B)$.

Now the polynomial $K_n^B(q, Y_1, Z) := {}^\ell B_n(q, 0, Y_1, Z)$ is the generating polynomial for the set K_n^B of the *signed desarrangements* by the statistic $(\ell, \mathrm{pix}^-, \mathrm{neg})$. From (1.7) we get

$$\text{(6.12)}\qquad \sum_{n\ge 0} \frac{u^n}{(-Zq;q)_n\,(q;q)_n} K_n^B(q, Y_1, Z) = \Bigl(1-\frac{u}{1-q}\Bigr)^{-1} \times (u;q)_\infty \Bigl(\sum_{n\ge 0} \frac{q^{\binom{n+1}{2}}(Y_1 Z u)^n}{(-Zq;q)_n\,(q;q)_n}\Bigr).$$

When the variable Z is given the zero value, the polynomials in the second column of Fig. 1 are mapped on generating polynomials for the *symmetric group*, listed in the third column. Also the variable Y_1 vanishes. Let $A_n(t, q, Y_0) := B_n(t^{1/2}, q^{1/2}, Y_0, 0, 0)$. Then

$$\text{(6.13)}\qquad A_n(t, q, Y_0) = \sum_{\sigma\in\mathfrak{S}_n} t^{\operatorname{des}\sigma} q^{\operatorname{maj}\sigma} Y_0^{\operatorname{fix}\sigma} \quad (\mathrm{fix} := \mathrm{fix}^+).$$

Identity (1.9) specializes into

$$\text{(6.14)}\qquad \sum_{n\ge 0} A_n(t, q, Y_0) \frac{u^n}{(t;q)_{n+1}} = \sum_{s\ge 0} t^s \Bigl(1 - u\sum_{i=0}^{s} q^i\Bigr)^{-1} \frac{(u;q)_{s+1}}{(uY_0;q)_{s+1}},$$

an identity derived by Gessel and Reutenauer ([GeRe93], Theorem 8.4) by means of a quasi-symmetric function technique. Note that they wrote their formula for "$1 + \mathrm{des}$" and not for "des."

Multiply (6.14) by $(1-t)$ and let $t := 1$, or let $Z := 0$ and q^2 be replaced by q in (1.8). Also, let $A_n(q, Y_0) := \sum_\sigma q^{\operatorname{maj}\sigma} Y^{\operatorname{fix}\sigma}$ $(\sigma \in \mathfrak{S}_n)$; we get

$$\text{(6.15)}\qquad \sum_{n\ge 0} \frac{u^n}{(q;q)_n} A_n(q, Y_0) = \Bigl(1-\frac{u}{1-q}\Bigr)^{-1} \frac{(u;q)_\infty}{(uY_0;q)_\infty},$$

an identity derived by Gessel and Reutenauer [GeRe93] and also by Clarke *et al.* [ClHaZe97] by means of a q-Seidel matrix approach.

We do not write the specialization of (6.14) when $Y_0 := 0$ to obtain the generating function for the polynomials $D_n(t, q) := \sum_{\sigma\in D_n} t^{\operatorname{des}\sigma} q^{\operatorname{maj}\sigma}$. As for the polynomial $D_n(q) := \sum_{\sigma\in D_n} q^{\operatorname{maj}\sigma}$, it has several analytical expressions, which can all be derived from (6.7)–(6.10) by letting $Z := 0$ and q^2 being replaced by q. We only write the identity which corresponds to (6.7)

$$\text{(6.16)}\qquad D_0(q) = 1 \quad\text{and}\quad \frac{(q;q)_n}{(1-q)^n} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q D_k(q) \quad \text{for } n \ge 1,$$

which is then equivalent to the identity

$$\text{(6.17)}\qquad e_q(u) \sum_{n\ge 0} \frac{u^n}{(q;q)_n} D_n(q) = \Bigl(1-\frac{u}{1-q}\Bigr)^{-1}.$$

The specialization of (6.8) for $Z := 0$ and q^2 replaced by q was originally proved by Wachs [Wa98] and again recently by Chen and Xu [ChXu06]. Those two authors make use of the now classical *MacMahon Verfahren*, that has been exploited in several papers and further extended to the case of signed permutation, as described in our previous paper [FoHa05].

In the next proposition we show that $D_n(q)$ can be expressed as a polynomial in q with *positive integral* coefficients. In the same manner, the usual derangement number d_n is an explicit sum of *positive* integers. To the best of the authors' knowledge those formulas have not appeared elsewhere. In (6.19) we make use of the traditional notation for the *ascending factorial*: $(a)_n = 1$ if $n = 0$ and $(a)_n = a(a+1)\cdots(a+n-1)$ if $n \geq 1$.

Proposition 6.2. *The following expressions hold:*

$$(6.18)\quad D_n(q) = \sum_{2\leq 2k\leq n-1} \frac{1-q^{2k}}{1-q}\,\frac{(q^{2k+2};q)_{n-2k-1}}{(1-q)^{n-2k-1}}\,q^{\binom{2k}{2}} + q^{\binom{n}{2}}\chi(n \text{ even}),$$

$$(6.19)\qquad d_n = \sum_{2\leq 2k\leq n-1} (2k)(2k+2)_{n-2k-1} + \chi(n \text{ even}).$$

Proof. When $q = 1$, then (6.18) is transformed into (6.19). As for (6.18), an easy q-calculation shows that its right-hand side satisfies (6.9) when $Z = 0$ and q replaced by $q^{1/2}$. As shown by Fu [Fu06], identity (6.18) can also be directly derived from (6.17) by a simple q-calculation. $\square$

Now, let ${}^{\mathrm{inv}}A_n(q, Y_0) := {}^{\ell}B_n(q, Y_0, 0, 0)$. Then

$$(6.20)\qquad {}^{\mathrm{inv}}A_n(q, Y_0) = \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{inv}\,\sigma}\, Y_0^{\mathrm{pix}\,\sigma} \quad (\mathrm{pix} := \mathrm{pix}^+).$$

Formula (1.7) specializes into

$$(6.21)\quad \sum_{n\geq 0} \frac{u^n}{(q;q)_n}\,{}^{\mathrm{inv}}A_n(q, Y_0) = \Bigl(1 - \frac{u}{1-q}\Bigr)^{-1} \frac{(u;q)_\infty}{(uY_0;q)_\infty};$$

In view of (6.15) we conclude that

$$(6.22)\qquad A_n(q, Y_0) = {}^{\mathrm{inv}}A_n(q, Y_0).$$

For each permutation $\sigma = \sigma(1)\cdots\sigma(n)$ let the *ligne of route* of σ be defined by $\mathrm{Ligne}\,\sigma := \{i : \sigma(i) > \sigma(i+1)\}$ and the *inverse ligne of route* by $\mathrm{Iligne}\,\sigma := \mathrm{Ligne}\,\sigma^{-1}$. Notice that $\mathrm{maj}\,\sigma = \sum_i i\,\chi(i \in \mathrm{Ligne}\,\sigma)$; we also let $\mathrm{imaj}\,\sigma := \sum_i i\,\chi(i \in \mathrm{Iligne}\,\sigma)$. Furthermore, let $\mathbf{i} : \sigma \mapsto \sigma^{-1}$. If Φ designates the *second fundamental transformation* described in [Fo68], [FoSc78], it is known that the bijection $\Psi := \mathbf{i}\,\Phi\,\mathbf{i}$ of $\mathfrak{S}_n$ onto itself has the following property: $(\mathrm{Ligne}, \mathrm{imaj})\,\sigma = (\mathrm{Ligne}, \mathrm{inv})\,\Psi(\sigma)$. Hence

$$(6.23)\qquad (\mathrm{pix}, \mathrm{imaj})\,\sigma = (\mathrm{pix}, \mathrm{inv})\,\Psi(\sigma)$$

and then $A_n(q, Y_0)$ has the other interpretation:

$$A_n(q, Y_0) = \sum_{\sigma \in \mathfrak{S}_n} q^{\operatorname{imaj} \sigma} Y_0^{\operatorname{pix} \sigma}. \tag{6.24}$$

Finally, let $K_n(q) := \sum_{\sigma \in K_n} q^{\operatorname{inv} \sigma}$. Then, with $Y_0 := 0$ in (6.22) we have:

$$K_n(q) = {}^{\operatorname{inv}}A_n(q, 0) = A_n(q, 0) = D_n(q). \tag{6.25}$$

However, it can be shown directly that $K_n(q)$ is equal to the right-hand side of (6.18), because the sum occurring in (6.18) reflects the geometry of the desarrangements. The running term is nothing but the generating polynomial for the desarrangements of order n whose leftmost trough is at position $2k$ by the number of inversions "inv."

The bijection Ψ also sends K_n onto itself, so that

$$\sum_{\sigma \in K_n} q^{\operatorname{inv} \sigma} = \sum_{\sigma \in K_n} q^{\operatorname{imaj} \sigma}, \tag{6.26}$$

a result obtained in this way by Désarménien and Wachs [DeWa90, 93], who also proved that for every subset $E \subset [n-1]$ we have

$$\#\{\sigma \in D_n : \operatorname{Ligne} \sigma = E\} = \#\{\sigma \in K_n : \operatorname{Iligne} \sigma = E\}. \tag{6.27}$$

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SIGNED WORDS AND PERMUTATIONS, V; A SEXTUPLE DISTRIBUTION

Dominique Foata and Guo-Niu Han

Abstract

We calculate the distribution of the sextuple statistic over the hyperoctahedral group B_n that involves the flag-excedance and flag-descent numbers "fexc" and "fdes," the flag-major index "fmaj," the positive and negative fixed point numbers "fix^+" and "fix^-" and the negative letter number "neg." Several specializations are considered. In particular, the joint distribution for the pair (fexc, fdes) is explicitly derived.

1. Introduction

As has been shown in our series of four papers ([FoHa07], [FoHa05], [FoHa06], [FoHa07a]), the *length function* "ℓ" (see [Bo68], p. 7, or [Hu90], p. 12) and the *flag major index* "fmaj" introduced by Adin and Roichman [AR01] have become the true q-analog makers for the calculation of various multivariable distributions on the hyperoctahedral group B_n of the signed permutations. The elements of B_n may be viewed as words $w = x_1x_2\cdots x_n$, where each x_i belongs to the set $\{-n,\ldots,-1,1,\ldots,n\}$ and $|x_1||x_2|\cdots|x_n|$ is a permutation of $12\ldots n$. The *set* (resp. the *number*) of *negative* letters among the x_i's is denoted by $\mathrm{Neg}\, w$ (resp. $\mathrm{neg}\, w$). A *positive fixed point* of the signed permutation $w = x_1x_2\cdots x_n$ is a (positive) integer i such that $x_i = i$. It is convenient to write $\bar{i} := -i$ for each integer i. If $x_i = \bar{i}$ with i positive, we say that $\bar{i}$ is a *negative fixed point* of w. The set of all positive (resp. negative) fixed points of w is denoted by $\mathrm{Fix}^+ w$ (resp. $\mathrm{Fix}^- w$). Notice that $\mathrm{Fix}^- w \subset \mathrm{Neg}\, w$. Also let

$$\mathrm{fix}^+ w := \#\,\mathrm{Fix}^+ w;\quad \mathrm{fix}^- w := \#\,\mathrm{Fix}^- w. \tag{1.1}$$

There are $2^n n!$ signed permutations of order n. The symmetric group $\mathfrak{S}_n$ may be considered as the subset of all w from B_n such that $\mathrm{Neg}\, w = \emptyset$. When w is an (ordinary) permutation from $\mathfrak{S}_n$, then $\mathrm{Fix}^- w = \emptyset$, so that we define $\mathrm{Fix}\, w := \mathrm{Fix}^+ w$ and $\mathrm{fix}\, w = \#\,\mathrm{Fix}\, w$.

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Now, for each statement A let $\chi(A) = 1$ or 0 depending on whether A is true or not. Besides the integer-valued statistics "fix^+," "fix^-" and "neg" the now classical *flag-descent number* "fdes", *flag-major index* "fmaj" and *flag-excedance number* "fexc" are also needed in the following study. They are defined for each signed permutation $w = x_1 x_2 \cdots x_n$ by

$$\begin{aligned} \mathrm{fdes}\, w &:= 2\,\mathrm{des}\, w + \chi(x_1 < 0); \\ \mathrm{fmaj}\, w &:= 2\,\mathrm{maj}\, w + \mathrm{neg}\, w; \\ \mathrm{fexc}\, w &:= 2\,\mathrm{exc}\, w + \mathrm{neg}\, w; \end{aligned}$$

where "des" is the *number of descents* $\mathrm{des}\, w := \sum_{i=1}^{n-1} \chi(x_i > x_{i+1})$, "maj" the *major index* $\mathrm{maj}\, w := \sum_{i=1}^{n-1} i\,\chi(x_i > x_{i+1})$ and "exc" the *number of excedances* $\mathrm{exc}\, w := \sum_{i=1}^{n-1} \chi(x_i > i)$.

Our intention is to calculate the distribution of the sextuple statistic $(\mathrm{fexc}, \mathrm{fdes}, \mathrm{fmaj}, \mathrm{fix}^+, \mathrm{fix}^-, \mathrm{neg})$ on the group B_n. This means that for each $n \geq 0$ the generating polynomial

$$(1.2)\quad B_n(s,t,q,Y_0,Y_1,Z) := \sum_{w \in B_n} s^{\mathrm{fexc}\, w} t^{\mathrm{fdes}\, w}\, q^{\mathrm{fmaj}\, w}\, Y_0^{\mathrm{fix}^+ w}\, Y_1^{\mathrm{fix}^- w}\, Z^{\mathrm{neg}\, w}$$

can be considered and, with a suitable normalization, the generating function for those polynomials be summed. Furthermore, the summation is to yield an appropriate closed form. Finally, the calculation must be compatible with the symmetric group $\mathfrak{S}_n$ in the sense that when the variable Z is given the zero value, earlier results derived for that group are to be recovered. As is shown below, this goal will be achieved by working in the algebra of q-series.

Our derivation has been motivated by the following recent statistical studies on B_n and $\mathfrak{S}_n$. In [FoHa07a] and [FoHa07b] we have respectively calculated the generating functions for the polynomials $B_n(1,t,q,Y_0,Y_1,Z)$ and $A_n(s,t,q,Y_0)$, where

$$(1.3)\qquad A_n(s,t,q,Y) := \sum_{\sigma \in \mathfrak{S}_n} s^{\mathrm{exc}\, \sigma} t^{\mathrm{des}\, \sigma} q^{\mathrm{maj}\, \sigma} Y^{\mathrm{fix}\, \sigma}$$

and shown that by giving certain variables specific values the former statistical results on $\mathfrak{S}_n$ and B_n could be reobtained. Note that, using the previous definitions of "fexc," "fdes" and "fmaj," the latter polynomial is nothing but $B_n(s^{1/2}, t^{1/2}, q^{1/2}, Y, 0, 0)$.

In the diagram of Fig. 1 the polynomials on the top (resp. bottom) level are specializations of the polynomial $B_n(s,t,q,Y_0,Y_1,Z)$ (resp. $A_n(s,t,q,Y)$). Each vertical arrow is given a $(Z=0)$-label. This means that when Z is given the 0-value, each polynomial $B_n(\cdots)$ is transformed into the corresponding polynomial $A_n(\cdots)$. The other arrows labelled s, t and Z indicate that each target polynomial is mapped onto the source polynomial when s (resp. t, resp. Z) is given the 1-value.

$$\begin{tikzcd}
 & & \mathbf{B_n(s,1,q,Y_0,Y_1,1)} \arrow[rr, "t"] \arrow[dl, "Z"'] & & \mathbf{B_n(s,t,q,Y_0,Y_1,1)} \arrow[dl, "Z"] & \\
 & \mathbf{B_n(s,1,q,Y_0,Y_1,Z)} \arrow[rr, "t"] & & \mathbf{B_n(s,t,q,Y_0,Y_1,Z)} & & \\
B_n(1,1,q,Y_0,Y_1,Z) \arrow[rr, "t"] \arrow[ur, "s"] & & B_n(1,t,q,Y_0,Y_1,Z) \arrow[ur, "s"] & & & \\
 & A_n(s,1,q,Y) \arrow[rr, "t"] \arrow[uu, "Z=0"] & & A_n(s,t,q,Y) \arrow[uu, "Z=0"'] & & \\
A_n(1,1,q,Y) \arrow[rr, "t"'] \arrow[uu, "Z=0"] \arrow[ur, "s"] & & A_n(1,t,q,Y) \arrow[uu, "Z=0"] \arrow[ur, "s"'] & & &
\end{tikzcd}$$

Fig. 1

The polynomials appearing on the top are all generating polynomials for the group B_n. Four of them are reproduced in boldface. Their factorial generating polynomials are derived in the present paper. The generating functions for the other six polynomials have been explicitly determined in earlier papers: $A_n(1,1,q,Y)$ and $A_n(1,t,q,Y)$ by Gessel and Reutenauer [GeRe03], then $A_n(s,1,q,Y)$ by Shareshian and Wachs [ShWa06], furthermore $A_n(s,t,q,Y)$ in [FoHa07b], finally $B_n(1,1,q,Y_0,Y_1,Z)$ and $B_n(1,t,q,Y_0,Y_1,Z)$ in [FoHa07a].

The classical notation on q-series will be used. First, the q-ascending factorials

$$(a;q)_n := \begin{cases} 1, & \text{if } n=0; \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & \text{if } n\geq 1. \end{cases}$$

$$(a;q)_\infty := \prod_{n\geq 1}(1-aq^{n-1});$$

then, the two q-exponentials (see [GaRa90, chap. 1])

$$e_q(u)=\sum_{n\geq 0}\frac{u^n}{(q;q)_n}=\frac{1}{(u;q)_\infty};\quad E_q(u)=\sum_{n\geq 0}q^{\binom{n}{2}}\frac{u^n}{(q;q)_n}=(-u;q)_\infty.$$

The main purpose of this paper is to prove the following theorem.

Theorem 1.1. *For each $n\geq 0$ let $B_n(s,t,q,Y_0,Y_1,Z)$ be the generating polynomial for the hyperoctahedral group B_n by the six-variable statistic* $(\mathrm{fexc},\mathrm{fdes},\mathrm{fmaj},\mathrm{fix}^+,\mathrm{fix}^-,\mathrm{neg})$ *as defined in* (1.2). *Then, the factorial generating function for the polynomials $B_n(s,t,q,Y_0,Y_1,Z)$ is given by:*

$$(1.4)\qquad \sum_{n\geq 0}(1+t)B_n(s,t,q,Y_0,Y_1,Z)\frac{u^n}{(t^2;q^2)_{n+1}}$$
$$=\sum_{r\geq 0}t^r\frac{(u;q^2)_{\lfloor r/2\rfloor+1}}{(uY_0;q^2)_{\lfloor r/2\rfloor+1}}\frac{(-usqY_1Z;q^2)_{\lfloor (r+1)/2\rfloor}}{(-usqZ;q^2)_{\lfloor (r+1)/2\rfloor}}F_r(u;s,q,Z),$$

where

$$(1.5)\quad F_r(u;s,q,Z) = \frac{(us^2q^2;q^2)_{\lfloor r/2\rfloor}\,(1-s^2q^2)\,(u;q^2)_{\lfloor (r+1)/2\rfloor}\,(u;q^2)_{\lfloor r/2\rfloor}}{(u;q^2)_{\lfloor r/2\rfloor+1}\Big((u;q^2)_{\lfloor (r+1)/2\rfloor}\big((u;q^2)_{\lfloor r/2\rfloor}-s^2q^2\,(us^2q^2;q^2)_{\lfloor r/2\rfloor}\big) + sqZ\,(u;q^2)_{\lfloor r/2\rfloor}\big((u;q^2)_{\lfloor (r+1)/2\rfloor}-(us^2q^2;q^2)_{\lfloor (r+1)/2\rfloor}\big)\Big)}.$$

The factorial generating functions for the other polynomials written in boldface in the diagram of Fig. 1 are shown to be specializations of the factorial generating function for the six-variable polynomials $B_n(s,t,q,Y_0,Y_1,Z)$, as stated in the next three Corollaries.

Corollary 1.2. *The factorial generating function for the polynomials $B_n(s,1,q,Y_0,Y_1,Z)$ is given by*

$$(1.6)\quad \sum_{n\ge 0} B_n(s,1,q,Y_0,Y_1,Z)\frac{u^n}{(q^2;q^2)_n} = \frac{e_{q^2}(uY_0)\,E_{q^2}(usqY_1Z)}{E_{q^2}(usqZ)} \times \frac{(1-s^2q^2)}{e_{q^2}(us^2q^2)-s^2q^2e_{q^2}(u)+sqZ(e_{q^2}\big(us^2q^2)-e_{q^2}(u)\big)}.$$

Corollary 1.3. *The factorial generating function for the polynomials $B_n(s,t,q,Y_0,Y_1,1)$ is given by*

$$(1.7)\quad \sum_{n\ge 0}(1+t)B_n(s,t,q,Y_0,Y_1,1)\frac{u^n}{(t^2;q^2)_{n+1}} = \sum_{r\ge 0} t^r\,\frac{(u;q^2)_{\lfloor r/2\rfloor+1}}{(uY_0;q^2)_{\lfloor r/2\rfloor+1}}\,\frac{(-usqY_1;q^2)_{\lfloor (r+1)/2\rfloor}}{(-usq;q^2)_{\lfloor (r+1)/2\rfloor}}\,F_r(u;s,q,1),$$

where $F_r(u;s,q,1)$ is given by

$$(1.8)\quad F_r(u;s,q,1) = \frac{(us^2q^2;q^2)_{\lfloor r/2\rfloor}}{(u;q^2)_{\lfloor r/2\rfloor+1}} \times \frac{(1-sq)\,(u;q^2)_{\lfloor (r+1)/2\rfloor}\,(usq;q^2)_{\lfloor r/2\rfloor}}{(u;q^2)_{\lfloor (r+1)/2\rfloor}\,(usq;q^2)_{\lfloor r/2\rfloor}-sq\,(usq;q^2)_{\lfloor (r+1)/2\rfloor}\,(us^2q^2;q^2)_{\lfloor r/2\rfloor}}.$$

Corollary 1.4. *The factorial generating function for the polynomials $B_n(s,1,q,Y_0,Y_1,1)$ is given by*

$$(1.9)\quad \sum_{n\ge 0} B_n(s,1,q,Y_0,Y_1,1)\frac{u^n}{(q^2;q^2)_n} = \frac{e_{q^2}(uY_0)\,E_{q^2}(usqY_1)}{E_{q^2}(usq)}\,\frac{(1-sq)}{e_{q^2}(us^2q^2)-sq\,e_{q^2}(u)}.$$

The proof of Theorem 1.1 requires several steps. First, the factorial generating function for the polynomials $B_n(s,t,q,Y_0,Y_1,Z)$, as it appears on the left-hand side of (1.4), is shown to be equal to a series $\sum\limits_{r\geq 0} t^r a_r$, where each a_r is the product of the rational function

$$\frac{(u;q^2)_{\lfloor r/2\rfloor+1}}{(uY_0;q^2)_{\lfloor r/2\rfloor+1}}\frac{(-usqY_1Z;q^2)_{\lfloor (r+1)/2\rfloor}}{(-usqZ;q^2)_{\lfloor (r+1)/2\rfloor}}$$

by the generating series $\sum u^{\lambda w}s^{\mathrm{fexc}\,w}q^{\mathrm{tot}\,c}Z^{\mathrm{neg}\,w}$ for the set $\mathrm{WSP}(r)$ of the so-called *weighted signed permutations* by a certain four-variable statistic $(\lambda,\mathrm{tot},\mathrm{fexc},\mathrm{neg})$ (see Theorem 4.2). The combinatorics of the weighted signed permutations was introduced in our previous paper [FoHa07a], but this time the extra variable "s" is to be added to make the calculation. All this derivation is developed in Sections 2 and 3. Noticeably, the variables Y_0 and Y_1 that carry the information on fixed points occur only in the above fraction, but not in the latter generating series. Also, the positive fixed point counter Y_0 occurs in the *denominator* of that fraction, while the negative fixed point counter Y_1 only in the *numerator*.

The next step is to show that this generating series for weighted signed permutations is also equal to the generating series for *words*, whose letters belong to the interval $[0,r]$, by another four-variable statistic $(\lambda,\mathrm{tot}\,v,\mathrm{evdec}+\mathrm{odd},\mathrm{odd})$. This is achieved by means of the construction of a *bijection* of the set of weighted signed permutations onto the set of those words having the adequate properties (see Theorem 5.1).

The crucial calculation is to evaluate the latter generating function and show that it is equal to the expression $F_r(u;s,q,Z)$ displayed in (1.5). Two proofs are given, the first one using a V-word *decomposition* theorem (see Theorem 6.1) derived in [FoHa07b] together with the traditional q-series *telescoping* technique, the second one taking advantage of a *word factorization*, which consists of cutting each word after every odd letter (see Section 8).

Formula (1.6) (resp. (1.9)) is deduced from (1.4) (resp. from (1.7)) by the traditional token that consists of multiplying the latter one by $(1-t)$ and letting r tend to $+\infty$, so that Corollaries 1.2 and 1.4 are easy consequences of Theorem 1.1 and Corollary 1.3.

We prove Corollary 1.3 in two different ways: first, as a specialization of Theorem 1.1 by an evaluation of the fraction $F_r(u;s,q,Z)$ for $Z=1$ (this requires some manipulations on q-series), second, by showing directly that $F_r(u;s,q,1)$ is the generating function for words by the two-variable statistic $(\mathrm{tot},2\,\mathrm{evdec}+\mathrm{odd})$, using a bijection constructed in our previous paper [FoHa07b]. We end the paper by deriving several specializations of Theorem 1.1, in particular, the joint distribution of the pair $(\mathrm{fexc},\mathrm{fdes})$ over the group B_n.

2. Weighted signed permutations

This section will appear to be an updated version of Section 4 of our previous paper [FoHa07a], where the notion of *weighted signed permutation* was introduced. With the addition of "fexc" it was essential to ascertain how that statistic behaved in the underlying combinatorial construction.

We use the following notations: if $c = c_1c_2\cdots c_n$ is a word, whose letters are nonnegative integers, let $\lambda(c) := n$ be the *length* of c, $\operatorname{tot} c := c_1+c_2+\cdots+c_n$ the *sum* of its letters and $\operatorname{odd} c$ the number of its *odd* letters. Furthermore, NIW_n (resp. $\mathrm{NIW}_n(r)$) designates the set of all *nonincreasing* monotonic words of length n, whose letters are nonnegative integers (resp. nonnegative integers at most equal to r). Also let $\mathrm{NIW}_n^e(r)$ (resp. $\mathrm{DW}_n^o(r)$) be the subset of $\mathrm{NIW}_n(r)$ of the monotonic nonincreasing (resp. strictly decreasing) words all letters of which are *even* (resp. *odd*).

Next, each pair $\binom{c}{w}$ is called a *weighted signed permutation* of order n if the four properties (*wsp1*)–(*wsp4*) hold:

(*wsp1*) c is a word $c_1c_2\cdots c_n$ from NIW_n;
(*wsp2*) w is a signed permutation $x_1x_2\cdots x_n$ from B_n;
(*wsp3*) $c_k = c_{k+1} \Rightarrow x_k < x_{k+1}$ for all $k = 1, 2, \ldots, n-1$;
(*wsp4*) x_k is positive (resp. negative) whenever c_k is even (resp. odd).

When w has no fixed points, either negative or positive, we say that $\binom{c}{w}$ is a *weighted signed derangement*. The set of weighted signed permutations (resp. derangements) $\binom{c}{w} = \binom{c_1c_2\cdots c_n}{x_1x_2\cdots x_n}$ of order n is denoted by WSP_n (resp. by WSD_n). The subset of all those weighted signed permutations (resp. derangements) such that $c_1 \le r$ is denoted by $\mathrm{WSP}_n(r)$ (resp. by $\mathrm{WSD}_n(r)$).

For example, the following pair

$$\binom{c}{w} = \left(\begin{array}{cc|c|ccc|ccc|c|cc|c} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ 10 & 10 & 9 & 7 & 7 & 7 & 4 & 4 & 4 & 3 & 2 & 2 & 1 \\ \mathbf{1} & \mathbf{2} & \overline{7} & \overline{6} & \overline{\mathbf{5}} & \overline{4} & 3 & \mathbf{8} & \mathbf{9} & \overline{\mathbf{10}} & \underline{12} & \underline{13} & \overline{11} \end{array}\right)$$

is a weighted signed permutation of order 13. It has four positive fixed points $(\mathbf{1}, \mathbf{2}, \mathbf{8}, \mathbf{9})$, two negative fixed points $(\overline{\mathbf{5}}, \overline{\mathbf{10}})$ and two excedances $(\underline{12}, \underline{13})$.

Proposition 2.1. *With each weighted signed permutation $\binom{c}{w}$ from the set $\mathrm{WSP}_n(r)$ can be associated a unique sequence $(i, j, k, \binom{c'}{w'}, v^e, v^o)$ such that*

(1) *i, j, k are nonnegative integers of sum n;*
(2) *$\binom{c'}{w'}$ is a weighted signed derangement from the set $\mathrm{WSD}_i(r)$;*
(3) *v^e is a nonincreasing word with even letters from the set $\mathrm{NIW}_j^e(r)$;*
(4) *v^o is a decreasing word with odd letters from the set $\mathrm{DW}_k^o(r)$;*

having the following properties:

$$\begin{aligned} &\operatorname{tot} c = \operatorname{tot} c' + \operatorname{tot} v^e + \operatorname{tot} v^o; \quad \operatorname{neg} w = \operatorname{neg} w' + \lambda(v^o); \\ &\operatorname{fix}^+ w = \lambda(v^e); \quad \operatorname{fix}^- w = \lambda(v^o); \quad \operatorname{fexc} w = \operatorname{fexc} w' + \lambda(v^0). \end{aligned} \tag{2.1}$$

The bijection $\binom{c}{w} \mapsto (\binom{c'}{w'}, v^e, v^o)$ is quite natural to define. Only its reverse requires some attention. To get the latter three-term sequence from $\binom{c}{w}$ proceed as follows:

(a) let $l_1, \ldots, l_\alpha$ (resp. $m_1, \ldots, m_\beta$) be the increasing sequence of the integers l_i (resp. m_i) such that x_{l_i} (resp. x_{m_i}) is a positive (resp. negative) fixed point of w;

(b) define: $v^e := c_{l_1} \cdots c_{l_\alpha}$ and $v^o := c_{m_1} \cdots c_{m_\beta}$;

(c) remove all the columns $\binom{c_{l_1}}{x_{l_1}}, \ldots, \binom{c_{l_\alpha}}{x_{l_\alpha}}$, $\binom{c_{m_1}}{x_{m_1}}, \ldots, \binom{c_{m_\beta}}{x_{m_\beta}}$ from $\binom{c}{w}$ and let c' be the nonincreasing word derived from c after the removal;

(d) once the letters $x_{l_1}, \ldots, x_{l_\alpha}, x_{m_1}, \ldots, x_{m_\beta}$ have been removed from the signed permutation w the remaining ones form a signed permutation of a subset A of $[n]$, of cardinality $n - \alpha - \beta$. Using the unique increasing bijection ϕ of A onto the interval $[n - \alpha - \beta]$ replace each remaining letter x_i by $\phi(x_i)$ if $x_i > 0$ or by $-\phi(-x_i)$ if $x_i < 0$. Let w' be the signed derangement of order $n - \alpha - \beta$ thereby obtained. There is no difficulty verifying that the properties listed in (2.1) hold.

For instance, with the above weighted signed permutation we have: $v^e = 10, 10, 4, 4$ and $v^o = 7, 3$. After removing the fixed point columns we obtain:

$$\left(\begin{array}{c|cc|c|cc|c} 3 & 4 & 6 & 7 & 11 & 12 & 13 \\ 9 & 7 & 7 & 4 & 2 & 2 & 1 \\ \overline{7} & \overline{6} & \overline{4} & 3 & \underline{12} & \underline{13} & \overline{11} \end{array}\right) \text{ and then } \begin{pmatrix} c' \\ w' \end{pmatrix} = \left(\begin{array}{c|cc|c|cc|c} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 9 & 7 & 7 & 4 & 2 & 2 & 1 \\ \overline{4} & \overline{3} & \overline{2} & 1 & \underline{6} & \underline{7} & \overline{5} \end{array}\right).$$

The signed permutation w (resp. w') has two excedances $\underline{12}$, $\underline{13}$ (resp. $\underline{6}, \underline{7}$) and six (resp. four) negative letters. Furthermore, $v^0 = 7, 3$ is of length 2. Hence $\operatorname{fexc} w = 2 \operatorname{exc} w + \operatorname{neg} w = 2 \times 2 + 6 = 10 = 2 \times 2 + 4 + 2 = \operatorname{fexc} w' + \lambda(v^0)$.

For reconstructing $\binom{c}{w}$ from the sequence $(\binom{c'}{w'}, v^e, v^o)$ consider the nonincreasing rearrangement of the juxtaposition product $v^e v^o$ in the form $b_1^{h_1} \cdots b_m^{h_m}$, where $b_1 > \cdots > b_m$ and $h_i \geq 1$ (resp. $h_i = 1$) if b_i is even (resp. odd). The pair $\binom{c'}{w'}$ being decomposed into matrix blocks, as shown in the example, each letter b_i indicates where the h_i fixed point columns are to be inserted. We do not give more details and simply illustrate the construction with the running example.

With the previous example $b_1^{h_1} \cdots b_m^{h_m} = 10^2\, 7\, 4^2\, 3$. First, implement 10^2:

$$\left(\begin{array}{cc|c|cc|c|cc|c} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \mathbf{10} & \mathbf{10} & 9 & 7 & 7 & 4 & 2 & 2 & 1 \\ 1 & 2 & \overline{6} & \overline{5} & \overline{4} & 3 & 8 & 9 & \overline{7} \end{array}\right);$$

then 7:

$$\left(\begin{array}{cc|c|ccc|c|cc|c} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 10 & 10 & 9 & 7 & \mathbf{7} & 7 & 4 & 2 & 2 & 1 \\ 1 & 2 & \overline{7} & \overline{6} & \overline{5} & \overline{4} & 3 & 9 & 10 & \overline{8} \end{array}\right);$$

notice that because of condition (*wsp3*) the letter **7** is to be inserted in *second* position in the third block; then insert 4^2:

$$\left(\begin{array}{cc|c|ccc|ccc|cc|c} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 10 & 10 & 9 & 7 & 7 & 7 & 4 & \mathbf{4} & \mathbf{4} & 2 & 2 & 1 \\ 1 & 2 & \overline{7} & \overline{6} & \overline{5} & \overline{4} & 3 & 8 & 9 & 11 & 12 & \overline{10} \end{array}\right).$$

The implementation of 3 gives back the original weighted signed permutation $\binom{c}{w}$.

3. A summation on weighted signed permutations

For $0 \le k \le n$ let $\begin{bmatrix} n \\ k \end{bmatrix}_q := \dfrac{(q;q)_n}{(q;q)_k\,(q;q)_{n-k}}$ be the usual q-binomial coefficient. It is q-routine (see, e.g., [An76, chap. 3]) to prove the following identities, where v_1 is the first letter of v:

$$\frac{1}{(u;q)_N} = \sum_{n\ge 0} \begin{bmatrix} N+n-1 \\ n \end{bmatrix}_q u^n; \qquad \begin{bmatrix} N+n \\ n \end{bmatrix}_q = \sum_{v\in \mathrm{NIW}_n(N)} q^{\mathrm{tot}\, v};$$

$$\frac{1}{(u;q)_{N+1}} = \sum_{n\ge 0} u^n \sum_{v\in \mathrm{NIW}_n(N)} q^{\mathrm{tot}\, v} = \frac{1}{1-u} \sum_{v\in \mathrm{NIW}_n} q^{\mathrm{tot}\, v} u^{v_1};$$

$$\frac{1}{(u;q^2)_{\lfloor r/2\rfloor+1}} = \sum_{n\ge 0} u^n \sum_{v^e\in \mathrm{NIW}_n^e(r)} q^{\mathrm{tot}\, v^e}; \tag{3.1}$$

$$(-uq;q^2)_{\lfloor (r+1)/2\rfloor} = \sum_{n\ge 0} u^n \sum_{v^o\in \mathrm{DW}_n^o(r)} q^{\mathrm{tot}\, v^o}. \tag{3.2}$$

The last two formulas and Proposition 2.1 are now used to calculate the generating function for the weighted signed permutations. The symbols $\mathrm{NIW}^e(r)$, $\mathrm{DW}^o(r)$, $\mathrm{WSP}(r)$, $\mathrm{WSD}(r)$ designate the unions for $n \ge 0$ of the corresponding symbols with an n-subscript.

Proposition 3.1. *The following identity holds:*

$$\sum_{n\ge 0} u^n \sum_{\binom{c}{w}\in \mathrm{WSP}_n(r)} q^{\mathrm{tot}\, c} s^{\mathrm{fexc}\, w} Y_0^{\mathrm{fix}^+ w} Y_1^{\mathrm{fix}^- w} Z^{\mathrm{neg}\, w} \tag{3.3}$$

$$= \frac{(u;q^2)_{\lfloor r/2\rfloor+1}}{(uY_0;q^2)_{\lfloor r/2\rfloor+1}} \frac{(-usqY_1Z;q^2)_{\lfloor (r+1)/2\rfloor}}{(-usqZ;q^2)_{\lfloor (r+1)/2\rfloor}} \times \sum_{n\ge 0} u^n \sum_{\binom{c}{w}\in \mathrm{WSP}_n(r)} q^{\mathrm{tot}\, c} s^{\mathrm{fexc}\, w} Z^{\mathrm{neg}\, w}.$$

Proof. First, summing over $(w^e, w^o, \binom{c}{w}) \in \mathrm{NIW}^e(r)\times\mathrm{DW}^o(r)\times\mathrm{WSP}(r)$, we have

$$\sum_{w^e,w^o,\binom{c}{w}} u^{\lambda(w^e)}q^{\mathrm{tot}\,w^e}\times(usZ)^{\lambda(w^o)}q^{\mathrm{tot}\,w^o}\times u^{\lambda(c)}q^{\mathrm{tot}\,c}s^{\mathrm{fexc}\,w}Y_0^{\mathrm{fix}^+\,w}Y_1^{\mathrm{fix}^-\,w}Z^{\mathrm{neg}\,w}$$
$$(3.4)\qquad = \frac{(-usqZ;q^2)_{\lfloor (r+1)/2\rfloor}}{(u;q^2)_{\lfloor r/2\rfloor+1}}\times\sum_{\binom{c}{w}} u^{\lambda(c)}q^{\mathrm{tot}\,c}s^{\mathrm{fexc}\,w}Y_0^{\mathrm{fix}^+\,w}Y_1^{\mathrm{fix}^-\,w}Z^{\mathrm{neg}\,w}$$

by (3.1) and (3.2). As $\mathrm{fexc}\,w = \mathrm{fexc}\,w' + \lambda(v^o)$, Proposition 2.1 implies that the initial expression can also be summed over five-term sequences $(\binom{c'}{w'}, v^e, v^o, w^e, w^o)$ from $\mathrm{WSD}(r)\times\mathrm{NIW}^e(r)\times\mathrm{DW}^o(r)\times\mathrm{NIW}^e(r)\times\mathrm{DW}^o(r)$ in the form

$$\sum_{\binom{c'}{w'},v^e,v^o,w^e,w^o} u^{\lambda(c')}q^{\mathrm{tot}\,c'}s^{\mathrm{fexc}\,w'}Z^{\mathrm{neg}\,w'}\times(uY_0)^{\lambda(v^e)}q^{\mathrm{tot}\,v^e}\times(usY_1Z)^{\lambda(v^o)}q^{\mathrm{tot}\,v^o}$$
$$\times\, u^{\lambda(w^e)}q^{\mathrm{tot}\,w^e}\times(usZ)^{\lambda(w^o)}q^{\mathrm{tot}\,w^o}$$
$$=\sum_{v^e,v^o}(uY_0)^{\lambda(v^e)}q^{\mathrm{tot}\,v^e}\times(usY_1Z)^{\lambda(v^o)}q^{\mathrm{tot}\,v^o}$$
$$\times\sum_{\binom{c'}{w'},w^e,w^o} u^{\lambda(c')}q^{\mathrm{tot}\,c'}s^{\mathrm{fexc}\,w'}Z^{\mathrm{neg}\,w'}\times u^{\lambda(w^e)}q^{\mathrm{tot}\,w^e}\times(usZ)^{\lambda(w^o)}q^{\mathrm{tot}\,w^o}.$$

The first summation can be evaluated by (3.1) and (3.2), while by Proposition 2.1 again the second sum can be expressed as a sum over weighted signed permutations $\binom{c}{w}\in\mathrm{WSP}(r)$. Therefore, the initial sum is also equal to

$$(3.5)\qquad \frac{(-usqY_1Z;q^2)_{\lfloor (r+1)/2\rfloor}}{(uY_0;q^2)_{\lfloor r/2\rfloor+1}}\times\sum_{\binom{c}{w}\in\mathrm{WSP}(s)} u^{\lambda(c)}q^{\mathrm{tot}\,c}s^{\mathrm{fexc}\,w}Z^{\mathrm{neg}\,w}.$$

Identity (3.3) follows by equating (3.4) with (3.5). □

4. A further evaluation

Proposition 4.1. *Let $B_n(s,t,q,Y_0,Y_1,Z)$ denote the generating polynomial for the group B_n by the statistic* $(\mathrm{fexc}, \mathrm{fdes}, \mathrm{maj}, \mathrm{fix}^+, \mathrm{fix}^-, \mathrm{neg})$. *Then*

$$(4.1)\qquad \frac{1+t}{(t^2;q^2)_{n+1}}B_n(s,t,q,Y_0,Y_1,Z)$$
$$=\sum_{r\ge 0}t^r\sum_{\binom{c}{w}\in\mathrm{WSP}_n(r)} q^{\mathrm{tot}\,c}s^{\mathrm{fexc}\,w}Y_0^{\mathrm{fix}^+\,w}Y_1^{\mathrm{fix}^-\,w}Z^{\mathrm{neg}\,w}.$$

Proof. A very similar calculation has been made in the proof of Theorem 4.1 in [FoHa05]. We also make use of the identities on the q-ascending factorials that were recalled in the previous section. First,

$$\begin{aligned}\frac{1+t}{(t^2;q^2)_{n+1}} &= \sum_{r'\geq 0}(t^{2r'}+t^{2r'+1})\begin{bmatrix} n+r' \\ r' \end{bmatrix}_{q^2}\\ &= \sum_{r\geq 0} t^r \begin{bmatrix} n+\lfloor r/2\rfloor \\ \lfloor r/2\rfloor \end{bmatrix}_{q^2} = \sum_{r\geq 0} t^r \sum_{b\in \mathrm{NIW}_n(\lfloor r/2\rfloor)} q^{2\,\mathrm{tot}\,b}.\end{aligned}$$

Then,

$$\begin{aligned}&\frac{1+t}{(t^2;q^2)_{n+1}}\, B_n(s,t,q,Y_0,Y_1,Z)\\ &\quad= \sum_{r\geq 0} t^r \sum_{\substack{b\in\mathrm{NIW}_n,\\ 2b_1\leq r}} q^{2\,\mathrm{tot}\,b} \sum_{w\in B_n} s^{\mathrm{fexc}\,w} t^{\mathrm{fdes}\,w} q^{\mathrm{fmaj}\,w} Y_0^{\mathrm{fix}^+\,w} Y_1^{\mathrm{fix}^-\,w} Z^{\mathrm{neg}\,w}\\ &\quad= \sum_{r\geq 0} t^r \sum_{\substack{b\in\mathrm{NIW}_n,\, w\in B_n\\ 2b_1+\mathrm{fdes}\,w\leq r}} s^{\mathrm{fexc}\,w} q^{2\,\mathrm{tot}\,b+\mathrm{fmaj}\,w} Y_0^{\mathrm{fix}^+\,w} Y_1^{\mathrm{fix}^-\,w} Z^{\mathrm{neg}\,w}.\end{aligned}$$

As proved in [FoHa05, § 4] to each $\binom{c}{w} = \binom{c_1\cdots c_n}{x_1\cdots x_n} \in \mathrm{WSP}_n(s)$ there corresponds a unique $b = b_1\cdots b_n \in \mathrm{NIW}_n$ such that $2b_1 + \mathrm{fdes}\,w = c_1$ and $2\,\mathrm{tot}\,b + \mathrm{fmaj}\,w = \mathrm{tot}\,c$. Moreover, the mapping $\binom{c}{w} \mapsto (b,w)$ is a bijection of $\mathrm{WSP}_n(r)$ onto the set of all pairs (b,w) such that $b = b_1\cdots b_n \in \mathrm{NIW}_n$ and $w \in B_n$ with the property that $2b_1 + \mathrm{fdes}\,w \leq r$.

The word b is determined as follows: write the signed permutation w as a linear word $w = x_1x_2\ldots x_n$ and for each $k = 1,2,\ldots,n$ let z_k be the number of descents ($x_i > x_{i+1}$) in the right factor $x_kx_{k+1}\cdots x_n$ and let ϵ_k be equal to 0 or 1 depending on whether x_k is positive or negative. Also for each $k = 1,2,\ldots,n$ define $a_k := (c_k - \epsilon_k)/2$, $b_k := (a_k - z_k)$ and form the word $b = b_1\cdots b_n$.

For example,

$$\begin{array}{rl} \mathrm{Id} &= 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ \ 9\ \ 10\\ c &= 9\ 7\ 7\ 4\ 4\ 4\ 2\ 2\ \ 1\ \ \ 1\\ w &= \overline{4}\ \overline{3}\ \overline{2}\ 1\ 5\ 6\ 8\ 9\ \overline{10}\ \overline{7}\\ z &= 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ \ 0\ \ \ 0\\ \epsilon &= 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ \ 1\ \ \ 1\\ a &= 4\ 3\ 3\ 2\ 2\ 2\ 1\ 1\ \ 0\ \ \ 0\\ b &= 3\ 2\ 2\ 1\ 1\ 1\ 0\ 0\ \ 0\ \ \ 0 \end{array}$$

Pursuing the above calculation we get (4.1). $\square$

The next theorem is then a consequence of Propositions 3.1 and 4.1.

Theorem 4.2. *The following identity holds:*

$$\sum_{n\ge 0}(1+t)B_n(s,t,q,Y_0,Y_1,Z)\frac{u^n}{(t^2;q^2)_{n+1}} \tag{4.2}$$
$$=\sum_{r\ge 0} t^r \frac{(u;q^2)_{\lfloor r/2\rfloor+1}}{(uY_0;q^2)_{\lfloor r/2\rfloor+1}}\frac{(-usqY_1Z;q^2)_{\lfloor (r+1)/2\rfloor}}{(-usqZ;q^2)_{\lfloor (r+1)/2\rfloor}}\sum_{\binom{c}{w}\in \mathrm{WSP}(r)} u^{\lambda w} q^{\mathrm{tot}\, c} s^{\mathrm{fexc}\, w} Z^{\mathrm{neg}\, w}.$$

In view of the statements of Theorems 1.1 and 4.2 we see that the former theorem will be proved if we can show that the following identity holds:

$$\sum_{\binom{c}{w}\in \mathrm{WSP}(r)} u^{\lambda w} q^{\mathrm{tot}\, c} s^{\mathrm{fexc}\, w} Z^{\mathrm{neg}\, w} = F_r(u;s,q,Z), \tag{4.3}$$

with $F_r(u;s,q,Z)$ given by (1.5). In our paper [FoHa06a] the sum $\sum q^{\mathrm{tot}\, c} Z^{\mathrm{neg}\, w}$ $(\binom{c}{w} \in \mathrm{WSP}_n(r))$ has been calculated, by setting up a bijection of $\mathrm{WSP}_n(r)$ onto the set $[0,r]^n$ of the words of length n, whose letters are taken from the interval $[0,r]$. Because of the presence of the new statistic "fexc" *another* bijection is to be constructed. If v is the image of $\binom{c}{w}$ under the new bijection, then the statistic "fexc" on signed permutations must have a counterpart on words. This role is played by the so-called *number of even decreases*, as shown in the next section.

5. Even decreases on words

Say that a letter y_i of a word $v = y_1y_2\cdots y_n$ from $[0,r]^n$ is a *decrease* in v if $1\le i\le n-1$ and $y_i \ge y_{i+1} \ge \cdots \ge y_j > y_{j+1}$ for some j such that $i\le j\le n-1$. Let $\mathrm{dec}\, v$ (resp. $\mathrm{evdec}\, v$) denote the *number* of decreases (resp. of *even* decreases) in v. Notice that $\mathrm{dec}\, v \ge \mathrm{des}\, v$ (the number of *descents* in v) and $\mathrm{dec}\, v = \mathrm{des}\, v$ whenever v is a permutation (without repetitions). Also, let $\mathrm{odd}\, v$ be the number of odd letters in v.

Recall that a nonempty word $v = y_1y_2\cdots y_n$ is a *Lyndon word*, if either $n=1$, or $n\ge 2$ and, with respect to the lexicographic order, the inequality $y_1y_2\cdots y_n > y_iy_{i+1}\cdots y_ny_1\cdots y_{i-1}$ holds for every i such that $2\le i\le n$. Let v, v' be two nonempty primitive words (none of them can be written as v_0^a for $a\ge 2$ and some word v_0). We write $v \preceq v'$ if and only if $v^a \le v'^a$ with respect to the lexicographic order for an integer a large enough. As shown for instance in [Lo83, Theorem 5.1.5] (also see [Ch58], [Sch65]) each nonempty word v can be written uniquely as a product $l_1l_2\cdots l_k$, called its *Lyndon factorization*, where each l_i is a Lyndon word and $l_1\preceq l_2\preceq\cdots\preceq l_k$. In the example below the Lyndon factorization of v has been materialized by vertical bars. The essential property of the Lyndon factorization needed in the sequel is the following:

$$\mathrm{dec}\, v = \mathrm{dec}\, l_1 + \mathrm{dec}\, l_2 + \cdots + \mathrm{dec}\, l_k.$$

Now, start with the Lyndon factorization $l_1 l_2 \cdots l_k$ of a word v from $[0,r]^n$. With such a v we associate a permutation σ from $\mathfrak{S}_n$ by means of a procedure developed by Gessel-Reutenauer [GeRe93]: each letter y_i of v belongs to a Lyndon word factor l_h, so that $l_h = v' y_i v''$. Then, form the infinite word $A(y_i) := y_i v'' v' y_i v'' v' \cdots$ If y_i and $y_{i'}$ are two letters of v, say that y_i *precedes* $y_{i'}$ if $A(y_i) > A(y_{i'})$ for the lexicographic order, or if $A(y_i) = A(y_{i'})$ and y_i is to the right of $y_{i'}$ in the word v. This precedence determines a total order on the n letters of v. The letter that precedes all the other ones is given label 1, the next one label 2, and so on. When each letter y_i of v is replaced by its label, say, $\mathrm{lab}(y_i)$, each Lyndon word factor l_j becomes a new word τ_j. The essential property is that each τ_j starts with its *minimum* element and those minimum elements read from left to right are in *decreasing order*. We can then interpret each τ_j as the *cycle* of a permutation and the (juxtaposition) product $\tau_1 \tau_2 \cdots \tau_k$ as the (functional) product of *disjoint* cycles. This product, said to be written *in canonical form*, defines a unique permutation σ from $\mathfrak{S}_n$ ([Lo83], § 10.2).

For example,

$$\begin{array}{l} v = 2 \mid 3\ \ 2\ \ 1\ \ 1 \mid 3 \mid 5 \mid 6\ 4\ 2\ \ 1\ \ 3\ \ 2\ 3 \mid 6\ 6\ 3\ \ 1\ 6\ 6\ 2 \mid 6 \\ \sigma = 16 \mid 12\ 18\ 22\ 21 \mid 10 \mid 7 \mid 4\ 8\ 17\ 20\ 11\ 15\ 9 \mid 2\ 5\ 13\ 19\ 3\ 6\ 14 \mid 1 \end{array}$$

The labels on the second row are obtained as follows: read the letters equal to 6 (the maximal letter) from left to right and form their associated infinite words: $64213236421\cdots$, $66316626631\cdots$, $6316621131\cdots$, $662663166\cdots$, $62663166\cdots$, $66666\cdots$ Those letters 6 read from left to right will be given the labels 4, 2, 5, 3, 6, 1. We continue the labellings by reading the letters equal to 5, then 4, ... in the above word v.

No decrease y_i in v can be the rightmost letter of a Lyndon word factor l_h. We have then $l_h = \cdots y_i y_{i+1} \cdots y_j y_{j+1} \cdots$ with $y_i \geq y_{i+1} \geq \cdots \geq y_j > y_{j+1}$. Consequently, $A(y_i) > A(y_{i+1})$ and $\mathrm{lab}(y_i) < \mathrm{lab}(y_{i+1})$. Conversely, if $\mathrm{lab}(y_i) < \mathrm{lab}(y_{i+1})$ and y_i, y_{i+1} belong to the same Lyndon factor, then y_i is a decrease in v. To each decrease y_i in v there corresponds a unique cycle τ_h of σ and a pair $\mathrm{lab}(y_i)\,\mathrm{lab}(y_{i+1})$ of *successive* letters of τ_h such that $\mathrm{lab}(y_i) < \mathrm{lab}(y_{i+1})$ and $\mathrm{lab}(y_{i+1}) = \sigma(\mathrm{lab}(y_i))$.

Consider the monotonic nonincreasing rearrangement $c = c_1 c_2 \cdots c_n$ of v and form the three-row matrix

$$\begin{array}{cccc} 1 & 2 & \cdots & n \\ c_1 & c_2 & \cdots & c_n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{array}$$

Then, if y_i is a decrease in v, the $\mathrm{lab}(y_i)$-th column of the previous matrix is of the form

$$\begin{array}{c} \mathrm{lab}(y_i) \\ y_i \\ \mathrm{lab}(y_{i+1}) \end{array} \quad \text{with} \quad \mathrm{lab}(y_i) < \mathrm{lab}(y_{i+1}).$$

At this stage we have $\operatorname{dec} v = \operatorname{exc} \sigma$. We then have to transform σ into a *signed* permutation w in such a way that only the excedances corresponding to the even decreases of v are preserved. We proceed as follows. The word c can be expressed as $a_1^{m_1} a_2^{m_2} \cdots a_k^{m_k}$ where $a_1 > a_2 > \cdots > a_k \ge 0$ and $m_1 \ge 1$, $m_2 \ge 1$, … , $m_k \ge 1$. We then define

$$\begin{aligned}
x_1 \cdots x_{m_1} &:= \begin{cases} \sigma(1)\cdots\sigma(m_1), & \text{if } a_1 \text{ is even;}\\ \overline{\sigma}(m_1)\cdots\overline{\sigma}(1), & \text{if } a_1 \text{ is odd;}\end{cases}\\
x_{m_1+1}\cdots x_{m_1+m_2} &:= \begin{cases} \sigma(m_1+1)\cdots\sigma(m_1+m_2), & \text{if } a_2 \text{ is even;}\\ \overline{\sigma}(m_1+m_2)\cdots\overline{\sigma}(m_1+1), & \text{if } a_2 \text{ is odd;}\end{cases}\\
\cdots\quad &\quad \cdots\\
x_{m_1+\cdots+m_{k-1}+1}\cdots x_n &:= \begin{cases} \sigma(m_1+\cdots+m_{k-1}+1)\cdots\sigma(n), \text{ if } a_k \text{ is even;}\\ \overline{\sigma}(n)\cdots\overline{\sigma}(m_1+\cdots+m_{k-1}+1), \text{ if } a_k \text{ is odd.}\end{cases}
\end{aligned}$$

The word $w := x_1 x_2 \cdots x_n$ is then a *signed* permutation. When going from σ to w, the excedances $\operatorname{lab}(y_i) < \operatorname{lab}(y_{i+1})$ of σ such that y_i is odd have vanished. The other ones have been preserved. Hence, $2 \operatorname{evdec} v + \operatorname{odd} v = 2 \operatorname{exc} w + \operatorname{neg} w = \operatorname{fexc} w$. Finally, as $\sigma(i) > \sigma(i+1) \Rightarrow c_i > c_{i+1}$, the pair $\binom{c}{w}$ is a *weighted signed permutation.* We then have the desired bijection $[0,r]^n \to \binom{c}{w}$.

Theorem 5.1. *The mapping $v \to \binom{c}{w}$ is a bijection of $[0,r]^n$ onto $\mathrm{WSP}_n(r)$ having the following properties:*

$$\operatorname{tot} c = \operatorname{tot} v; \quad \operatorname{fexc} w = 2 \operatorname{evdec} v + \operatorname{odd} v; \quad \operatorname{neg} w = \operatorname{odd} v.$$

With the running example we have

$$\begin{array}{r@{\,=\,}l}
\mathrm{Id} & 1\ 2\ 3\ 4\ \ 5\ \ \ 6\ |7|\ 8\ |\ 9\ \ 10\ 11\ 12\ 13\,|\,14\ 15\ 16\ 17\ 18\,|\,19\ 20\ 21\ 22\\
c & 6\ 6\ 6\ 6\ \ 6\ \ \ 6\ |5|\ 4\ |\ 3\ \ \ 3\ \ \ 3\ \ \ 3\ \ \ 3\ |\ 2\ \ \ 2\ \ \ 2\ \ \ 2\ \ \ 2\ |\ 1\ \ \ 1\ \ \ 1\ \ \ 1\\
\sigma & 1\ 5\ 6\ 8\ 13\ 14\,|7|\,17\,|\ 4\ \ 10\ 15\ 18\ 19\,|\ 2\ \ \ 9\ \ 16\ 20\ 22\,|\ 3\ \ 11\ 12\ 21\\
w & 1\ \mathbf{5}\ \mathbf{6}\ \mathbf{8}\ \mathbf{13}\ \mathbf{14}\,|\overline{7}|\,\mathbf{17}\,|\,\overline{19}\ \overline{18}\ \overline{15}\ \overline{10}\ \ \overline{4}\ |\ 2\ \ \ 9\ \ 16\ \mathbf{20}\ \mathbf{22}\,|\,\overline{21}\ \overline{12}\ \overline{11}\ \ \overline{3}
\end{array}$$

The signed permutation w has eight excedances (reproduced in boldface) and ten negative letters. Therefore, $\operatorname{fexc} w = 2 \times 8 + 10 = 26$. There are eight even decreases of the word v: $2 \to 1$, $6 \to 4$, $4 \to 2$, $2 \to 1$, $6 \to 6$, $6 \to 3$, $6 \to 6$, $6 \to 2$. Moreover, $\operatorname{odd} v = 10$. Hence $2 \operatorname{evdec} v + \operatorname{odd} v = 26 = \operatorname{fexc} w$ and of course $\operatorname{neg} w = \operatorname{odd} v = 10$.

In view of (4.3) and Theorem 5.1 the proof of Theorem 1.1 will be completed if the identy

$$\sum_{v \in [0,r]^*} u^{\lambda v} q^{\operatorname{tot} v} s^{2 \operatorname{evdec} v + \operatorname{odd} v} Z^{\operatorname{odd} v} = F_r(u; s, q, Z) \tag{5.1}$$

holds, the sum being over all words whose letters are in $[0,r]$.

6. The calculation of the latter sum

As introduced in our previous paper [FoHa07b] a word $w = x_1x_2\dots x_n$ of length $n \geq\ 2$ is said to be a V-*word*, if for some integer i such that $1 \leq i \leq n-1$ we have $x_1 \geq x_2 \geq \cdots \geq x_i > x_{i+1}$ and $x_{i+1} \leq x_{i+2} \leq \cdots \leq x_n < x_i$ whenever $i+1 < n$. The pair (x_i, x_{i+1}) is called the *critical biletter* of w. The V-word decomposition for permutations was introduced by Kim and Zeng [KiZe01]. The following theorem is a simple consequence of Theorem 3.4 proved in our previous paper [FoHa07b].

Theorem 6.1 (V-word decomposition)**.** *To each word $v = y_1y_2\cdots y_n$ whose letters are nonnegative integers there corresponds a unique sequence $(v_0, v_1, v_2, \dots, v_k)$, where v_0 is a monotonic nondecreasing word and v_1, v_2, ... , v_k are V-words with the further property that $v_0v_1v_2\cdots v_k$ is a rearrangement of v and*

$$\operatorname{evdec} v = \operatorname{evdec} v_1 + \operatorname{evdec} v_2 + \cdots + \operatorname{evdec} v_k. \tag{6.1}$$

Let X_0, X_1, ... , X_r be $(r+1)$ commuting variables. The *even weight*, evweight v, of each word $v = y_1y_2\dots y_n$ is defined to be

$$\operatorname{evweight} v := X_{y_1}X_{y_2}\cdots X_{y_n}\, s^{2\operatorname{evdec} v}.$$

Now, consider the infinite series:

$$U(l) := \prod_{l\leq j\leq r}(1-s^{2\chi(j\ \mathrm{even})}X_j)^{-1}, \quad (1\leq l\leq r);$$
$$M(k,l) := \prod_{k\leq j\leq l-1}(1-X_j)^{-1}, \quad (0\leq k<l).$$

Then, the generating function for V-words, whose critical biletter is (l,k) $(0\leq k<l\leq r)$, by "evweight" is equal to:

$$U(l)\, s^{2\chi(l\ \mathrm{even})}\, X_lX_k\, M(k,l).$$

The following theorem is then a consequence of Theorem 6.1.

Theorem 6.2. *We have the identity:*

$$\sum_v \operatorname{evweight} v = \frac{M(0,r+1)}{1-\sum\limits_{0\leq k<l\leq r} U(l)\, s^{2\chi(l\ \mathrm{even})}\, X_lX_k\, M(k,l)},$$

where the sum is over all words whose letters belong to the interval $[0,r]$.

Consider the homomorphism ϕ generated by:

$$\phi(X_k) := uq^k(sZ)^{\chi(k\ \mathrm{odd})} \quad (0\leq k\leq r).$$

If $v = y_1y_2\cdots y_n$, then $\phi(\operatorname{evweight} v) = u^{\lambda v}q^{\operatorname{tot} v}s^{2\operatorname{evdec} v+\operatorname{odd} v}Z^{\operatorname{odd} v}$.

Also

$$\phi(U(l)) = \prod_{l\le j\le r}(1-uq^j s^{2\chi(j\,\text{even})}(sZ)^{\chi(j\,\text{odd})})^{-1};$$
$$\phi(M(k,l)) = \prod_{k\le j\le l-1}(1-uq^j (sZ)^{\chi(j\,\text{odd})})^{-1}.$$

Now define

$$(u)'_k := \begin{cases} 1, & \text{if } k=0;\\ \prod_{0\le j\le k-1}(1-uq^j(sZ)^{\chi(j\,\text{odd})}), & \text{if } k\ge 1;\end{cases}$$
$$(u)''_l := \begin{cases} 1, & \text{if } l=0;\\ \prod_{1\le j\le l}(1-uq^j s^{2\chi(j\,\text{even})}(sZ)^{\chi(j\,\text{odd})}), & \text{if } l\ge 1.\end{cases}$$

Then $\phi(U(l)) = \dfrac{(u)''_{l-1}}{(u)''_r}$ $(1\le l)$; $\phi(M(k,l)) = \dfrac{(u)'_k}{(u)'_l}$ $(0\le k<l)$. Now, take the image of the identity of Theorem 6.2 under ϕ:

$$\sum_{w\in[0,r]^*} u^{\lambda w} q^{\text{tot}\,w} s^{(2\,\text{evdec}\,+\,\text{odd})\,w} Z^{\text{odd}\,w} = \sum_{w\in[0,r]^*}\phi(\text{evweight}\,w) = \frac{1}{(u)'_{r+1}}\frac{1}{S},$$
where
$$S = 1-\sum_{0\le k<l\le r}\frac{(u)''_{l-1}}{(u)''_r}s^{2\chi(l\,\text{even})}uq^l(sZ)^{\chi(l\,\text{odd})}uq^k(sZ)^{\chi(k\,\text{odd})}\frac{(u)'_k}{(u)'_l}.$$

As $-uq^k(sZ)^{\chi(k\,\text{odd})}(u)'_k = (u)'_{k+1}-(u)'_k$, we have:

$$S = 1+\sum_{1\le l\le r}\frac{(u)''_{l-1}}{(u)''_r}s^{2\chi(l\,\text{even})}uq^l(sZ)^{\chi(l\,\text{odd})}\frac{1}{(u)'_l}((u)'_l-1).$$

Now, $-uq^l s^{2\chi(l\,\text{even})}(sZ)^{\chi(l\,\text{odd})}(u)''_{l-1} = (u)''_l-(u)''_{l-1}$. Hence,

$$S = \frac{1}{(u)''_r}-\frac{1}{(u)''_r}\sum_{1\le l\le r}\frac{(u)''_{l-1}}{(u)'_l}uq^l s^{2\chi(l\,\text{even})}(sZ)^{\chi(l\,\text{odd})}.$$

We are then left to prove:

$$\text{(6.2)}\quad \frac{(u)''_r}{(u)'_{r+1}}\Big(1-\sum_{1\le l\le r}\frac{(u)''_{l-1}}{(u)'_l}uq^l s^{2\chi(l\,\text{even})}(sZ)^{\chi(l\,\text{odd})}\Big)^{-1} = F_r(u;s,q,Z)$$

We can also write:

$$(u)'_r = (u;q^2)_{\lfloor (r+1)/2\rfloor}\,(usqZ;q^2)_{\lfloor r/2\rfloor},$$
$$(u)''_r = (usqZ;q^2)_{\lfloor (r+1)/2\rfloor}\,(us^2q^2;q^2)_{\lfloor r/2\rfloor},$$

so that

$$(6.3)\qquad \frac{(u)''_r}{(u)'_{r+1}} = \frac{(us^2q^2;q^2)_{\lfloor r/2\rfloor}}{(u;q^2)_{\lfloor r/2\rfloor+1}} \quad\text{and}\quad \frac{(u)''_{l-1}}{(u)'_l} = \frac{(us^2q^2;q^2)_{\lfloor (l-1)/2\rfloor}}{(u;q^2)_{\lfloor (l+1)/2\rfloor}}.$$

Identity (6.2) may be rewritten as

$$(6.4)\qquad \frac{(us^2q^2;q^2)_{\lfloor r/2\rfloor}}{(u;q^2)_{\lfloor r/2\rfloor+1}}(1-T)^{-1} = F_r(u;s,q,Z),$$

where

$$T := \sum_{1\le l\le r} \frac{(us^2q^2;q^2)_{\lfloor (l-1)/2\rfloor}}{(u;q^2)_{\lfloor (l+1)/2\rfloor}} uq^l s^{2\chi(l \text{ even})}(sZ)^{\chi(l\text{ odd})}$$
$$= \sum_{0\le l\le \lfloor (r-1)/2\rfloor} \frac{(us^2q^2;q^2)_l}{(u;q^2)_{l+1}} uq^{2l} sqZ + \sum_{0\le l\le \lfloor r/2\rfloor -1} \frac{(us^2q^2;q^2)_l}{(u;q^2)_{l+1}} uq^{2l}s^2q^2.$$

We can then introduce

$$G(m) := \sum_{0\le l\le m} \frac{(us^2q^2;q^2)_l}{(u;q^2)_{l+1}} uq^{2l}$$

and try to sum it. As

$$\frac{(us^2q^2;q^2)_{l+1}}{(u;q^2)_{l+1}} - \frac{(us^2q^2;q^2)_l}{(u;q^2)_l} = \frac{(us^2q^2;q^2)_l}{(u;q^2)_{l+1}} uq^{2l}(1-s^2q^2),$$

we get

$$G(m) = \frac{1}{1-s^2q^2}\Bigl(\frac{(us^2q^2;q^2)_{m+1}}{(u;q^2)_{m+1}} - 1\Bigr),$$

so that

$$T = sqZ\,G(\lfloor (r-1)/2\rfloor) + s^2q^2\,G(\lfloor r/2\rfloor -1)$$
$$= \frac{1}{1-s^2q^2}\Bigl(sqZ\frac{(us^2q^2;q^2)_{\lfloor (r+1)/2\rfloor}}{(u;q^2)_{\lfloor (r+1)/2\rfloor}} - sqZ + s^2q^2\frac{(us^2q^2;q^2)_{\lfloor r/2\rfloor}}{(u;q^2)_{\lfloor r/2\rfloor}} - s^2q^2\Bigr).$$

By reporting this value of T into (6.4) we exactly find the expression of $F_r(u;s,q,Z)$ displayed in (1.5). ∎

7. Second proof of (5.1)

When $Z=0$ in $B_n(s,t,q,Y_0,Y_1,Z)$, then Y_1 is also null. Furthermore, each polynomial $B_n(s,t,q,Y_0,0,0)$ is a polynomial in s^2, t^2, q^2, Y_0. The summation on the right-hand side of (1.6) only involves even powers of t. We then have

$$F_{2r}(u;s,q,0)=\frac{(us^2q^2;q^2)_r\,(1-s^2q^2)\,(u;q^2)_r}{(u;q^2)_{r+1}\big((u;q^2)_r-s^2q^2\,(us^2q^2;q^2)_r\big)} \tag{7.1}$$

and

$$\sum_{n\ge 0}B_n(s,t,q,Y_0,0,0)\frac{u^n}{(t^2;q^2)_{n+1}}=\sum_{r\ge 0}t^{2r}\frac{(u;q^2)_{r+1}}{(uY_0;q^2)_{r+1}}F_{2r}(u;s,q,0). \tag{7.2}$$

But $B_n(s^{1/2},t^{1/2},q^{1/2},Y,0,0)$ is the generating polynomial $A_n(s,t,q,Y)$ for the symmetric group $\mathfrak{S}_n$ by the statistic $(\mathrm{exc},\mathrm{des},\mathrm{maj},\mathrm{fix})$ and it was proved in our previous paper [FoHa06b] that

$$\sum_{n\ge 0}A_n(s,t,q,Y)\frac{u^n}{(t;q)_{n+1}}=\sum_{r\ge 0}t^r\frac{(u;q)_{r+1}}{(uY;q)_{r+1}}\frac{(usq;q)_r\,(1-sq)\,(u;q)_r}{(u;q)_{r+1}\big((u;q)_r-sq(usq;q)_r\big)}.$$

Accordingly, (1.4) holds when $Z=0$ and consequently (5.1) holds when r is even and Z null. To show that identity (5.1), which we shall rewrite as

$$\sum_{v\in[0,r]^*}\mathrm{EV}(v)=F_r(u;s,q,Z), \tag{7.3}$$

is true for all values of r and when Z is not necessarily 0 we proceed as follows.

Cut each word $v\in[0,r]^*$ in the (5.1) summation *after* every occurrence of an *odd* letter. This defines a unique factorization:

$$v=p_0i_0p_1i_1\cdots p_ki_kp_{k+1} \tag{7.4}$$

of v having the following properties

(C1) $k\ge 0$,

(C2) $p_0,\,p_1,\,\ldots\,,\,p_k,\,p_{k+1}$ all from P_r^* with $P_r:=\{0,2,4,\ldots,2\lfloor r/2\rfloor\}$,

(C3) $i_0,\,i_1,\,\ldots,\,i_k$ all from I_r with $I_r:=\{1,3,5,\ldots,2\lfloor (r+1)/2\rfloor-1\}$.

The following properties of this factorization are easy to verify:

(P1) $\mathrm{tot}\,v=\mathrm{tot}(p_0i_0)+\mathrm{tot}(p_1i_1)+\cdots+\mathrm{tot}(p_ki_k)+\mathrm{tot}(p_{k+1})$,

(P2) $\mathrm{evdec}\,v=\mathrm{evdec}(p_0i_0)+\mathrm{evdec}(p_1i_1)+\cdots+\mathrm{evdec}(p_ki_k)+\mathrm{evdec}(p_{k+1})$,

(P3) $\mathrm{odd}\,v=\mathrm{odd}(p_0i_0)+\mathrm{odd}(p_1i_1)+\cdots+\mathrm{odd}(p_ki_k)+\mathrm{odd}(p_{k+1})$,

(P4) $\lambda v=\lambda(p_0i_0)+\lambda(p_1i_1)+\cdots+\lambda(p_ki_k)+\lambda(p_{k+1})$,

(P5) $\mathrm{EV}(p_0 i_0 p_1 i_1 \cdots p_k i_k p_{k+1}) = \mathrm{EV}(p_0 i_0)\,\mathrm{EV}(p_1 i_1) \cdots \mathrm{EV}(p_k i_k)\,\mathrm{EV}(p_{k+1})$,
(P6) $\mathrm{EV}(pi) = sqZ\,\mathrm{EV}(p(i-1))$ for $p \in P_r^*$ and $i \in I_r$,
(P7) $\mathrm{EV}(p(2k)) = uq^{2k}\,\mathrm{EV}(p)$ for $p \in P_{2k}^*$.

Hence

$$\sum_{v \in [0,r]^*} v = \Bigl(1 - \sum_{p \in P_r^*, i \in I_r} pi\Bigr)^{-1} \sum_{p \in P_r^*} p;$$

$$(7.5)\qquad \sum_{v \in [0,r]^*} \mathrm{EV}(v) = \frac{1}{1 - \sum_{p \in P_r^*, i \in I_r} \mathrm{EV}(pi)} \sum_{p \in P_r^*} \mathrm{EV}(p).$$

On the other hand,

$$\sum_{p \in P_r^*, i \in I_r} \mathrm{EV}(pi) = sqZ \sum_{p \in P_r^*, i \in I_r} \mathrm{EV}(p(i-1)) = sqZ \sum_{p \in P_r^*, j \in P_{r-1}} \mathrm{EV}(pj).$$

If $r = 2k+1$, then $P_r = P_{r-1} = P_{2k}$, so that

$$\sum_{p \in P_r^*, j \in P_{r-1}} \mathrm{EV}(pj) = \sum_{p \in P_{2k}^*, \lambda(p) \geq 1} \mathrm{EV}(p) = F_{2k}(u; s, q, 0) - 1,$$

because (7.3) holds for r even and $Z = 0$. It follows from (7.5) that

$$(7.6)\qquad \sum_{v \in [0,2k+1]^*} \mathrm{EV}(v) = \frac{F_{2k}(u; s, q, 0)}{1 - sqZ\bigl(F_{2k}(u; s, q, 0) - 1\bigr)}.$$

If $r = 2k$, then $P_r = P_{2k}$ and $P_{r-1} = P_{2k-2}$, so that

$$\begin{aligned}
\sum_{p \in P_r^*, j \in P_{r-1}} \mathrm{EV}(pj) &= \sum_{p \in P_{2k}^*, j \in P_{2k-2}} \mathrm{EV}(pj) \\
&= \sum_{p \in P_{2k}^*, \lambda(p) \geq 1} \mathrm{EV}(p) - \sum_{p \in P_{2k}^*} \mathrm{EV}(p(2k)) \\
&= (F_{2k}(u; s, q, 0) - 1) - uq^{2k} F_{2k}(u; s, q, 0) \\
&= (1 - uq^{2k}) F_{2k}(u; s, q, 0) - 1.
\end{aligned}$$

Hence (7.5) implies

$$(7.7)\qquad \sum_{v \in [0,2k]^*} \mathrm{EV}(v) = \frac{F_{2k}(u; s, q, 0)}{1 - sqZ\bigl((1 - uq^{2k}) F_{2k}(u; s, q, 0) - 1\bigr)}.$$

We can then report the value of $F_{2k}(u; s, q, 0)$ obtained in (7.1) in both expressions (7.6) and (7.7). By combining them in a single formula we exactly get the formula displayed in (1.5) for $F_r(u; s, q, Z)$. $\square$

8. Two proofs of Corollary 1.3

For the first proof we proceed as follows. Let $Z=1$ in (1.5) and write the formula as:

$$\begin{aligned}
&\frac{(us^2q^2;q^2)_{\lfloor r/2\rfloor}\,(1-s^2q^2)\,(u;q^2)_{\lfloor (r+1)/2\rfloor}}{(u;q^2)_{\lfloor r/2\rfloor+1}F_r(u;s,q,1)}\\
&\qquad\qquad = (u;q^2)_{\lfloor (r+1)/2\rfloor} - s^2q^2\,\frac{(u;q^2)_{\lfloor (r+1)/2\rfloor}}{(u;q^2)_{\lfloor r/2\rfloor}}(us^2q^2;q^2)_{\lfloor r/2\rfloor}\\
&\qquad\qquad\qquad + sq\,(u;q^2)_{\lfloor (r+1)/2\rfloor} - sq(us^2q^2;q^2)_{\lfloor (r+1)/2\rfloor}.
\end{aligned}$$

We have

$$\begin{aligned}
&s^2q^2\,\frac{(u;q^2)_{\lfloor (r+1)/2\rfloor}}{(u;q^2)_{\lfloor r/2\rfloor}}(us^2q^2;q^2)_{\lfloor r/2\rfloor} + sq(us^2q^2;q^2)_{\lfloor (r+1)/2\rfloor}\\
&\qquad = sq(us^2q^2;q^2)_{\lfloor r/2\rfloor}\big(sq(1-uq^{2\lfloor (r-1)/2\rfloor}) + 1 - us^2q^{2\lfloor (r-1)/2\rfloor+2}\big)\\
&\qquad = sq(us^2q^2;q^2)_{\lfloor r/2\rfloor}(1+sq)(1-usq^{2\lfloor (r-1)/2\rfloor+1})\\
&\qquad = sq(us^2q^2;q^2)_{\lfloor r/2\rfloor}(1+sq)\frac{(usq;q^2)_{\lfloor (r+1)/2\rfloor}}{(usq;q^2)_{\lfloor r/2\rfloor}},
\end{aligned}$$

so that

$$\begin{aligned}
&\frac{(us^2q^2;q^2)_{\lfloor r/2\rfloor}\,(1-s^2q^2)\,(u;q^2)_{\lfloor (r+1)/2\rfloor}}{(u;q^2)_{\lfloor r/2\rfloor+1}F_r(u;s,q,1)}\\
&\qquad = (1+sq)\big((u;q^2)_{\lfloor (r+1)/2\rfloor} - sq(us^2q^2;q^2)_{\lfloor r/2\rfloor}\frac{(usq;q^2)_{\lfloor (r+1)/2\rfloor}}{(usq;q^2)_{\lfloor r/2\rfloor}}.
\end{aligned}$$

By dividing both sides by $(1+sq)$ we recover (1.8). $\square$

For the second proof we again use the notations introduced in Section 7, namely,

$$\begin{aligned}
(u)'_r &:= \begin{cases} 1, & \text{if } r=0;\\ \displaystyle\prod_{0\le j\le r-1}(1-uq^j(sZ)^{\chi(j\ \text{odd})}), & \text{if } r\ge 1;\end{cases}\\
&= (u;q^2)_{\lfloor (r+1)/2\rfloor}\,(usqZ;q^2)_{\lfloor r/2\rfloor},
\end{aligned}$$

and directly prove the identity

$$\sum_{v\in[0,r]^n} q^{\mathrm{tot}\,v}s^{2\,\mathrm{evdec}\,v+\mathrm{odd}\,v} = F_r(u;s,q,1),$$

where $F_r(u;s,q,1)$ is given by (1.8), an identity that may be rewritten as:

$$\sum_{v\in[0,r]^n} q^{\mathrm{tot}\,v}s^{2\,\mathrm{evdec}\,v+\mathrm{odd}\,v} = \frac{1}{(u)'_{r+1}}\times\frac{(1-sq)(u)'_r(usq)'_r}{(u)'_r - sq(usq)'_r}. \tag{8.1}$$

For each $r \geq 0$ let $D(r)$ be the set of all pairs (w, i) such that $w \in \mathrm{NIW}(r-1)$, $1 \leq i \leq \lambda w - 1$, $\lambda w \geq 2$. In our previous paper ([FoHa07b], Theorem 2.1) we have constructed a bijection mapping each word v onto a sequence

$$(w_0, (w_1, i_1), (w_2, i_2), \ldots, (w_k, i_k)),$$

such that $w_0 \in \mathrm{NIW}(r)$ and each pair $(w_l, i_l) \in D(r)$ $(1 \leq l \leq k)$ and $w_0 w_1 \cdots w_k$ is a rearrangement of v. Furthermore, the bijection has the following property:

$$(8.2)\quad 2\,\mathrm{evdec}\, v + \mathrm{odd}\, v = i_1 + i_2 + \cdots i_k + \mathrm{odd}\, w_0 + \mathrm{odd}\, w_1 + \cdots + \mathrm{odd}\, w_k.$$

Hence, $\sum\limits_{v\in[0,r]^n} q^{\mathrm{tot}\, v} s^{2\,\mathrm{evdec}\, v + \mathrm{odd}\, v} = \dfrac{A}{1-B}$, where $A = \sum\limits_{w\in\mathrm{NIW}(r)} q^{\mathrm{tot}\, w} s^{\mathrm{odd}\, w} u^{\lambda w}$ and $B = \sum\limits_{(w,i)\in D(r)} (sq)^i q^{\mathrm{tot}\, w} s^{\mathrm{odd}\, w} u^{\lambda w}$. But $A = \dfrac{1}{(u)'_{r+1}}$ and

$$\begin{aligned}
B &= \sum_{(w,i)\in D(r)} (sq)^i q^{\mathrm{tot}\, w} s^{\mathrm{odd}\, w} u^{\lambda w}\\
&= \sum_{n\geq 2} \sum_{i=1}^{n-1} (sq)^i \sum_{w\in\mathrm{NIW}_n(r-1)} q^{\mathrm{tot}\, w} s^{\mathrm{odd}\, w} u^{\lambda w}\\
&= \sum_{n\geq 2} \frac{sq - (sq)^n}{1-sq} \sum_{w\in\mathrm{NIW}_n(r-1)} q^{\mathrm{tot}\, w} s^{\mathrm{odd}\, w} u^{\lambda w}\\
&= \frac{sq}{1-sq} \sum_{n\geq 2} \sum_{w\in\mathrm{NIW}_n(r-1)} q^{\mathrm{tot}\, w} s^{\mathrm{odd}\, w} u^{\lambda w}\\
&\qquad + \frac{1}{1-sq} \sum_{n\geq 2} \sum_{w\in\mathrm{NIW}_n(r-1)} q^{\mathrm{tot}\, w} s^{\mathrm{odd}\, w} (usq)^{\lambda w}\\
&= \frac{sq}{1-sq}\Bigl(\frac{1}{(u)'_r} - (1+cu)\Bigr) + \frac{1}{1-sq}\Bigl(\frac{1}{(usq)'_r} - (1+cusq)\Bigr)\\
&\qquad\qquad [\text{where } c \text{ is the coefficient of } u \text{ in } 1/(u)'_r]\\
&= 1 + \frac{1}{1-sq}\Bigl(\frac{sq}{(u)'_r} - \frac{1}{(usq)'_r}\Bigr).
\end{aligned}$$

Finally, $\sum\limits_{v\in[0,r]^n} q^{\mathrm{tot}\, v} s^{2\,\mathrm{evdec}\, v + \mathrm{odd}\, v} = \dfrac{A}{1-B} = \dfrac{1}{(u)'_{r+1}} \times \dfrac{(1-sq)(u)'_r (usq)'_r}{(u)'_r - sq(usq)'_r}$, which is the expression written in (8.1). $\square$

Remark. The method used in the latter proof cannot be applied when $Z \neq 1$. Although identity (8.2) always holds, we do not have

$$\mathrm{odd}\, v = \mathrm{odd}\, w_0 + \mathrm{odd}\, w_1 + \cdots + \mathrm{odd}\, w_k$$

in general.

9. Specializations

In Section 7 we have seen that the generating function for the polynomials $A_n(s,t,q,Y)$ can be derived from (1.6) by letting $Z=0$ and replace the triplet (s,t,q,Y_0) by $(s^{1/2},t^{1/2},q^{1/2},Y)$. We just comment the specializations $s=1$ and $Y_0=Y_1=Z=1$.

9.1. *Case* $s=1$. We have

$$F_r(u;1,q,Z) \\ = \frac{(uq^2;q^2)_{\lfloor r/2\rfloor}\,(1-q^2)\,(u;q^2)_{\lfloor (r+1)/2\rfloor}\,(u;q^2)_{\lfloor r/2\rfloor}}{(u;q^2)_{\lfloor r/2\rfloor+1}\Bigl((u;q^2)_{\lfloor (r+1)/2\rfloor}\bigl((u;q^2)_{\lfloor r/2\rfloor}-q^2\,(uq^2;q^2)_{\lfloor r/2\rfloor}\bigr) + qZ\,(u;q^2)_{\lfloor r/2\rfloor}\bigl((u;q^2)_{\lfloor (r+1)/2\rfloor}-(uq^2;q^2)_{\lfloor (r+1)/2\rfloor}\bigr)\Bigr)},$$

so that we can divide both numerator and denominator by the product $(u;q^2)_{\lfloor (r+1)/2\rfloor}\,(u;q^2)_{\lfloor r/2\rfloor}$. We then get

$$\begin{aligned}F_r(u;1,q,Z) &= \frac{1-q^2}{1-u}\Bigl(1-q^2\frac{1-uq^{2\lfloor r/2\rfloor}}{1-u}+qZ-qZ\frac{1-uq^{2\lfloor (r+1)/2\rfloor}}{1-u}\Bigr)^{-1}\\ &= \Bigl(1-u\frac{1-q^{2\lfloor r/2\rfloor+2}}{1-q^2}-uqZ\frac{1-q^{2\lfloor (r+1)/2\rfloor}}{1-q^2}\Bigr)^{-1}\\ &= \Bigl(1-u\sum_{0\le i\le r}q^iZ^{\chi(i\text{ odd})}\Bigr)^{-1}.\end{aligned}$$

We then recover identity (1.9) from our paper [FoHa07a] in the form:

$$\sum_{n\ge 0}(1+t)B_n(1,t,q,Y_0,Y_1,Z)\frac{u^n}{(t^2;q^2)_{n+1}} \\ = \sum_{r\ge 0}t^r\frac{(u;q^2)_{\lfloor r/2\rfloor+1}}{(uY_0;q^2)_{\lfloor r/2\rfloor+1}}\,\frac{(-uqY_1Z;q^2)_{\lfloor (r+1)/2\rfloor}}{(-uqZ;q^2)_{\lfloor (r+1)/2\rfloor}}\Bigl(1-u\sum_{i=0}^{r}q^iZ^{\chi(i\text{ odd})}\Bigr)^{-1}. \tag{9.1}$$

9.2. *Case* $Y_0=Y_1=Z=1$. Identity (1.7) simply becomes

$$\sum_{n\ge 0}(1+t)B_n(s,t,q,1,1,1)\frac{u^n}{(t^2;q^2)_{n+1}}=\sum_{r\ge 0}t^rF_r(u;s,q,1), \tag{9.2}$$

where $F_r(u;s,q,1)$ is given by (1.8). Note that $B_n(s,t,q,1,1,1)$ is the generating polynomial for the group B_n by (fexc, fdes, fmaj). Write $B_n(s,t)$ for $B_n(s,t,1,1,1,1)$. From (9.2) we deduce

$$\sum_{n\ge 0}B_n(s,t)\frac{u^n}{(1-t^2)^n}=\sum_{r\ge 0}\Bigl(\frac{(1-s)(1-t)t^{2r}}{(1-u)^{r+1}(1-us^2)^{-r}-s(1-u)} + \frac{(1-s)(1-t)t^{2r+1}}{(1-u)^{r+1}(1-us^2)^{-r}-s(1-us)}\Bigr). \tag{9.3}$$

To obtain the marginal distribution $B_n(s,1) = \sum_{w\in B_n} s^{\operatorname{fexc} w}$ it is convenient to specialize formula (1.9). We obtain

$$\sum_{n\ge 0} B_n(s,1)\frac{u^n}{n!} = \frac{1-s}{-s+\exp(u(s^2-1))}. \tag{9.4}$$

As for $B_n(1,t) = \sum_{w\in B_n} t^{\operatorname{fdes} w}$ we specialize (9.1) and find:

$$\sum_{n\ge 0} B_n(1,t)\frac{u^n}{(1-t^2)^n} = \sum_{r\ge 0} t^r \frac{1-t}{1-u(r+1)}. \tag{9.5}$$

Hence, $B_n(1,t)/(1-t^2)^n = (1-t)\sum_{r\ge 0} t^r(r+1)^n$ and

$$\sum_{n\ge 0} B_n(1,t)\frac{u^n}{n!} = \frac{1-t}{-t+\exp(u(t^2-1))}. \tag{9.6}$$

It follows from (9.4) and (9.6) that $B_n(s,1) = B_n(1,s)$, so that "fexc" and "fdes" are equally distributed on the group B_n. For $k,n\ge 0$ let $B_{n,k}$ be the number of signed permutations w from B_n such that $\operatorname{fdes} w = k$. In our paper [FoHa06] we have derived the recurrence formula for the $B_{n,k}$'s in a more general context. The recurrence reads as follows

$$\begin{gathered}
B_{0,0}=1,\quad B_{0,k}=0 \text{ for all } k\ne 0;\\
B_{1,0}=1,\quad B_{1,1}=1,\quad B_{1,k}=0 \text{ for all } k\ne 0,1;\\
B_{n,k}=(k+1)B_{n-1,k}+B_{n-1,k-1}+(2n-k)B_{n-1,k-2};
\end{gathered}$$

for $n\ge 2$ and $0\le k\le 2n-1$. Let $B'_{n,k} := \#\{w\in B_n : \operatorname{fexc} w = k\}$. The coefficients $B'_{n,k}$'s satisfy the same recurrence as the $B_{n,k}$'s. The induction is easy, so that we can fabricate a bijection ψ of B_n onto itself such that $\operatorname{fexc} w = \operatorname{fdes} \psi(w)$.

10. Concluding remarks

The statistical study of the group B_n and some other Weyl groups has been initiated by Reiner ([Re93a], [Re93b], [Re93c], [Re95a], [Re95b]) and continued by the Roman school ([Br94], [Bi03], [BiCa04]). It has been rejuvenated by Adin, Roichman [AR01] with their definition of the *flag major index* for signed permutations and the first proof of the fact that length function and flag-major index were equidistributed over B_n ([ABR01], [ABR05], [ABR06]). In our series "Signed words and permutations; I–V" we have tried to work out analytical expressions for the *multivariable* distributions on the group B_n that were natural extensions of the expressions already derived for the symmetric group $\mathfrak{S}_n$. Other works along those lines are due to Gessel and his school ([ChGe07], [Ch03]). Further algebraic extensions have recently been done by the Minnesota school [BRS07].

We should like to thank Christian Krattenthaler [Kr07] who urged us to use the telescoping technique to shorten our q-calculations (see Section 6).

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Fix-Mahonian Calculus, I: two transformations

Dominique Foata and Guo-Niu Han

Tu es Petrus, et super hanc petram,
aedificavisti Lacim Uqam tuam.

To Pierre Leroux, Montreal, Sept. 2006,
on the occasion of the LerouxFest.

Abstract. We construct two bijections of the symmetric group $\mathfrak{S}_n$ onto itself that enable us to show that three new three-variable statistics are equidistributed with classical statistics involving the number of fixed points. The first one is equidistributed with the triplet $(\mathrm{fix}, \mathrm{des}, \mathrm{maj})$, the last two with $(\mathrm{fix}, \mathrm{exc}, \mathrm{maj})$, where "fix," "des," "exc" and "maj" denote the number of fixed points, the number of descents, the number of excedances and the major index, respectively.

1. Introduction

In this paper *Fix-Mahonian Calculus* is understood to mean the study of multivariable statistics on the symmetric group $\mathfrak{S}_n$, which involve the number of fixed points "fix" as a marginal component. As for the two transformations mentioned in the title, they make it possible to show that the new statistics defined below are equidistributed with the classical ones.

The *descent set*, $\mathrm{DES}\, w$, and *rise set*, $\mathrm{RISE}\, w$, of a word $w = x_1 x_2 \cdots x_n$, whose letters are nonnegative integers, are respectively defined as being the *subsets*:

$$\mathrm{DES}\, w := \{i : 1 \le i \le n-1, x_i > x_{i+1}\}; \tag{1.1}$$

$$\mathrm{RISE}\, w := \{i : 1 \le i \le n, x_i \le x_{i+1}\}. \tag{1.2}$$

By convention, $x_0 = x_{n+1} = +\infty$. If $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ is a permutation of $12\cdots n$, we can then consider $\mathrm{RISE}\,\sigma$, but also a new statistic $\mathrm{RIZE}\,\sigma$, which is simply the rise set of the word w derived from σ by replacing each *fixed point* $\sigma(i) = i$ by 0. For instance, with $\sigma = 3\,2\,5\,4\,1$ having the two fixed points 2, 4, we get $w = 3\,0\,5\,0\,1$, so that $\mathrm{RISE}\,\sigma = \{2, 5\}$ and $\mathrm{RIZE}\,\sigma = \mathrm{RISE}\, w = \{2, 4, 5\}$. It is quite unexpected to notice that the two set-theoretic statistics "RISE" and "RIZE" are equidistributed on each symmetric group $\mathfrak{S}_n$. Such an equidistribution property, which is a consequence of our Theorem 1.4, has become a powerful tool, as it has enabled

Xin and the second author [HaXi07] to give an immediate proof of a conjecture of Stanley [St06] on alternating permutations. More importantly, the equidistribution properties are proved by means of new transformations of the symmetric group, the bijections Φ and F_3 introduced in the sequel, whose properties will be fully exploited in our next paper [FoHa07]. Those transformations will be described not directly on $\mathfrak{S}_n$, but on classes of *shuffles*, as now introduced.

Let $0 \le m \le n$ and let v be a nonempty word of length m, whose letters are *positive* integers (with possible repetitions). Designate by $\mathrm{Sh}(0^{n-m}v)$ the set of all *shuffles* of the words 0^{n-m} and v, that is, the set of all rearrangements of the juxtaposition product $0^{n-m}v$, whose longest *subword* of positive letters is v. Let $w = x_1x_2\cdots x_n$ be a word from $\mathrm{Sh}(0^{n-m}v)$. It is convenient to write: $\mathrm{Pos}\, w := v$, $\mathrm{Zero}\, w := \{i : 1 \le i \le n,\, x_i = 0\}$, $\mathrm{zero}\, w := \#\,\mathrm{Zero}\, w$ $(= n - m)$, so that w is completely characterized by the pair $(\mathrm{Zero}\, w, \mathrm{Pos}\, w)$.

The *major index* of w is defined by

$$\mathrm{maj}\, w := \sum_{i\ge 1} i \quad (i \in \mathrm{DES}\, w), \tag{1.3}$$

and a new integral-valued statistic "mafz" by

$$\mathrm{mafz}\, w := \sum_{i\in \mathrm{Zero}\, w} i - \sum_{i=1}^{\mathrm{zero}\, w} i + \mathrm{maj}\, \mathrm{Pos}\, w. \tag{1.4}$$

Note that the first three definitions are also valid for each arbitrary word with nonnegative letters. The link of "mafz" with the statistic "maf" introduced in [CHZ97] for permutations will be further mentioned.

We shall also be interested in shuffle classes $\mathrm{Sh}(0^{n-m}v)$ when the word v is a *derangement* of the set $[\,m\,] := \{1, 2, \ldots, m\}$, that is, when the word $v = y_1y_2\cdots y_m$ is a permutation of $12\cdots m$ and $y_i \ne i$ for all i. For short, v is a *derangement of order* m. Let $w = x_1x_2\cdots x_n$ a be word from the shuffle class $\mathrm{Sh}(0^{n-m}v)$. Then $v = y_1y_2\cdots y_m = x_{j_1}x_{j_2}\cdots x_{j_m}$ for a certain sequence $1 \le j_1 < j_2 < \cdots < j_m \le n$. Let "red" be the increasing bijection of $\{j_1, j_2, \ldots, j_m\}$ onto $[\,m\,]$. Say that each positive letter x_k of w is *excedent* (resp. *subexcedent*) if and only if $x_k > \mathrm{red}\, k$ (resp. $x_k < \mathrm{red}\, k$). Another kind of rise set, denoted by $\mathrm{RISE}^\bullet\, w$, can then be introduced as follows.

Say that $i \in \mathrm{RISE}^\bullet\, w$ if and only if $1 \le i \le n$ and if one of the following conditions holds (assuming that $x_{n+1} = +\infty$):

(1) $0 < x_i < x_{i+1}$;
(2) $x_i = x_{i+1} = 0$;
(3) $x_i = 0$ and x_{i+1} is excedent;
(4) x_i is subexcedent and $x_{i+1} = 0$.

Note that if $x_i = 0$ and x_{i+1} is subexcedent, then $i \in \textsc{rise}\, w \setminus \textsc{rise}^\bullet\, w$, while if x_i is subexcedent and $x_{i+1} = 0$, then $i \in \textsc{rise}^\bullet\, w \setminus \textsc{rise}\, w$.

Example. Let $v = 5\,1\,2\,3\,6\,4$ be a derangement of order 6. Its excedent letters are 5, 6. Let $w = 5\,0\,1\,2\,0\,0\,3\,6\,4 \in \mathrm{Sh}(0^3 v)$. Then, $\textsc{rise}\, w = \{2, 3, 5, 6, 7, 9\}$ and $\textsc{rise}^\bullet\, w = \{3, 4, 5, 7, 9\}$. Also $\operatorname{mafz} w = (2 + 5 + 6) - (1 + 2 + 3) + \operatorname{maj}(512364) = 7 + 6 = 13$.

Theorem 1.1. *For each derangement v of order m and each integer $n \geq m$ the transformation $\mathbf{\Phi}$ constructed in Section 2 is a bijection of* $\mathrm{Sh}(0^{n-m} v)$ *onto itself having the property that*

$$\textsc{rise}\, w = \textsc{rise}^\bullet\, \mathbf{\Phi}(w) \tag{1.5}$$

holds for every $w \in \mathrm{Sh}(0^{n-m} v)$.

Theorem 1.2. *For each arbitrary word v of length m with positive letters and each integer $n \geq m$ the transformation $\mathbf{F}_3$ constructed in Section 4 is a bijection of $\mathrm{Sh}(0^{n-m} v)$ onto itself having the property that*

$$\operatorname{maj} w = \operatorname{mafz} \mathbf{F}_3(w); \tag{1.6}$$

$$L\, w = L\, \mathbf{F}_3(w) \quad (\text{“}L\text{” for “last” or rightmost letter}); \tag{1.7}$$

hold for every $w \in \mathrm{Sh}(0^{n-m} v)$.

We emphasize the fact that Theorem 1.1 is restricted to the case where v is a derangement, while Theorem 1.2 holds for an arbitrary word v with possible repetitions. In Fig. 1 we can see that “RISE” and “RISE$^\bullet$” (resp. “maj” and “mafz”) are equidistributed on the shuffle class $\mathrm{Sh}(0^2 312)$ (resp. $\mathrm{Sh}(0^2 121)$).

$\textsc{rise}\, w$	w	$\mathbf{\Phi}(w)$	$\textsc{rise}^\bullet \mathbf{\Phi}(w)$
1, 2, 4, 5	0 0 3 1 2	0 0 3 1 2	1, 2, 4, 5
1, 3, 4, 5	0 3 0 1 2	0 3 1 2 0	1, 3, 4, 5
1, 4, 5	0 3 1 0 2	0 3 0 1 2	1, 4, 5
1, 3, 5	0 3 1 2 0	0 3 1 0 2	1, 3, 5
2, 3, 4, 5	3 0 0 1 2	3 1 2 0 0	2, 3, 4, 5
2, 4, 5	3 0 1 0 2 3 1 2 0 0	3 0 0 1 2 3 1 0 2 0	2, 4, 5
2, 3, 5	3 0 1 2 0	3 0 1 0 2	2, 3, 5
3, 4, 5	3 1 0 0 2	3 0 1 2 0	3, 4, 5
3, 5	3 1 0 2 0	3 1 0 0 2	3, 5

$\mathrm{Sh}(0^2 312)$

$\operatorname{maj} w$	w	$\mathbf{F}_3(w)$	$\operatorname{mafz} \mathbf{F}_3(w)$
2	1 2 0 0 1	0 0 1 2 1	2
3	0 1 2 0 1	0 1 0 2 1	3
4	0 0 1 2 1 1 0 2 0 1	1 0 0 2 1 0 1 2 0 1	4
5	1 0 0 2 1 1 2 1 0 0	1 0 2 0 1 0 1 2 1 0	5
6	0 1 0 2 1 1 2 0 1 0	1 2 0 0 1 1 0 2 1 0	6
7	0 1 2 1 0	1 2 0 1 0	7
8	1 0 2 1 0	1 2 1 0 0	8

$\mathrm{Sh}(0^2 121)$

Fig. 1

Those two transformations are fully exploited once we know how to map those shuffle classes onto the symmetric groups. The permutations from the symmetric group $\mathfrak{S}_n$ will be regarded as linear words $\sigma =$

$\sigma(1)\sigma(2)\cdots\sigma(n)$. If σ is such a permutation, let $\mathrm{FIX}\,\sigma$ denote the set of its fixed points, i.e., $\mathrm{FIX}\,\sigma := \{i : 1 \le i \le n,\, \sigma(i) = i\}$ and let $\mathrm{fix}\,\sigma := \#\,\mathrm{FIX}\,\sigma$. Let $(j_1, j_2, \ldots, j_m)$ be the increasing sequence of the integers k such that $1 \le k \le n$ and $\sigma(k) \ne k$ and "red" be the increasing bijection of $\{j_1, j_2, \ldots, j_m\}$ onto $[m]$. The word $w = x_1x_2\cdots x_n$ derived from $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ by replacing each fixed point by 0 and each other letter $\sigma(j_k)$ by $\mathrm{red}\,\sigma(j_k)$ will be denoted by $\mathrm{ZDer}(\sigma)$. Also let

$$\mathrm{Der}\,\sigma := \mathrm{red}\,\sigma(j_1)\ \mathrm{red}\,\sigma(j_2)\ \cdots\ \mathrm{red}\,\sigma(j_m), \tag{1.8}$$

so that $\mathrm{Der}\,\sigma$ is the word derived from $\mathrm{ZDer}(\sigma)$ by deleting all the zeros. Accordingly, $\mathrm{Der}\,\sigma = \mathrm{Pos}\ \mathrm{ZDer}(\sigma)$.

It is important to notice that $\mathrm{Der}\,\sigma$ is a *derangement* of order m. Also $\sigma(j_k)$ is excedent in σ (*i.e.* $\sigma(j_k) > j_k$) if and only if $\mathrm{red}\,\sigma(j_k)$ is excedent in $\mathrm{Der}\,\sigma$ (*i.e.* $\mathrm{red}\,\sigma(j_k) > \mathrm{red}\,j_k$)

Recall that the statistics "DES," "RISE" and "maj" are also valid for permutations $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ and that the statistics "des" (*number of descents*) and "exc" (*number of excedances*) are defined by

$$\mathrm{des}\,\sigma := \#\,\mathrm{DES}\,\sigma; \tag{1.9}$$

$$\mathrm{exc}\,\sigma := \#\{i : 1 \le i \le n-1,\ \sigma(i) > i\}. \tag{1.10}$$

We further define:

$$\mathrm{DEZ}\,\sigma := \mathrm{DES}\ \mathrm{ZDer}(\sigma); \tag{1.11}$$

$$\mathrm{RIZE}\,\sigma := \mathrm{RISE}\ \mathrm{ZDer}(\sigma); \tag{1.12}$$

$$\mathrm{dez}\,\sigma := \#\,\mathrm{DEZ}\,\sigma = \mathrm{des}\ \mathrm{ZDer}(\sigma); \tag{1.13}$$

$$\mathrm{maz}\,\sigma := \mathrm{maj}\ \mathrm{ZDer}(\sigma); \tag{1.14}$$

$$\mathrm{maf}\,\sigma := \mathrm{mafz}\ \mathrm{ZDer}(\sigma). \tag{1.15}$$

As the zeros of $\mathrm{ZDer}(\sigma)$ correspond to the fixed points of σ, we also have

$$\mathrm{maf}\,\sigma := \sum_{i\in \mathrm{FIX}\,\sigma} i - \sum_{i=1}^{\mathrm{fix}\,\sigma} i + \mathrm{maj}\,\mathrm{Der}\,\sigma. \tag{1.16}$$

Example. Let $\sigma = 8\,\mathbf{2}\,1\,3\,\mathbf{5}\,\mathbf{6}\,4\,9\,7$; then $\mathrm{DES}\,\sigma = \{1,2,6,8\}$, $\mathrm{des}\,\sigma = 4$, $\mathrm{maj}\,\sigma = 17$, $\mathrm{exc}\,\sigma = 2$. Furthermore, $\mathrm{ZDer}(\sigma) = w = 5\,0\,1\,2\,0\,0\,3\,6\,4$ and $\mathrm{Pos}\,w = \mathrm{Der}\,\sigma = 5\,1\,2\,3\,6\,4$ is a derangement of order 6. We have $\mathrm{FIX}\,\sigma = \{2,5,6\}$, $\mathrm{fix}\,\sigma = 3$, $\mathrm{DEZ}\,\sigma = \{1,4,8\}$, $\mathrm{RIZE}\,w = \{2,3,5,6,7,9\}$, $\mathrm{dez} = 3$, $\mathrm{maz}\,\sigma = 13$ and $\mathrm{maf}\,\sigma = (2{+}5{+}6){-}(1{+}2{+}3){+}\mathrm{maj}(512364) = 7{+}6 = 13$.

For each $n \ge 0$ let D_n be the set of all derangements of order n and $\mathfrak{S}_n^{\mathrm{Der}}$ be the union: $\mathfrak{S}_n^{\mathrm{Der}} := \bigcup_{m,v} \mathrm{Sh}(0^{n-m}v) \quad (0 \le m \le n,\ v \in D_m)$.

Proposition 1.3. *The map* ZDer *is a bijection of* $\mathfrak{S}_n$ *onto* $\mathfrak{S}_n^{\mathrm{Der}}$ *having the following properties:*

$$\mathrm{RIZE}\,\sigma = \mathrm{RISE}\ \mathrm{ZDer}(\sigma) \quad \text{and} \quad \mathrm{RISE}\,\sigma = \mathrm{RISE}^{\bullet}\ \mathrm{ZDer}(\sigma). \tag{1.17}$$

Proof. It is evident to verify that ZDer is bijective and to define its inverse ZDer^{-1}. On the other hand, we have $\mathrm{RIZE} = \mathrm{RISE}\,\mathrm{ZDer}$ by definition. Finally, let $w = x_1x_2\cdots x_n = \mathrm{ZDer}(\sigma)$ and $\sigma(i) < \sigma(i+1)$ for $1 \le i \le n-1$. Four cases are to be considered:

(1) both i and $i+1$ are not fixed by σ and $0 < x_i < x_{i+1}$;

(2) both i and $i+1$ are fixed points and $x_i = x_{i+1} = 0$;

(3) $\sigma(i) = i$ and $\sigma(i+1)$ is excedent; then $x_i = 0$ and x_{i+1} is also excedent;

(4) $\sigma(i) < i < i+1 = \sigma(i+1)$; then x_i is subexcedent and $x_{i+1} = 0$.

We recover the four cases considered in the definition of $\mathrm{RISE}^{\bullet}$. The case $i = n$ is banal to study. $\square$

We next form the two chains:

$$\Phi : \sigma \overset{\mathrm{ZDer}}{\mapsto} w \overset{\boldsymbol{\Phi}}{\mapsto} w' \overset{\mathrm{ZDer}^{-1}}{\mapsto} \sigma'; \tag{1.18}$$

$$\mathrm{F}_3 : \sigma \overset{\mathrm{ZDer}}{\mapsto} w \overset{\mathbf{F}_3}{\mapsto} w'' \overset{\mathrm{ZDer}^{-1}}{\mapsto} \sigma''. \tag{1.19}$$

The next theorem is then a consequence of Theorems 1.1 and 1.2 and Propositions 1.3.

Theorem 1.4. *The mappings* Φ, F_3 *defined by* (1.18) *and* (1.19) *are bijections of* $\mathfrak{S}_n$ *onto itself and have the following properties*

$$(\mathrm{fix}, \mathrm{RIZE}, \mathrm{Der})\,\sigma = (\mathrm{fix}, \mathrm{RISE}, \mathrm{Der})\,\Phi(\sigma); \tag{1.20}$$

$$(\mathrm{fix}, \mathrm{maz}, \mathrm{Der})\,\sigma = (\mathrm{fix}, \mathrm{maf}, \mathrm{Der})\,\mathrm{F}_3(\sigma); \tag{1.21}$$

$$(\mathrm{fix}, \mathrm{maj}, \mathrm{Der})\,\sigma = (\mathrm{fix}, \mathrm{maf}, \mathrm{Der})\,\mathrm{F}_3 \circ \Phi^{-1}(\sigma); \tag{1.22}$$

for every σ *from* $\mathfrak{S}_n$.

It is evident that if $\mathrm{Der}\,\sigma = \mathrm{Der}\,\tau$ holds for a pair of permutations σ, τ of order n, then $\mathrm{exc}\,\sigma = \mathrm{exc}\,\tau$. Since $\mathrm{DES}\,\sigma = [\,n\,] \setminus \mathrm{RISE}\,\sigma$ and $\mathrm{DEZ}\,\sigma = [\,n\,] \setminus \mathrm{RIZE}\,\sigma$ it follows from (1.20) that

$$(\mathrm{fix}, \mathrm{DEZ}, \mathrm{exc})\,\sigma = (\mathrm{fix}, \mathrm{DES}, \mathrm{exc})\,\Phi(\sigma); \tag{1.23}$$

$$(\mathrm{fix}, \mathrm{dez}, \mathrm{maz}, \mathrm{exc})\,\sigma = (\mathrm{fix}, \mathrm{des}, \mathrm{maj}, \mathrm{exc})\,\Phi(\sigma). \tag{1.24}$$

On the other hand, (1.21) implies that

$$(\mathrm{fix}, \mathrm{maz}, \mathrm{exc})\,\sigma = (\mathrm{fix}, \mathrm{maf}, \mathrm{exc})\,\mathrm{F}_3(\sigma). \tag{1.25}$$

As a consequence we obtain the following Corollary.

Corollary 1.5. *The two triplets* (fix, dez, maz) *and* (fix, des, maj) *are equidistributed over* $\mathfrak{S}_n$. *Moreover, the three triplets* (fix, exc, maz), (fix, exc, maj) *and* (fix, exc, maf) *are also equidistributed over* $\mathfrak{S}_n$.

The distributions of (fix, des, maj) and (fix, exc, maj) have been calculated by Gessel-Reutenauer ([GeRe93], Theorem 8.4) and by Shareshian and Wachs [ShWa06], respectively, using the algebra of the q-series (see, e.g., Gasper and Rahman ([GaRa90], chap. 1). Let

$$A_n(s,t,q,Y) := \sum_{\sigma\in\mathfrak{S}_n} s^{\operatorname{exc}\sigma} t^{\operatorname{des}\sigma} q^{\operatorname{maj}\sigma} Y^{\operatorname{fix}\sigma} \quad (n\ge 0).$$

Then, they respectively derived the identities:

$$(1.26)\quad \sum_{n\ge 0} A_n(1,t,q,Y)\frac{u^n}{(t;q)_{n+1}} = \sum_{r\ge 0} t^r \Bigl(1-u\sum_{i=0}^{r} q^i\Bigr)^{-1} \frac{(u;q)_{r+1}}{(uY;q)_{r+1}};$$

$$(1.27)\quad \sum_{n\ge 0} A_n(s,1,q,Y)\frac{u^n}{(q;q)_n} = \frac{(1-sq)e_q(Yu)}{e_q(squ)-sqe_q(u)}.$$

In our third paper [FoHa07a] we have shown that the factorial generating function for the four-variable polynomials $A_n(s,t,q,Y)$ could be evaluated under the form

$$(1.28)\quad \sum_{n\ge 0} A_n(s,t,q,Y)\frac{u^n}{(t;q)_{n+1}} = \sum_{r\ge 0} t^r \frac{(1-sq)\,(u;q)_r\,(usq;q)_r}{((u;q)_r - sq(usq;q)_r)(uY;q)_{r+1}},$$

the two identities (1.26) and (1.27) becoming simple specializations. We then know the distributions over $\mathfrak{S}_n$ of the triplets (fix, dez, maz), (fix, exc, maz) and (fix, exc, maf) and also the distribution of the quadruplet (fix, dez, maz, exc). Note that the statistic "maf" was introduced by Clarke *et al.* [CHZ97]. Although it was not explicitly stated, their bijection "CHZ" of $\mathfrak{S}_n$ onto itself satisfies identity (1.22) when $\mathrm{F}_3\circ\Phi^{-1}$ is replaced by "CHZ."

As is shown in Section 2, the transformation $\mathbf{\Phi}$ is described as a composition product of bijections ϕ_l. The image $\phi_l(w)$ of each word w from a shuffle class $\mathrm{Sh}(0^{n-m}v)$ is obtained by moving its l-th zero, to the right or to the left, depending on its preceding and following letters. The description of the inverse bijection $\mathbf{\Psi}$ of $\mathbf{\Phi}$ follows an analogous pattern. The verification of identity (1.5) requires some attention and is made in Section 3. The construction of the transformation $\mathbf{F}_3$ is given in Section 4. Recall that $\mathbf{F}_3$ maps each shuffle class $\mathrm{Sh}(0^{n-m}v)$ onto itself, the word v being an *arbitrary* word with nonnegative letters. Very much like the *second fundamental transformation* (see, e.g., [Lo83], p. 201, Algorithm 10.6.1) the construction of $\mathbf{F}_3$ is defined by induction on the length of the words and preserves the *rightmost* letter.

2. The bijection $\mathbf{\Phi}$

Let v be a *derangement* of order m and $w = x_1x_2\cdots x_n$ be a word from the shuffle class $\mathrm{Sh}(0^{n-m}v)$ $(0 \le n \le m)$, so that $v = x_{j_1}x_{j_2}\cdots x_{j_m}$ for $1 \le j_1 < j_2 < \cdots < j_m \le n$. Let "red" ("reduction") be the increasing bijection of $\{j_1, j_2, \ldots, j_m\}$ onto the interval $[\,m\,]$. Remember that a positive letter x_k of w is said to be excedent (resp. subexcedent) if and only if $x_k > \mathrm{red}\,k$ (resp. $x_k < \mathrm{red}\,k$). Accordingly, a letter is non-subexcedent if it is either equal to 0 or excedent.

We define n bijections ϕ_l $(1 \le l \le n)$ of $\mathrm{Sh}(0^{n-m}v)$ onto itself in the following manner: for each l such that $n-m+1 \le l \le n$ let $\phi_l(w) := w$. When $1 \le l \le n-m$, let x_j denote the l-th letter of w, equal to 0, when w is read *from left to right*. Three cases are next considered (by convention, $x_0 = x_{n+1} = +\infty$):

(1) x_{j-1}, x_{j+1} both non-subexcedent;

(2) x_{j-1} non-subexcedent, x_{j+1} subexcedent; or x_{j-1}, x_{j+1} both subexcedent with $x_{j-1} > x_{j+1}$;

(3) x_{j-1} subexcedent, x_{j+1} non-subexcedent; or x_{j-1}, x_{j+1} both subexcedent with $x_{j-1} < x_{j+1}$.

When case (1) holds, let $\phi_l(w) := w$.

When case (2) holds, determine the *greatest* integer $k \ge j+1$ such that

$$x_{j+1} < x_{j+2} < \cdots < x_k < \mathrm{red}(k),$$

so that

$$w = x_1 \cdots x_{j-1}\, 0\, x_{j+1} \cdots x_k\, x_{k+1} \cdots x_n$$

and define:

$$\phi_l(w) := x_1 \cdots x_{j-1}\, x_{j+1} \cdots x_k\, 0\, x_{k+1} \cdots x_n.$$

When case (3) holds, determine the *smallest* integer $i \le j-1$ such that

$$\mathrm{red}(i) > x_i > x_{i+1} > \cdots > x_{j-1},$$

so that

$$w = x_1 \cdots x_{i-1}\, x_i \cdots x_{j-1}\, 0\, x_{j+1} \cdots x_n$$

and define:

$$\phi_l(w) := x_1 \cdots x_{i-1}\, 0\, x_i \cdots x_{j-1}\, x_{j+1} \cdots x_n.$$

It is important to note that ϕ_l has no action on the 0's other than the l-th one. Then the mapping $\mathbf{\Phi}$ in Theorem 1.1 is defined to be the composition product

$$\mathbf{\Phi} := \phi_1\phi_2\cdots\phi_{n-1}\phi_n.$$

Example. The following word w has four zeros, so that $\mathbf{\Phi}(w)$ can be reached in four steps:

$$\begin{array}{rl}
\mathrm{Id} & = 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11 \\
w & = 5\ \mathbf{0}\ 1\ 2\ \mathbf{0}\ \mathbf{0}\ 3\ 6\ \mathbf{0}\ 7\ 4 \quad j=9, \text{apply } \phi_4, \text{case } (1); \\
 & \ 5\ \mathbf{0}\ 1\ 2\ \mathbf{0}\ \mathbf{0}\ 3\ 6\ \mathbf{0}\ 7\ 4 \quad j=6, \text{apply } \phi_3, \text{case } (2),\ k=7; \\
 & \ 5\ \mathbf{0}\ 1\ 2\ \mathbf{0}\ 3\ \mathbf{0}\ 6\ \mathbf{0}\ 7\ 4 \quad j=5, \text{apply } \phi_2, \text{case } (3),\ i=4; \\
 & \ 5\ \mathbf{0}\ 1\ \mathbf{0}\ 2\ 3\ \mathbf{0}\ 6\ \mathbf{0}\ 7\ 4 \quad j=2, \text{apply } \phi_1, \text{case } (2),\ k=3; \\
\mathbf{\Phi}(w) & = 5\ 1\ \mathbf{0}\ \mathbf{0}\ 2\ 3\ \mathbf{0}\ 6\ \mathbf{0}\ 7\ 4.
\end{array}$$

We have: $\mathrm{RISE}\, w = \mathrm{RISE}^\bullet\, \mathbf{\Phi}(w) = \{2,3,5,6,7,9,11\}$, as desired.

To verify that $\mathbf{\Phi}$ is bijective, we introduce a class of bijections ψ_l, whose definitions are parallel to the definitions of the ϕ_l's. Let $w = x_1 x_2 \cdots x_n \in \mathrm{Sh}(0^{n-m}v)$ $(0 \le m \le n)$. For each l such that $n-m+1 \le l \le n$ let $\psi_l(w) := w$. When $1 \le l \le n-m$, let x_j denote the l-th letter of w, equal to 0, when w is read *from left to right.* Consider the following three cases (remember that $x_0 = x_{n+1} = +\infty$ by convention):

$(1') = (1)$ x_{j-1}, x_{j+1} both non-subexcedent;

$(2')$ x_{j-1} subexcedent, x_{j+1} non-subexcedent; or x_{j-1}, x_{j+1} both subexcedent with $x_{j-1} > x_{j+1}$;

$(3')$ x_{j-1} non-subexcedent, x_{j+1} subexcedent; or x_{j-1}, x_{j+1} both subexcedent with $x_{j-1} < x_{j+1}$.

When case $(1')$ holds, let $\psi_l(w) := w$.

When case $(2')$ holds, determine the *smallest* integer $i \le j-1$ such that

$$x_i < x_{i+1} < \cdots < x_{j-1} < \mathrm{red}(j-1),$$

so that

$$w = x_1 \cdots x_{i-1}\ x_i \cdots x_{j-1}\ 0\ x_{j+1} \cdots x_n$$

and define:

$$\psi_l(w) := x_1 \cdots x_{i-1}\ 0\ x_i \cdots x_{j-1}\ x_{j+1} \cdots x_n.$$

When case $(3')$ holds, determine the *greatest* integer $k \ge j+1$ such that

$$\mathrm{red}(j+1) > x_{j+1} > x_{j+2} > \cdots > x_k,$$

so that

$$w = x_1 \cdots x_{j-1}\ 0\ x_{j+1} \cdots x_k\ x_{k+1} \cdots x_n$$

and define:

$$\psi_l(w) := x_1 \cdots x_{j-1}\ x_{j+1} \cdots x_k\ 0\ x_{k+1} \cdots x_n.$$

We now observe that when case (2) (resp. (3)) holds for w, then case (2′) (resp. (3′)) holds for $\phi_l(w)$. Also, when case (2′) (resp. (3′)) holds for w, then case (2) (resp. (3)) holds for $\psi_l(w)$. Therefore

$$\phi_l\psi_l = \psi_l\phi_l = \text{Identity map}$$

and the product $\mathbf{\Psi} := \psi_n\psi_{n-1}\cdots\psi_2\psi_1$ is the inverse bijection of Φ.

3. Verification of $\mathrm{RISE}\, w = \mathrm{RISE}^\bullet\, \mathbf{\Phi}(w)$

Let us introduce an alternate definition for $\mathbf{\Phi}$. Let w belong to $\mathrm{Sh}(0^{n-m}v)$ and w' be a nonempty left factor of w, of length n'. Let w' have p' letters equal to 0. If $p' \geq 1$, write

$$w' = x_1 \cdots x_{j-1}\, 0^h x_{j+h}x_{j+h+1}\cdots x_{n'},$$

where $1 \leq h \leq p'$, $x_{j-1} \neq 0$ and where the right factor $x_{j+h}x_{j+h+1}\cdots x_{n'}$ contains no 0. By convention $x_{j+h} := +\infty$ if $j+h = n'+1$. If x_{j+h} is subexcedent let k be the *greatest* integer $k \geq j+h$ such that $x_{j+h} < x_{j+h+1} < \cdots < x_k < \mathrm{red}(k)$. If x_{j-1} is subexcedent let i be the *smallest* integer $i \leq j-1$ such that $\mathrm{red}(i) > x_i > x_{i+1} > \cdots > x_{j-1}$. Examine four cases:

(1) if x_{j-1} and x_{j+h} are both excedent, let

$$u := x_1\cdots x_{j-1}, \quad u' := 0^h x_{j+h}x_{j+h+1}\cdots x_{n'}$$

and define

$$\theta(u') := u'.$$

(2) if x_{j-1} is excedent and x_{j+h} subexcedent, or if x_{j-1}, x_{j+h} are both subexcedent with $x_{j-1} > x_{j+h}$, let

$$u := x_1\cdots x_{j-1}, \quad u' := 0^h\, x_{j+h}\cdots x_k\, x_{k+1}\cdots x_{n'}$$

and define

$$\theta(u') := x_{j+h}\cdots x_k\, 0^h\, x_{k+1}\cdots x_{n'}.$$

(3) if x_{j-1}, x_{j+h} are both subexcedent with $x_{j-1} < x_{j+h}$, let

$$u := x_1\cdots x_{i-1}, \quad u' := x_i\cdots x_{j-1}0^h x_{j+h}\cdots x_k\, x_{k+1}\cdots x_{n'}$$

and define

$$\theta(u') := 0\, x_i\cdots x_{j-1}x_{j+h}\cdots x_k\, 0^{h-1}\, x_{k+1}\cdots x_{n'}.$$

(4) if x_{j-1} is subexcedent and x_{j+h} excedent, let

$$u := x_1\cdots x_{i-1}, \quad u' := x_i\cdots x_{j-1}0^h x_{j+h}\cdots x_{n'}$$

and define

$$\theta(u') := 0\, x_i\cdots x_{j-1}\, 0^{h-1}\, x_{j+h}\cdots x_{n'}.$$

By construction $w' = uu'$. Call it the *canonical factorization* of w'. In the four cases we evidently have:

$$\mathrm{RISE}\, u' = \mathrm{RISE}^\bullet\, \theta(u'). \tag{3.1}$$

Define $\Theta(w')$ to be the three-term sequence:

$$\Theta(w') := (u, u', \theta(u')). \tag{3.2}$$

Let q be the length of u. Then $q = j-1$ in cases (1) and (2) and $q = i-1$ in cases (3) and (4).

Lemma 3.1. *We have*

$$\mathrm{RISE}\, x_q u' = \mathrm{RISE}^\bullet\, x_q \theta(u'), \tag{3.3}$$

$$\mathrm{RISE}\, x_q u' = \mathrm{RISE}^\bullet\, 0\theta(u'), \quad \text{if } x_q \text{ is subexcedent.} \tag{3.4}$$

Proof. Let α (resp. β) be the leftmost letter of u' (resp. of $\theta(u')$). Because of (3.1) we only have to prove that $\mathrm{RISE}\, x_q\alpha = \mathrm{RISE}^\bullet\, x_q\beta$ for (3.3) and $\mathrm{RISE}\, x_q\alpha = \mathrm{RISE}^\bullet\, 0\beta$ when x_q is subexcedent for (3.4). Let us prove identity (3.3). There is nothing to do in case (1). In case (2) we have to verify $\mathrm{RISE}\, x_{j-1}0 = \mathrm{RISE}^\bullet\, x_{j-1}x_{j+h}$. When x_{j-1} is excedent and x_{j+h} subexcedent, then $1 \notin \mathrm{RISE}\, x_{j-1}0$ and $1 \notin \mathrm{RISE}^\bullet\, x_{j-1}x_{j+h}$. When x_{j-1}, x_{j+h} are both subexcedent with $x_{j-1} > x_{j+h}$, then $1 \notin \mathrm{RISE}\, x_{j-1}0$ and $1 \notin \mathrm{RISE}^\bullet\, x_{j-1}x_{j+h}$.

In cases (3) and (4) we have to verify $\mathrm{RISE}\, x_{i-1}x_i = \mathrm{RISE}^\bullet\, x_{i-1}0$. If $x_{i-1} = 0$ (resp. excedent), then $1 \in \mathrm{RISE}\, x_{i-1}x_i$ (resp. $1 \notin \mathrm{RISE}\, x_{i-1}x_i$) and $1 \in \mathrm{RISE}^\bullet\, x_{i-1}0$ (resp. $1 \notin \mathrm{RISE}^\bullet\, x_{i-1}0$). When x_{i-1} is subexcedent, then $x_{i-1} < x_i$ by definition of i. Hence $1 \in \mathrm{RISE}\, x_{i-1}x_i$ and $1 \in \mathrm{RISE}^\bullet\, x_{i-1}0$.

We next prove identity (3.4). In case (1) x_q is always excedent, so that identity (3.4) need not be considered. In case (2) we have to verify $\mathrm{RISE}\, x_{j-1}0 = \mathrm{RISE}^\bullet\, 0x_{j+h}$. But if x_{j-1} is subexcedent, then x_{j+h} is also subexcedent, so that the above two sets are empty. In cases (3) and (4) we have to verify $\mathrm{RISE}\, x_{i-1}x_i = \mathrm{RISE}^\bullet\, 00 = \{1\}$. But if x_{i-1} is subexcedent, then $x_{i-1} < x_i$ by definition of i, so that $1 \in \mathrm{RISE}\, x_{i-1}x_i$. $\square$

Now, if w has p letters equal to 0 with $p \geq 1$, it may be expressed as the juxtaposition product

$$w = w_1\, 0^{h_1}\, w_2\, 0^{h_2} \cdots w_r\, 0^{h_r}\, w_{r+1},$$

where $h_1 \geq 1$, $h_2 \geq 1, \ldots, h_r \geq 1$ and where the factors $w_1, w_2, \ldots, w_r$, w_{r+1} contain no 0 and $w_2, \ldots, w_r$ are nonempty. We may define: $\Theta(w) := (u_r, u'_r, \theta(u'_r))$, where $w = u_r u'_r$ is the canonical factorization of w. As u_r

is a left factor of w, we next define $\Theta(u_r) := (u_{r-1}, u'_{r-1}, \theta(u'_{r-1}))$, where $u_r = u_{r-1}u'_{r-1}$ is the canonical factorization of u_r, and successively $\Theta(u_{r-1}) := (u_{r-2}, u'_{r-2}, \theta(u'_{r-2}))$ with $u_{r-1} = u_{r-2}u'_{r-2}$, ... , $\Theta(u_2) := (u_1, u'_1, \theta(u'_1))$ with $u_2 = u_1u'_1$, so that $w = u_1u'_1u'_2\cdots u'_r$ and $\mathbf{\Phi}(w) = u_1\,\theta(u'_1)\theta(u'_2)\cdots\theta(u'_r)$.

It can be verified that $u_r\,\theta(u'_r) = \phi_{p-h_r+1}\cdots\phi_{p-1}\phi_p(w)$, $u_{r-1}\theta(u'_{r-1})\theta(u'_r) = \phi_{p-h_r-h_{r-1}+1}\cdots\phi_{p-1}\phi_p(w)$, etc.

With identities (3.1), (3.3) and (3.4) the proof of (1.5) is now completed.

Again, consider the word w of the preceding example

$$w = 5\,0\,1\,2\,0\,0\,3\,6\,0\,7\,4,$$

so that $r = 3$, $h_1 = 1$, $h_2 = 2$, $h_3 = 1$, $w_1 = 5$, $w_2 = 1\,2$ and $w_3 = 3\,6$, $w_4 = 7\,4$. We have

$$\begin{aligned}
\Theta(w) &= (u_3, u'_3, \theta(u'_3)) = (5\,0\,1\,2\,0\,0\,3\,6;\ 0\,7\,4;\ 0\,7\,4); && \text{case (1)}\\
\Theta(u_3) &= (u_2, u'_2, \theta(u'_2)) = (5\,0\,1;\ 2\,0\,0\,3\,6;\ 0\,2\,3\,0\,6); && \text{case (3)}\\
\Theta(u_2) &= (u_1, u'_1, \theta(u'_1)) = (5;\ 0\,1;\ 1\,0); && \text{case (2)}\\
\mathbf{\Phi}(w) &= u_1\,\theta(u'_1)\theta(u'_2)\theta(u'_3) = 5 \mid 1\,0 \mid 0\,2\,3\,0\,6 \mid 0\,7\,4.
\end{aligned}$$

4. The transformation $\mathbf{F}_3$

The bijection $\mathbf{F}_3$ we are now defining maps each shuffle class $\mathrm{Sh}(0^{n-m}v)$ with v an arbitrary word of length m ($0 \le m \le n$) onto itself. When $n = 1$ the unique element of the shuffle class is sent onto itself. Also let $\mathbf{F}_3(w) = w$ when $\mathrm{des}(w) = 0$. Let $n \ge 2$ and assume that $\mathbf{F}_3(w')$ has been defined for all words w' with nonnegative letters, of length $n' \le n-1$. Further assume that (1.6) and (1.7) hold for all those words. Let w be a word of length n such that $\mathrm{des}(w) \ge 1$. We may write

$$w = w'a0^rb,$$

where $a \ge 1$, $b \ge 0$ and $r \ge 0$. Three cases are considered:
(1) $a \le b$; (2) $a > b$, $r \ge 1$; (3) $a > b$, $r = 0$.

In case (1) define: $\mathbf{F}_3(w) = \mathbf{F}_3(w'a0^rb) := (\mathbf{F}_3(w'a0^r))b$.

In case (2) we may write $\mathbf{F}_3(w'a0^r) = w''0$ by Property (1.7). We then define

$$\begin{gathered}
\gamma\,\mathbf{F}_3(w'a0^r) := 0w'';\\
\mathbf{F}_3(w) = \mathbf{F}_3(w'a0^rb) := (\gamma\,\mathbf{F}_3(w'a0^r))b = 0w''b.
\end{gathered}$$

In short, add one letter "0" to the left of $\mathbf{F}_3(w'a0^r)$, then delete the rightmost letter "0" and add b to the right.

In case (3) remember that $r = 0$. Write

$$\mathbf{F}_3(w'a) = 0^{m_1} x_1 v_1 0^{m_2} x_2 v_2 \cdots 0^{m_k} x_k v_k,$$

where $m_1 \geq 0, m_2, \ldots, m_k$ are all positive, then $x_1, x_2, \ldots, x_k$ are positive letters and $v_1, v_2, \ldots, v_k$ are words with positive letters, possibly empty. Then define:

$$\begin{aligned} \delta\,\mathbf{F}_3(w'a) &:= x_1 0^{m_1} v_1 x_2 0^{m_2} v_2 x_3 \cdots x_k 0^{m_k} v_k; \\ \mathbf{F}_3(w) = \mathbf{F}_3(w'ab) &:= (\delta\,\mathbf{F}_3(w'a))b. \end{aligned}$$

In short, move each positive letter occurring just after a 0-factor of $\mathbf{F}_3(w'a)$ to the beginning of that 0-factor and add b to the right.

Example.

$$\begin{aligned} w &= 0\,0\,0\,3\,1\,2\,2\,0\,0\,1\,3 \\ \mathbf{F}_3(0\,0\,0\,3) &= 0\,0\,0\,3 && \text{no descent} \\ \mathbf{F}_3(0\,0\,0\,3\,1) &= \delta(0\,0\,0\,3)1 = 3\,0\,0\,0\,1 && \text{case (3)} \\ \mathbf{F}_3(0\,0\,0\,3\,1\,2\,2) &= 3\,0\,0\,0\,1\,2\,2 && \text{case (1)} \\ \mathbf{F}_3(0\,0\,0\,3\,1\,2\,2\,0) &= \delta(3\,0\,0\,0\,1\,2\,2)0 = 3\,1\,0\,0\,0\,2\,2\,0 && \text{case (3)} \\ \mathbf{F}_3(0\,0\,0\,3\,1\,2\,2\,0\,0) &= \gamma(3\,1\,0\,0\,0\,2\,2\,0)0 = 0\,3\,1\,0\,0\,0\,2\,2\,0 && \text{case (2)} \\ \mathbf{F}_3(0\,0\,0\,3\,1\,2\,2\,0\,0\,1) &= \gamma(0\,3\,1\,0\,0\,0\,2\,2\,0)1 = 0\,0\,3\,1\,0\,0\,0\,2\,2\,1 && \text{case (2)} \\ \mathbf{F}_3(0\,0\,0\,3\,1\,2\,2\,0\,0\,1\,3) &= 0\,0\,3\,1\,0\,0\,0\,2\,2\,1\,3. \end{aligned}$$

We have: $\operatorname{maj} w = \operatorname{maj}(0\,0\,0\,3\,1\,2\,2\,0\,0\,1\,3) = 4 + 7 = 11$ and $\operatorname{mafz} \mathbf{F}_3(w) = \operatorname{mafz}(0\,0\,3\,1\,0\,0\,0\,2\,2\,1\,3) = (1+2+5+6+7) - (1+2+3+4+5) + (1+4) = 11$.

By construction the rightmost letter is preserved by $\mathbf{F}_3$. To prove (1.6) proceed by induction. Assume that $\operatorname{mafz} w'a0^r = \operatorname{mafz} \mathbf{F}_3(w'a0^r)$ holds. In case (1) "maj" and "mafz" remain invariant when b is juxtaposed at the end. In case (2) we have $\operatorname{maj} w = \operatorname{maj}(w'a0^r b) = \operatorname{maj}(w'a0^r)$, but $\operatorname{mafz} \gamma\,\mathbf{F}_3(w'a0^r) = \operatorname{mafz} \mathbf{F}_3(w'a0^r) - |w'a0^r|_{\geq 1}$ and $\operatorname{mafz}\,(\gamma\,\mathbf{F}_3(w'a0^r))b = \operatorname{mafz} \gamma\,\mathbf{F}_3(w'a0^r) + |w'a0^r|_{\geq 1}$, where $|w'a0^r|_{\geq 1}$ denotes the number of *positive* letters in $w'a0^r$. Hence (1.6) holds. In case (3) remember $r = 0$. We have $\operatorname{maj}(w'ab) = \operatorname{maj}(w'a) + |w'a|$, where $|w'a|$ denotes the length of the word $w'a$. But $\operatorname{mafz} \delta\mathbf{F}_3(w'a) = \operatorname{mafz} \mathbf{F}_3(w'a) + \operatorname{zero}(w'a)$ and $\operatorname{mafz}(\delta\mathbf{F}_3(w'a))b = \operatorname{mafz} \delta\mathbf{F}_3(w'a) + |w'a|_{\geq 1}$. The equality holds for $b = 0$ and $b \geq 1$, as easily verified. As $\operatorname{zero}(w'a) + |w'a|_{\geq 1} = |w'a|$, we have $\operatorname{mafz} \mathbf{F}_3(w) = \operatorname{mafz}\,(\delta\mathbf{F}_3(w'a))b = \operatorname{mafz} \mathbf{F}_3(w'a) + |w'a| = \operatorname{maj} w'ab = \operatorname{maj} w$. Thus (1.6) holds in the three cases.

To define the inverse bijection $\mathbf{F}_3^{-1}$ of $\mathbf{F}_3$ we first need the inverses $\gamma^{-1}(w)$ and $\delta^{-1}(w)$ for each word w. Let $w = 0w'$ be a word, whose first letter is 0. Define $\gamma^{-1}(w)$ to be the word derived from w by deleting the first letter 0 and adding one letter "0" to the right of w. Clearly, $\gamma^{-1}\gamma = \gamma\gamma^{-1}$ is the identity map.

Next, let w be a word, whose first letter is positive. Define $\delta^{-1}(w)$ to be the word derived from w by moving each positive letter occurring just before a 0-factor of w to the end of that 0-factor. Again $\delta^{-1}\delta = \delta\delta^{-1}$ is the identity map.

We may write

$$w = cw'a0^r b,$$

where $a \geq 1$, $b \geq 0$, $c \geq 0$ and $r \geq 0$. Three cases are considered:

(1) $a \leq b$; (2) $a > b$, $c = 0$; (3) $a > b$, $c \geq 1$.

In case (1) define: $\mathbf{F}_3^{-1}(w) := (\mathbf{F}_3^{-1}(cw'a0^r))b$.

In case (2) define: $\mathbf{F}_3^{-1}(w) := (\gamma^{-1}(\mathbf{F}_3^{-1}(cw'a0^r)))b$.

In case (3) define: $\mathbf{F}_3^{-1}(w) := (\delta^{-1}(\mathbf{F}_3^{-1}(cw'a0^r)))b$.

We end this section by proving a property of the transformation $\mathbf{F}_3$, which will be used in our next paper [FoHa07].

Proposition 4.1. *Let w, w'' be two words with nonnegative letters, of the same length. If* $\mathrm{Zero}\, w = \mathrm{Zero}\, w''$ *and* $\mathrm{DES}\,\mathrm{Pos}\, w = \mathrm{DES}\,\mathrm{Pos}\, w''$, *then* $\mathrm{Zero}\,\mathbf{F}_3(w) = \mathrm{Zero}\,\mathbf{F}_3(w'')$.

Proof. To derive $\mathbf{F}_3(w)$ (resp. $\mathbf{F}_3(w'')$) from w (resp. w'') we have to consider one of the three cases (1), (2) or (3), described above, at each step. Because of the two conditions $\mathrm{Zero}\, w = \mathrm{Zero}\, w''$ and $\mathrm{DES}\,\mathrm{Pos}\, w = \mathrm{DES}\,\mathrm{Pos}\, w''$, case (i) $(i = 1, 2, 3)$ is used at the j-th step in the calculation of $\mathbf{F}_3(w)$, if and only if the same case is used at that j-th step for the calculation of $\mathbf{F}_3(w'')$. Consequently the letters equal to 0 are in the same places in both words $\mathbf{F}_3(w)$ and $\mathbf{F}_3(w'')$. □

By the very definition of $\Phi : \sigma \overset{\mathrm{ZDer}}{\mapsto} w \overset{\mathbf{\Phi}}{\mapsto} w' \overset{\mathrm{ZDer}^{-1}}{\mapsto} \sigma'$, given in (1.18) and of $\mathrm{F}_3 : \sigma \overset{\mathrm{ZDer}}{\mapsto} w \overset{\mathbf{F}_3}{\mapsto} w'' \overset{\mathrm{ZDer}^{-1}}{\mapsto} \sigma''$, given in (1.19) we have $\Phi(\sigma) = \sigma$ and $\mathrm{F}_3(\sigma) = \sigma$ if σ is a derangement. In the next two tables we have calculated $\Phi(\sigma) = \sigma'$ and $\mathrm{F}_3(\sigma) = \sigma''$ for the fifteen non-derangement permutations σ of order 4.

fix σ	Der σ	RIZE σ	σ	w	w'	σ'	RISE σ'	Der σ'	fix σ'
4	e	$1,2,3,4$	1234	**0000**	**0000**	1234	$1,2,3,4$	e	4
		$1,2,4$	1243	**00**21	**00**21	1243	$1,2,4$		
		$1,4$	1324	**0**21**0**	**0**2**0**1	1432	$1,4$		
		$1,3,4$	1432	**0**2**0**1	**0**21**0**	1324	$1,3,4$		
2	21	$3,4$	2134	21**00**	2**0**1**0**	3214	$3,4$	21	2
		$2,4$	3214	2**0**1**0**	2**00**1	4231	$2,4$		
		$2,3,4$	4231	2**00**1	21**00**	2134	$2,3,4$		
		$1,2,4$	1342	**0**231	**0**231	1342	$1,2,4$		
		$1,4$	2314	231**0**	23**0**1	2431	$1,4$		
1	231	$1,3,4$	2431	23**0**1	231**0**	2314	$1,3,4$	231	1
		$2,4$	3241	2**0**31	2**0**31	3241	$2,4$		
		$1,3,4$	1423	**0**312	**0**312	1423	$1,3,4$		
1	312	$2,4$	3124	312**0**	31**0**2	4132	$2,4$	312	1
		$3,4$	4132	31**0**2	3**0**12	4213	$3,4$		
		$2,3,4$	4213	3**0**12	312**0**	3124	$2,3,4$		

Calculation of $\sigma' = \Phi(\sigma)$

fix σ	Der σ	maz σ	σ	w	w''	σ''	maf σ''	Der σ''	fix σ''
4	e	0	1234	**0000**	**0000**	1234	0	e	4
		3	1243	**00**21	2**00**1	4231	3		
		5	1324	**0**21**0**	21**00**	2134	5		
		2	1432	**0**2**0**1	**0**2**0**1	1432	2		
2	21	3	2134	21**00**	**0**21**0**	1324	3	21	2
		4	3214	2**0**1**0**	2**0**1**0**	3214	4		
		1	4231	2**00**1	**00**21	1243	1		
		3	1342	**0**231	2**0**31	3241	3		
		5	2314	231**0**	231**0**	2314	5		
1	231	2	2431	23**0**1	**0**231	1342	2	231	1
		4	3241	2**0**31	23**0**1	2431	4		
		2	1423	**0**312	3**0**12	4213	2		
1	312	4	3124	312**0**	312**0**	3124	4	312	1
		3	4132	31**0**2	31**0**2	4213	3		
		1	4213	3**0**12	**0**312	3124	1		

Calculation of $\sigma'' = \mathrm{F}_3(\sigma)$

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Fix-Mahonian Calculus, II: further statistics

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ABSTRACT. Using classical transformations on the symmetric group and two transformations constructed in Fix-Mahonian Calculus I, we show that several multivariable statistics are equidistributed either with the triplet (fix,des,maj), or the pair (fix,maj), where "fix," "des" and "maj" denote the number of fixed points, the number of descents and the major index, respectively.

1. Introduction

First, recall the traditional notations for the q-ascending factorials

$$(a;q)_n := \begin{cases} 1, & \text{if } n=0; \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & \text{if } n\ge 1; \end{cases}$$
$$(a;q)_\infty := \prod_{n\ge 1}(1-aq^{n-1});$$

and the q-exponential (see [GaRa90, chap. 1])

$$e_q(u)=\sum_{n\ge 0}\frac{u^n}{(q;q)_n}=\frac{1}{(u;q)_\infty}.$$

Furthermore, let $(A_n(Y,t,q))$ and $(A_n(Y,q))$ $(n\ge 0)$ be the sequences of polynomials respectively defined by the factorial generating functions

$$\sum_{n\ge 0}A_n(Y,t,q)\frac{u^n}{(t;q)_{n+1}} := \sum_{s\ge 0}t^s\Bigl(1-u\sum_{i=0}^{s}q^i\Bigr)^{-1}\frac{(u;q)_{s+1}}{(uY;q)_{s+1}}; \tag{1.1}$$
$$\sum_{n\ge 0}A_n(Y,q)\frac{u^n}{(q;q)_n} := \Bigl(1-\frac{u}{1-q}\Bigr)^{-1}\frac{(u;q)_\infty}{(uY;q)_\infty}. \tag{1.2}$$

Of course, (1.2) can be derived from (1.1) by letting the variable t tend to 1, so that $A_n(Y,q)=A_n(Y,1,q)$. The classical combinatorial interpretation for those classes of polynomials has been found by Gessel and Reutenauer [GeRe93] (see Theorem 1.1 below). For each permutation $\sigma=\sigma(1)\sigma(2)\cdots\sigma(n)$ from the symmetric group $\mathfrak{S}_n$ let $\mathbf{i}\,\sigma := \sigma^{-1}$ denote the inverse of σ; then let its *set of fixed points*, $\mathrm{FIX}\,\sigma$, *descent set*, $\mathrm{DES}\,\sigma$, *idescent set*, $\mathrm{IDES}\,\sigma$, be defined as the subsets:

$$\begin{aligned}\mathrm{FIX}\,\sigma &:= \{i : 1 \le i \le n,\ \sigma(i) = i\};\\ \mathrm{DES}\,\sigma &:= \{i : 1 \le i \le n-1,\ \sigma(i) > \sigma(i+1)\};\\ \mathrm{IDES}\,\sigma &:= \mathrm{DES}\,\sigma^{-1}.\end{aligned}$$

Note that $\mathrm{IDES}\,\sigma$ is also the set of all i such that $i+1$ is on the left of i in the linear representation $\sigma(1)\sigma(2)\cdots\sigma(n)$ of σ. Also let $\mathrm{fix}\,\sigma := \#\,\mathrm{FIX}\,\sigma$ (the *number of fixed points*), $\mathrm{des}\,\sigma := \#\,\mathrm{DES}\,\sigma$ (the *number of descents*), $\mathrm{maj}\,\sigma := \sum_i i\ (i \in \mathrm{DES}\,\sigma)$ (the *major index*), $\mathrm{imaj}\,\sigma := \sum_i i\ (i \in \mathrm{IDES}\,\sigma)$ (the *inverse major index*).

Theorem 1.1 (Gessel, Reutenauer). *For each $n \ge 0$ the generating polynomial for $\mathfrak{S}_n$ by* $(\mathrm{fix}, \mathrm{des}, \mathrm{maj})$ *(resp. by* $(\mathrm{fix}, \mathrm{maj})$*) is equal to $A_n(Y,t,q)$ (resp. to $A_n(Y,q)$). Accordingly,*

$$A_n(Y,t,q) = \sum_{\sigma\in\mathfrak{S}_n} Y^{\mathrm{fix}\,\sigma} t^{\mathrm{des}\,\sigma} q^{\mathrm{maj}\,\sigma}; \tag{1.3}$$

$$A_n(Y,q) = \sum_{\sigma\in\mathfrak{S}_n} Y^{\mathrm{fix}\,\sigma} q^{\mathrm{maj}\,\sigma}. \tag{1.4}$$

The purpose of this paper is to show that there are several other three-variable (resp. two-variable) statistics on $\mathfrak{S}_n$, whose distribution is given by the generating polynomial $A_n(Y,t,q)$ (resp. $A_n(Y,q)$). For proving that those statistics are equidistributed with $(\mathrm{fix}, \mathrm{des}, \mathrm{maj})$ (resp. $(\mathrm{fix}, \mathrm{maj})$) we make use the properties of the classical bijections $\mathrm{F}_2^{\mathrm{loc}}$, F_2', CHZ, $\mathrm{DW}^{\mathrm{glo}}$, $\mathrm{DW}^{\mathrm{loc}}$, plus two transformations F_3, Φ, constructed in our previous paper [FoHa07], finally a new transformation F_3' described in Section 3.

All those bijections appear as arrows in the diagram of Fig. 1. The nodes of the diagram are pairs or triplets of statistics, whose definitions have been, or will be, given in the paper. The integral-valued statistics are written in lower case, such as "fix" or "maj", while the set-valued ones appear in capital letters, such as "FIX" or "DES". We also introduce two mappings "Der" and "Desar" of $\mathfrak{S}_n$ into $\mathfrak{S}_m$ with $m \le n$.

Each arrow goes from one node to another node with the following meaning that we shall explain by means of an example: the vertical arrow $(\mathrm{fix}, \mathrm{maz}, \mathrm{Der}) \xrightarrow{\mathrm{F}_3} (\mathrm{fix}, \mathrm{maf}, \mathrm{Der})$ (here rewritten horizontally for typographical reasons!) indicates that the bijection F_3 maps $\mathfrak{S}_n$ onto itself and has the property: $(\mathrm{fix}, \mathrm{maz}, \mathrm{Der})\,\sigma = (\mathrm{fix}, \mathrm{maf}, \mathrm{Der})\,\mathrm{F}_3(\sigma)$ for all σ. The remaining statistics are now introduced together with the main two decompositions of permutations: the *fixed* and *pixed decompositions.*

$$
\begin{tikzcd}
(\mathrm{fix}, \mathrm{maf}, \mathrm{Der}) \arrow[dr, "\mathrm{CHZ}"] & (\mathrm{FIX}, \mathrm{maf}) \arrow[r, "\mathrm{DW}^{\mathrm{loc}}"] & (\mathrm{PIX}, \mathrm{mag}) \arrow[d, "\mathrm{F}_2^{\mathrm{loc}}"] & (\mathrm{pix}, \mathrm{mag}, \mathrm{Desar}) \\
 & \begin{array}{c}(\mathrm{fix}, \mathrm{maj}, \mathrm{Der})\\ (\mathrm{fix}, \mathrm{DES})\end{array} & \begin{array}{c}(\mathrm{PIX}, \mathrm{inv})\\ (\mathrm{pix}, \mathrm{inv})\end{array} \arrow[d, "(\mathrm{F}_2')^{-1}"] & \\
(\mathrm{fix}, \mathrm{maz}, \mathrm{Der}) \arrow[uu, "\mathrm{F}_3"] & \begin{array}{c}(\mathrm{fix}, \mathrm{DEZ})\\ (\mathrm{FIX}, \mathrm{DEZ})\end{array} \arrow[u, "\Phi"] \arrow[r, "\mathrm{DW}^{\mathrm{loc}}"'] & \begin{array}{c}(\mathrm{pix}, \mathrm{imaj})\\ (\mathrm{pix}, \mathrm{IDES})\\ (\mathrm{PIX}, \mathrm{IDES})\end{array} \arrow[ul, "\mathrm{DW}^{\mathrm{glo}}"] & (\mathrm{pix}, \mathrm{imaj}, \mathrm{Desar}) \arrow[uu, "\mathrm{F}_3'"']
\end{tikzcd}
$$

Fig. 1

1.1. *The fixed decomposition.* Let $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ be a permutation and let $(i_1, i_2, \ldots, i_{n-m})$ (resp. $(j_1, j_2, \ldots, j_m)$) be the increasing sequence of the integers k (resp. k') such that $1 \le k \le n$ and $\sigma(k) = k$ (resp. $1 \le k' \le n$ and $\sigma(k') \ne k'$). Also let "red" denote the increasing bijection of $\{j_1, j_2, \ldots, j_m\}$ onto $[\,m\,]$. Let $\mathrm{ZDer}(\sigma) = x_1 x_2 \cdots x_n$ be the word derived from $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ by replacing each fixed point $\sigma(i_k)$ by 0 and each other letter $\sigma(j_{k'})$ by $\mathrm{red}\,\sigma(j_{k'})$. As "DES," "des" and "maj" can also be defined for arbitrary words with nonnegative letters, we further introduce:

$$\mathrm{DEZ}\,\sigma := \mathrm{DES}\ \mathrm{ZDer}(\sigma); \tag{1.5}$$

$$\mathrm{dez}\,\sigma := \mathrm{des}\ \mathrm{ZDer}(\sigma), \quad \mathrm{maz}\,\sigma := \mathrm{maj}\ \mathrm{ZDer}(\sigma); \tag{1.6}$$

$$\mathrm{Der}\,\sigma := \mathrm{red}\,\sigma(j_1)\ \mathrm{red}\,\sigma(j_2)\ \cdots\ \mathrm{red}\,\sigma(j_m); \tag{1.7}$$

$$\mathrm{maf}\,\sigma := \sum_{k=1}^{n-m} (i_k - k) + \mathrm{maj} \circ \mathrm{Der}\,\sigma. \tag{1.8}$$

The subword $\mathrm{Der}\,\sigma$ of $\mathrm{ZDer}(\sigma)$ can be regarded as a permutation from $\mathfrak{S}_m$ (with the above notations). It is important to note that $\mathrm{FIX}\,\mathrm{Der}\,\sigma = \emptyset$, so that $\mathrm{Der}\,\sigma$ is a *derangement* of order m. Also note that σ is fully characterized by the pair $(\mathrm{FIX}\,\sigma, \mathrm{Der}\,\sigma)$, which is called the *fixed decomposition* of σ. Finally, we can rewrite $\operatorname{maf}\sigma$ as

$$\operatorname{maf}\sigma := \sum_{i\in \mathrm{FIX}\,\sigma} i - \sum_{i=1}^{\operatorname{fix}\sigma} i + \operatorname{maj}\circ \operatorname{Der}\sigma. \tag{1.9}$$

Example. With $\sigma = \begin{pmatrix} 1\,\mathbf{2}\,3\,4\,\mathbf{5}\,\mathbf{6}\,7\,8\,9 \\ 8\,\mathbf{2}\,1\,3\,\mathbf{5}\,\mathbf{6}\,4\,9\,7 \end{pmatrix}$ we have $\operatorname{red} = \begin{pmatrix} 1\,3\,4\,7\,8\,9 \\ 1\,2\,3\,4\,5\,6 \end{pmatrix}$, $\operatorname{ZDer}(\sigma) = 5\,0\,1\,2\,0\,0\,3\,6\,4$, $\mathrm{DEZ}\,\sigma = \{1,4,8\}$, $\operatorname{dez}\sigma = 3$, $\operatorname{maz}\sigma = 13$, $\operatorname{Der}\sigma = 5\,1\,2\,3\,6\,4$, $\mathrm{FIX}\,\sigma = \{\mathbf{2},\mathbf{5},\mathbf{6}\}$ and $\operatorname{maf}\sigma = (2-1)+(5-2)+(6-3)+\operatorname{maj}(5\,1\,2\,3\,6\,4) = 7+6 = 13$.

1.2. *The pixed decomposition.* Let $w = y_1y_2\cdots y_n$ be a word having no repetitions, without necessarily being a permutation of $12\cdots n$. Say that w is a *desarrangement* if $y_1 > y_2 > \cdots > y_{2k}$ and $y_{2k} < y_{2k+1}$ for some $k \geq 1$. By convention, $y_{n+1} = \infty$. We could also say that the *leftmost trough* of w occurs at an *even* position. This notion was introduced by Désarménien [De84] and elegantly used in a subsequent paper [DW88]. A further refinement is due to Gessel [Ge91].

Let $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ be a permutation. Unless σ is increasing, there is always a nonempty right factor of σ which is a desarrangement. It then makes sense to define σ^d as the *longest* such a right factor. Hence, σ admits a unique factorization $\sigma = \sigma^p\sigma^d$, called the *pixed factorization*, where σ^p is *increasing* and σ^d is the longest right factor of σ which is a desarrangement. The set (resp. number) of the letters in σ^p is denoted by $\mathrm{PIX}\,\sigma$ (resp. $\operatorname{pix}\sigma$).

If $\sigma^d = \sigma(n-m+1)\sigma(n-m+2)\cdots\sigma(n)$ and if "red" is the increasing bijection mapping the set $\{\sigma(n-m+1), \sigma(n-m+2), \ldots, \sigma(n)\}$ onto $\{1,2,\ldots,m\}$, define

$$\operatorname{Desar}\sigma := \operatorname{red}\sigma(n-m+1)\ \operatorname{red}\sigma(n-m+2)\ \ldots\ \operatorname{red}\sigma(n); \tag{1.10}$$

$$\operatorname{mag}\sigma := \sum_{i\in \mathrm{PIX}\,\sigma} i - \sum_{i=1}^{\operatorname{pix}\sigma} i + \operatorname{imaj}\circ \operatorname{Desar}\sigma. \tag{1.11}$$

Note that $\operatorname{Desar}\sigma$ is a desarrangement and belongs to $\mathfrak{S}_m$, for short, a desarrangement of order m. Also note that σ is fully characterized by the pair $(\mathrm{PIX}\,\sigma, \operatorname{Desar}\sigma)$, which will called the *pixed decomposition* of σ. Form the inverse $(\operatorname{Desar}\sigma)^{-1} = y_1y_2\cdots y_m$ of $\operatorname{Desar}\sigma$ and define $\operatorname{ZDesar}(\sigma)$ to be the unique shuffle $x_1x_2\cdots x_n$ of 0^{n-m} and $y_1y_2\cdots y_m$, where $x_i = 0$ if and only if $i \in \mathrm{PIX}\,\sigma$.

Example. With $\sigma = 3\,5\,7\,4\,2\,8\,1\,9\,6$, then $\sigma^p = 3\,5\,7$, $\sigma^d = 4\,2\,8\,1\,9\,6$, $\mathrm{PIX}\,\sigma = \{3,5,7\}$, $\operatorname{pix}\sigma = 3$. Also, $\operatorname{Desar}\sigma = 3\,2\,5\,1\,6\,4$, $\operatorname{imaj}\circ\operatorname{Desar}\sigma = 1+2+4 = 7$, $(\operatorname{Desar}\sigma)^{-1} = 4\,2\,1\,6\,3\,5$, $\operatorname{ZDesar}\sigma = 4\,2\,0\,1\,0\,6\,0\,3\,5$ and $\operatorname{mag}\sigma = (3+5+7)-(1+2+3)+7 = 16$.

Referring to the diagram in Fig. 1 the purpose of this paper is to prove the next two theorems.

Theorem 1.2. *In each of the following four groups the pairs of statistics are equidistributed on* $\mathfrak{S}_n$*:*

(1) $(\mathrm{fix}, \mathrm{maj})$, $(\mathrm{fix}, \mathrm{maf})$, $(\mathrm{fix}, \mathrm{maz})$, $(\mathrm{pix}, \mathrm{mag})$, $(\mathrm{pix}, \mathrm{inv})$, $(\mathrm{pix}, \mathrm{imaj})$*;*
(2) $(\mathrm{FIX}, \mathrm{maf})$, $(\mathrm{PIX}, \mathrm{mag})$, $(\mathrm{PIX}, \mathrm{inv})$*;*
(3) $(\mathrm{fix}, \mathrm{DEZ})$, $(\mathrm{fix}, \mathrm{DES})$, $(\mathrm{pix}, \mathrm{IDES})$*;*
(4) $(\mathrm{FIX}, \mathrm{DEZ})$, $(\mathrm{PIX}, \mathrm{IDES})$.

Theorem 1.3. *In each of the following two groups the triplets of statistics are equidistributed on* $\mathfrak{S}_n$*:*

(1) $(\mathrm{fix}, \mathrm{maf}, \mathrm{Der})$ *and* $(\mathrm{fix}, \mathrm{maz}, \mathrm{Der})$*;*
(2) $(\mathrm{pix}, \mathrm{mag}, \mathrm{Desar})$ *and* $(\mathrm{pix}, \mathrm{imaj}, \mathrm{Desar})$.

Furthermore, the following diagram, involving the bijections $\mathrm{DW}^{\mathrm{loc}}$, F_3 *and* F_3' *is commutative.*

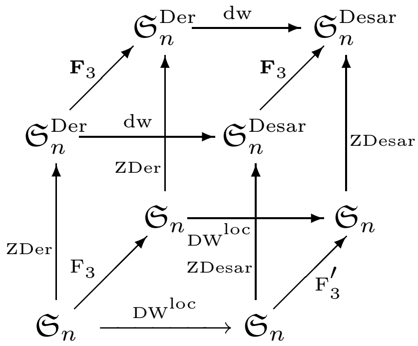

Fig. 2

2. The bijections

2.1. *The transformations* Φ *and "*CHZ*".* In our preceding paper [FoHa07] we have given the constructions of two bijections Φ, F_3 of $\mathfrak{S}_n$ onto itself. The latter one will be re-studied and used in Section 3. As was shown in our previous paper [FoHa07], the first one has the following property:

$$(\mathrm{fix}, \mathrm{DEZ}, \mathrm{Der})\,\sigma = (\mathrm{fix}, \mathrm{DES}, \mathrm{Der})\,\Phi(\sigma) \quad (\sigma \in \mathfrak{S}_n). \tag{2.1}$$

This shows that over $\mathfrak{S}_n$ the pairs $(\mathrm{fix}, \mathrm{maz})$ and $(\mathrm{fix}, \mathrm{maj})$ are equidistributed over $\mathfrak{S}_n$, their generating polynomial being given by the polynomial $A_n(Y, q)$ introduced in (1.2). Also the triplets $(\mathrm{fix}, \mathrm{dez}, \mathrm{maz})$ and $(\mathrm{fix}, \mathrm{des}, \mathrm{maj})$ are equidistributed, with generating polynomial $A_n(Y, t, q)$ introduced in (1.1).

In [CHZ97] the authors have constructed a bijection, here called "CHZ", satisfying

$$(\mathrm{fix}, \mathrm{maf}, \mathrm{Der})\,\sigma = (\mathrm{fix}, \mathrm{maj}, \mathrm{Der})\,\mathrm{CHZ}(\sigma) \quad (\sigma \in \mathfrak{S}_n). \tag{2.2}$$

2.2. *The Désarménien-Wachs bijection.* For each $n \geq 0$ let D_n denote the set of permutations σ from $\mathfrak{S}_n$ such that $\mathrm{FIX}\,\sigma = \emptyset$. The elements of D_n are referred to as the *derangements* of order n. Let K_n be the set of permutations σ from $\mathfrak{S}_n$ such that $\mathrm{PIX}\,\sigma = \emptyset$. The class K_n was introduced by Désarménien [De84], who called its elements *desarrangements* of order n. He also set up a one-to-one correspondence between D_n and K_n. Later, by means of a symmetric function argument Désarménien and Wachs [DW88] proved that for every subset $J \subset [n-1]$ the equality

$$\#\{\sigma \in D_n : \mathrm{DES}\,\sigma = J\} = \#\{\sigma \in K_n : \mathrm{IDES}\,\sigma = J\} \tag{2.3}$$

holds. In a subsequent paper [DW93] they constructed a bijection $\mathrm{DW} : D_n \to K_n$ having the expected property, that is,

$$\mathrm{IDES} \circ \mathrm{DW}(\sigma) = \mathrm{DES}\,\sigma. \tag{2.4}$$

Although their bijection is based on an inclusion-exclusion argument, leaving the door open to the discovery of an explicit correspondence, we use it as such in the sequel. For the very definition of "DW" we refer the reader to their original paper [DW93].

We now make a full use of the fixed and pixed decompositions introduced in §§ 1.1 and 1.2. Let $\tau \in \mathfrak{S}_n$ and consider the chain

$$\tau \mapsto (\mathrm{FIX}\,\tau, \mathrm{Der}\,\tau) \mapsto (\mathrm{FIX}\,\tau, \mathrm{DW} \circ \mathrm{Der}\,\tau) \mapsto \sigma, \tag{2.5}$$

where σ is the permutation defined by

$$(\mathrm{PIX}\,\sigma, \mathrm{Desar}\,\sigma) := (\mathrm{FIX}\,\tau, \mathrm{DW} \circ \mathrm{Der}\,\tau). \tag{2.6}$$

Then, the mapping $\mathrm{DW}^{\mathrm{loc}}$ defined by

$$\mathrm{DW}^{\mathrm{loc}}(\tau) := \sigma, \tag{2.7}$$

is a bijection of $\mathfrak{S}_n$ onto itself satisfying $\mathrm{FIX}\,\tau = \mathrm{PIX}\,\sigma$ and $\mathrm{DES} \circ \mathrm{Der}\,\tau = \mathrm{IDES} \circ \mathrm{Desar}\,\sigma$. In particular, $\mathrm{fix}\,\tau = \mathrm{pix}\,\sigma$. Taking the definitions of "maf" and "mag" given in (1.9) and (1.11) into account we have:

$$\begin{aligned}\mathrm{maf}\,\tau &= \sum_{i \in \mathrm{FIX}\tau} i - \sum_{i=1}^{\mathrm{fix}\,\tau} i + \mathrm{maj} \circ \mathrm{Der}\,\tau \\ &= \sum_{i \in \mathrm{PIX}\sigma} i - \sum_{i=1}^{\mathrm{pix}\,\sigma} i + \mathrm{imaj} \circ \mathrm{Desar}\,\sigma = \mathrm{mag}\,\sigma.\end{aligned}$$

We have then proved the following proposition.

Proposition 2.1. *Let* $\sigma := \mathrm{DW}^{\mathrm{loc}}(\tau)$. *Then*

$$\text{(2.8)} \qquad \mathrm{FIX}\,\tau = \mathrm{PIX}\,\sigma, \quad \mathrm{DES}\circ\mathrm{Der}\,\tau = \mathrm{IDES}\circ\mathrm{Desar}\,\sigma, \quad \mathrm{maf}\,\tau = \mathrm{mag}\,\sigma.$$

Corollary 2.2. *The pairs* $(\mathrm{FIX}, \mathrm{maf})$ *and* $(\mathrm{PIX}, \mathrm{mag})$ *are equidistributed over* $\mathfrak{S}_n$.

Example. Assume that the bijection "DW" maps the derangement 512364 onto the desarrangement 623145. On the other hand, the fixed decomposition of $\tau = 182453697$ is equal to $(\{1,4,5\}, 512364)$ and $(\{1,4,5\}, 623145)$ is the pixed decomposition of the permutation $\sigma = 145936278$. Hence $\mathrm{DW}^{\mathrm{loc}}(182453697) = 145936278$.

We verify that $\mathrm{DES}\circ\mathrm{Der}\,\tau = \mathrm{DES}(512364) = \{1,5\} = \mathrm{IDES}(623145) = \mathrm{IDES}\circ\mathrm{Desar}\,\sigma$. Also $\mathrm{maf}\,\tau = (1{+}4{+}5) - (1{+}2{+}3) + (1{+}5) = 10 = \mathrm{mag}\,\sigma$.

Proposition 2.3. *Let* $\sigma := \mathrm{DW}^{\mathrm{loc}}(\tau)$. *Then*

$$\text{(2.9)} \qquad (\mathrm{FIX}, \mathrm{DEZ})\,\tau = (\mathrm{PIX}, \mathrm{IDES})\,\sigma;$$

$$\text{(2.10)} \qquad (\mathrm{fix}, \mathrm{maz})\,\tau = (\mathrm{pix}, \mathrm{imaj})\,\sigma.$$

Proof. It suffices to prove (2.9) and in fact only $\mathrm{DEZ}\,\tau = \mathrm{IDES}\,\sigma$. Let $\sigma^p\sigma^d$ be the pixed factorization of σ. Then σ^p is the *increasing* sequence of the elements of $\mathrm{PIX}\,\sigma = \mathrm{FIX}\,\tau$. We have $i \in \mathrm{DEZ}\,\tau$ if and only if $\tau(i) \neq i$ and one of the following conditions holds:

(1) $\tau(i) > \tau(i+1)$ and $\tau(i+1) \neq i+1$;

(2) $\tau(i+1) = i+1$.

In case (1) the letters $\mathrm{red}\,\tau(i)$ and $\mathrm{red}\,\tau(i{+}1)$ are adjacent letters in $\mathrm{Der}\,\tau$ and $\mathrm{red}\,\tau(i) > \mathrm{red}\,\tau(i+1)$. As $\mathrm{DES}\circ\mathrm{Der}\,\tau = \mathrm{IDES}\circ\mathrm{Desar}\,\sigma$, the letter $\mathrm{red}(i+1)$ is to the left of the letter $\mathrm{red}(i)$ in $\mathrm{Desar}\,\sigma$ and then $(i+1)$ is to the left of i is σ, so that $i \in \mathrm{IDES}\,\sigma$.

In case (2) we have $(i{+}1) \in \mathrm{FIX}\,\tau = \mathrm{PIX}\,\sigma$ and $\mathrm{red}\,i$ is a letter of $\mathrm{Desar}\,\sigma$. Again $i \in \mathrm{IDES}\,\sigma$. $\square$

Corollary 2.4. *The pairs* $(\mathrm{FIX}, \mathrm{DEZ})$ *and* $(\mathrm{PIX}, \mathrm{IDES})$ *are equidistributed over* $\mathfrak{S}_n$.

Using the same example as above we have: $\tau = 182453697$, $\mathrm{FIX}\,\tau = \{1,4,5\}$, $\mathrm{ZDer}\,\tau = 082003697$, so that $\mathrm{DEZ}\,\tau = \{2,3,8\}$. Moreover, $\sigma = \mathrm{DW}^{\mathrm{loc}}(182453697) = 145 \mid 936278$. Hence $2, 8 \in \mathrm{IDES}\,\sigma$ (case (1)) and $3 \in \mathrm{IDES}\,\sigma$ (case (2)).

2.3. *The second fundamental transformation.* As described in [Lo83, p. 201, Algorithm 10.6.1] by means of an algorithm, the second fundamental transformation, further denoted by F_2, can be defined on permutations as well as on words. Here we need only consider the case of permutations. As usual, the *number of inversions* of a permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ is defined by $\operatorname{inv}\sigma := \#\{(i,j) \mid 1 \le i < j \le n, \sigma(i) > \sigma(j)\}$. Its construction was given in [Fo68]. Further properties have been proved in [FS78], [BjW88]. Here we need the following result.

Theorem 2.5 [FS78]. *The transformation* F_2 *defined on the symmetric group* $\mathfrak{S}_n$ *is bijective and the following identities hold for every permutation* $\sigma \in \mathfrak{S}_n$*:* $\operatorname{inv}\mathrm{F}_2(\sigma) = \operatorname{maj}\sigma$; $\quad \mathrm{IDES}\,\mathrm{F}_2(\sigma) = \mathrm{IDES}\,\sigma$.

Using the composition product $\mathrm{F}_2' := \mathbf{i} \circ \mathrm{F}_2 \circ \mathbf{i}$ we therefore have:

$$\operatorname{inv}\mathrm{F}_2'(\sigma) = \operatorname{imaj}\sigma; \quad \mathrm{DES}\,\mathrm{F}_2'(\sigma) = \mathrm{DES}\,\sigma \tag{2.11}$$

for every $\sigma \in \mathfrak{S}_n$.

As the descent set "DES" is preserved under the transformation F_2', each desarrangement is mapped onto another desarrangement. It then makes sense to consider the chain:

$$\sigma \mapsto (\mathrm{PIX}\,\sigma, \operatorname{Desar}\sigma) \mapsto (\mathrm{PIX}\,\sigma, \mathrm{F}_2' \circ \operatorname{Desar}\sigma) \mapsto \rho,$$

where $(\mathrm{PIX}\,\rho, \operatorname{Desar}\rho) := (\mathrm{PIX}\,\sigma, \mathrm{F}_2' \circ \operatorname{Desar}\sigma)$. The mapping $\mathrm{F}_2^{\mathrm{loc}} : \sigma \mapsto \rho$ is a bijection of $\mathfrak{S}_n$ onto itself. Moreover, $\mathrm{PIX}\,\sigma = \mathrm{PIX}\,\rho$, $\operatorname{imaj} \circ \operatorname{Desar}\sigma = \operatorname{inv} \circ \operatorname{Desar}\rho$ and $\mathrm{DES} \circ \operatorname{Desar}\sigma = \mathrm{DES} \circ \operatorname{Desar}\rho$. Hence,

$$\begin{aligned}
\operatorname{mag}\sigma &= \sum_{i \in \mathrm{PIX}\,\sigma} i - \sum_{i=1}^{\operatorname{pix}\sigma} i + \operatorname{imaj} \circ \operatorname{Desar}\sigma \\
&= \sum_{i \in \mathrm{PIX}\,\rho} i - \sum_{i=1}^{\operatorname{pix}\rho} i + \operatorname{inv} \circ \operatorname{Desar}\rho \\
&= \#\{(i,j) : 1 \le i \le \operatorname{pix}\rho < j \le n,\ \rho(i) > \rho(j)\} \\
&\qquad + \#\{(i,j) : \operatorname{pix}\rho < i < j \le n,\ \rho(i) > \rho(j)\} \\
&= \operatorname{inv}\rho.
\end{aligned}$$

We have then proved the following proposition.

Proposition 2.6. *Let* $\rho := \mathrm{F}_2^{\mathrm{loc}}(\sigma)$. *Then*

$$(\mathrm{PIX}, \operatorname{mag})\,\sigma = (\mathrm{PIX}, \operatorname{inv})\,\rho. \tag{2.12}$$

Corollary 2.7. *The pairs* $(\mathrm{PIX}, \operatorname{mag})$ *and* $(\mathrm{PIX}, \operatorname{inv})$ *are equidistributed over* $\mathfrak{S}_n$.

Finally, go back to Properties (2.11) and let $\xi := \mathrm{F}_2'(\rho)$. Also let $\xi^p\xi^d$ and $\rho^p\rho^d$ be the pixed factorizations of ξ and ρ, respectively. We do not have $\xi^p = \rho^p$ necessarily, but as ρ and ξ have the same descent set, the factors ξ^p and ρ^p have the same length, i.e., $\operatorname{pix}\xi = \operatorname{pix}\rho$. Let us state this result in the next proposition.

Proposition 2.8. *Let* $\xi := \mathrm{F}_2'(\rho)$. *Then*

$$(\mathrm{pix}, \mathrm{inv})\,\xi = (\mathrm{pix}, \mathrm{imaj})\,\rho. \tag{2.13}$$

Corollary 2.9. *The pairs* $(\mathrm{pix}, \mathrm{inv})$ *and* $(\mathrm{pix}, \mathrm{imaj})$ *are equidistributed over* $\mathfrak{S}_n$.

Finally, $\mathrm{DW}^{\mathrm{glo}}$ attached to the unique oblique arrow in Fig. 1 refers to the global bijection constructed by Désarménien and Wachs ([DW93], § 5). It has the property: $(\mathrm{fix}, \mathrm{DES})\,\mathrm{DW}^{\mathrm{glo}}(\sigma) = (\mathrm{pix}, \mathrm{IDES})\,\sigma$ for all σ in $\mathfrak{S}_n$. The big challenge is to find two explicit bijections f and g, replacing $\mathrm{DW}^{\mathrm{glo}}$ and $\mathrm{DW}^{\mathrm{loc}}$, such that $(\mathrm{fix}, \mathrm{DES})\,g(\sigma) = (\mathrm{pix}, \mathrm{IDES})\,\sigma$ and $(\mathrm{PIX}, \mathrm{IDES})\,f(\sigma) = (\mathrm{FIX}, \mathrm{DEZ})\,\sigma$, which would make the bottom triangle commutative, that is, $\Phi = g \circ f$.

3. The bijections F_3 and F_3'

Let $0 \le m \le n$ and let v be a nonempty word of length m, whose letters are *positive* integers (with possible repetitions). Designate by $\mathrm{Sh}(0^{n-m}v)$ the set of all *shuffles* of the words 0^{n-m} and v, that is, the set of all rearrangements of the juxtaposition product $0^{n-m}v$, whose longest *subword* of positive letters is v. Let $w = x_1x_2\cdots x_n$ be a word from $\mathrm{Sh}(0^{n-m}v)$. It is convenient to write: $\mathrm{Pos}\,w := v$, $\mathrm{Zero}\,w := \{i : 1 \le i \le n,\, x_i = 0\}$, $\mathrm{zero}\,w := \#\,\mathrm{Zero}\,w\ (= n - m)$, so that w is completely characterized by the pair $(\mathrm{Zero}\,w, \mathrm{Pos}\,w)$. Besides the statistic "maj" we will need the statistic "mafz" that associates the number

$$\mathrm{mafz}\,w := \sum_{i \in \mathrm{Zero}\,w} i - \sum_{i=1}^{\mathrm{zero}\,w} i + \mathrm{maj}\,\mathrm{Pos}\,w. \tag{3.1}$$

with each word from $\mathrm{Sh}(0^{n-m}v)$. In ([FoHa07], §4) we gave the construction of a bijection $\mathbf{F}_3$ of $\mathrm{Sh}(0^{n-m}v)$ onto itself having the following property:

$$\mathrm{maj}\,w = \mathrm{mafz}\,\mathbf{F}_3(w) \quad (w \in \mathrm{Sh}(0^{n-m}v)). \tag{3.2}$$

The bijection $\mathbf{F}_3$ is now applied to each shuffle class $\mathrm{Sh}(0^{n-m}v)$, when v is a derangement, or the *inverse* of a desarrangement. Let

$$\mathfrak{S}_n^{\mathrm{Der}} := \bigcup_{m,v} \mathrm{Sh}(0^{n-m}v) \quad (0 \le m \le n,\, v \in D_m);$$
$$\mathfrak{S}_n^{\mathrm{Desar}} := \bigcup_{m,v} \mathrm{Sh}(0^{n-m}v) \quad (0 \le m \le n,\, v^{-1} \in K_m).$$

As already seen in § 1.1, the mapping ZDer is a bijection of $\mathfrak{S}_n$ onto $\mathfrak{S}_n^{\mathrm{Der}}$ satisfying

$$\begin{aligned} &\mathrm{FIX}\,\sigma = \mathrm{Zero}\ \mathrm{ZDer}(\sigma); \quad \mathrm{Der}\,\sigma = \mathrm{Pos}\ \mathrm{ZDer}(\sigma);\\ &\mathrm{maf}\,\sigma = \mathrm{mafz}\ \mathrm{ZDer}(\sigma); \quad \mathrm{DEZ}\,\sigma = \mathrm{DES}\ \mathrm{ZDer}(\sigma). \end{aligned} \tag{3.3}$$

Example 3.2. Let $\sigma = 1\,7\,3\,5\,2\,6\,4$. Then $w := \mathrm{ZDer}(\sigma) = 0\,4\,0\,3\,1\,0\,2$. We have $\mathrm{FIX}\,\sigma = \mathrm{Zero}\,w = \{1,3,6\}$; $\mathrm{Der}\,\sigma = \mathrm{Pos}\,w = 4\,3\,1\,2$; $\mathrm{maf}\,\sigma = \mathrm{mafz}\,w = (1{+}3{+}6){-}(1{+}2{+}3){+}(1{+}2{+}3) = 10$, $\mathrm{DEZ}\,\sigma = \mathrm{DES}\,w = \{2,4,5\}$.

Now define the bijection F_3 of $\mathfrak{S}_n$ onto itself by the chain

$$\mathrm{F}_3 : \sigma \overset{\mathrm{ZDer}}{\mapsto} w \overset{\mathbf{F}_3}{\mapsto} w' \overset{\mathrm{ZDer}^{-1}}{\mapsto} \sigma'. \tag{3.4}$$

Then, by (3.2),

$$\begin{aligned} (\mathrm{fix},\mathrm{maz},\mathrm{Der})\,\sigma &= (\mathrm{zero},\mathrm{maj},\mathrm{Pos})\,w\\ &= (\mathrm{zero},\mathrm{mafz},\mathrm{Pos})\,w'\\ &= (\mathrm{fix},\mathrm{maf},\mathrm{Der})\,\sigma', \end{aligned}$$

$$(\mathrm{fix},\mathrm{maz},\mathrm{Der})\,\sigma = (\mathrm{fix},\mathrm{maf},\mathrm{Der})\,\mathrm{F}_3(\sigma). \tag{3.5}$$

The map "Desar" has been defined in (1.10) and it was noticed that each permutation σ was fully characterized by the pair $(\mathrm{PIX}\,\sigma, \mathrm{Desar}\,\sigma)$. Another way of deriving $\mathrm{ZDesar}(\sigma)$ introduced in §1.2 is to form the inverse $\sigma^{-1} = \sigma^{-1}(1)\sigma^{-1}(2)\cdots\sigma^{-1}(n)$ of σ. As $\sigma^{-1}(i) \ge \mathrm{pix}\,\sigma + 1$ if and only if $i \in [\,n\,] \setminus \mathrm{PIX}\,\sigma$, we see that $\mathrm{ZDesar}(\sigma)$ is also the word $w = x_1x_2\cdots x_n$, where

$$x_i := \begin{cases} 0, & \text{if } i \in \mathrm{PIX}\,\sigma;\\ \sigma^{-1}(i) - \mathrm{pix}\,\sigma, & \text{if } i \in [\,n\,]\setminus \mathrm{PIX}\,\sigma. \end{cases}$$

The word σ^{-1} contains the *subword* $1\,2\,\cdots\,\mathrm{pix}\,\sigma$. We then have: $i \in \mathrm{IDES}\,\sigma \Leftrightarrow i \in \mathrm{DES}\,\sigma^{-1} \Leftrightarrow \sigma^{-1}(i) \ge \mathrm{pix}\,\sigma + 1$ and $\sigma^{-1}(i) > \sigma^{-1}(i+1) \Leftrightarrow x_i \ge 1$ and $x_i > x_{i+1} \Leftrightarrow i \in \mathrm{DES}\,w$, so that $\mathrm{IDES}\,\sigma = \mathrm{DES}\,w$.

On the other hand, as $\mathrm{PIX}\,\sigma = \mathrm{Zero}\,w$ and $(\mathrm{Desar}\,\sigma)^{-1} = \mathrm{Pos}\,w$, we also have, by (1.11)

$$\begin{aligned} \mathrm{mag}\,\sigma &= \sum_{i\in \mathrm{PIX}\,\sigma} i - \sum_{i=1}^{\mathrm{pix}\,\sigma} i + \mathrm{imaj}\circ\mathrm{Desar}\,\sigma\\ &= \sum_{i\in \mathrm{Zero}\,w} i - \sum_{i=1}^{\mathrm{zero}\,w} i + \mathrm{maj}\circ\mathrm{Pos}\,w = \mathrm{mafz}\,w. \end{aligned}$$

As a summary,

$$\begin{aligned} &\mathrm{PIX}\,\sigma = \mathrm{Zero}\,\mathrm{ZDesar}(\sigma); \quad \mathrm{Desar}\,\sigma = \mathrm{Pos}\,\mathrm{ZDesar}(\sigma);\\ &\mathrm{mag}\,\sigma = \mathrm{mafz}\,\mathrm{ZDesar}(\sigma); \quad \mathrm{IDES}\,\sigma = \mathrm{DES}\,\mathrm{ZDesar}(\sigma). \end{aligned} \tag{3.6}$$

Example 3.3. Let $\sigma = 1\,3\,6\,5\,4\,7\,2$. Then $\sigma^{-1} = 1\,7\,2\,5\,4\,3\,6$; $w := \mathrm{ZDesar}(\sigma) = 0\,4\,0\,2\,1\,0\,3$, $\mathrm{PIX}\,\sigma = \mathrm{Zero}\,w = \{1,3,6\}$; $\mathrm{Desar}\,\sigma = 3\,2\,4\,1$, $(\mathrm{Desar}\,\sigma)^{-1} = \mathrm{Pos}\,w = 4\,2\,1\,3$; $\mathrm{mag}\,\sigma = \mathrm{mafz}\,w = (1+3+6)-(1+2+3)+(1+2) = 7$, $\mathrm{IDES}\,\sigma = \mathrm{DES}\,w = \{2,4,5\}$.

Next define the bijection F_3' of $\mathfrak{S}_n$ onto itself by the chain

$$\mathrm{F}_3' : \sigma \overset{\mathrm{ZDesar}}{\longmapsto} w \overset{\mathbf{F}_3}{\longmapsto} w' \overset{\mathrm{ZDesar}^{-1}}{\longmapsto} \sigma'. \tag{3.7}$$

Then, by (3.2),

$$\begin{aligned}(\mathrm{pix}, \mathrm{imaj}, \mathrm{Desar})\,\sigma &= (\mathrm{zero}, \mathrm{maj}, \mathrm{Pos})\,w\\ &= (\mathrm{zero}, \mathrm{mafz}, \mathrm{Pos})\,w'\\ &= (\mathrm{pix}, \mathrm{mag}, \mathrm{Desar})\,\sigma',\end{aligned}$$

$$(\mathrm{pix}, \mathrm{imaj}, \mathrm{Desar})\,\sigma = (\mathrm{pix}, \mathrm{mag}, \mathrm{Desar})\,\mathrm{F}_3'(\sigma). \tag{3.8}$$

With (3.5) and (3.8) the first part of Theorem 1.3 is proved.

The second part of Theorem 1.3 is proved as follows. Remember that, if $\mathrm{zero}\,w = n-m$, the pair $(\mathrm{Zero}\,w, \mathrm{Pos}\,w)$ uniquely determines the shuffle of $(n-m)$ letters equal to 0 into the letters of $\mathrm{Pos}\,w$. The bijection "dw" defined by $\mathrm{dw} := \mathrm{ZDesar} \circ \mathrm{DW}^{\mathrm{loc}} \circ \mathrm{ZDer}^{-1}$ can also be derived by the chain

$$\begin{aligned}\mathrm{dw} : w &\mapsto (\mathrm{Zero}\,w, \mathrm{Pos}\,w)\\ &\mapsto (\mathrm{Zero}\,w, (\mathrm{DW} \circ \mathrm{Pos}\,w)^{-1}) = (\mathrm{Zero}\,w', \mathrm{Pos}\,w') \mapsto w',\end{aligned} \tag{3.9}$$

where DW denotes the Désarménien-Wachs bijection. It maps $\mathfrak{S}_n^{\mathrm{Der}}$ onto $\mathfrak{S}_n^{\mathrm{Desar}}$. In particular,

$$\mathrm{Pos} \circ \mathrm{dw}\,w = (\mathrm{DW} \circ \mathrm{Pos}\,w)^{-1}; \quad \mathrm{Zero} \circ \mathrm{dw}(w) = \mathrm{Zero}\,w.$$

Because of (2.4) we also have $\mathrm{DES} \circ \mathrm{Pos}\,w = \mathrm{DES} \circ \mathrm{Pos}\,w'$. As shown in our previous paper ([FoHa07], Proposition 4.1), the latter property implies that

$$\mathrm{Zero} \circ \mathbf{F}_3(w) = \mathrm{Zero} \circ \mathbf{F}_3(w').$$

Furthermore,

$$\mathrm{Pos} \circ \mathbf{F}_3(w) = \mathrm{Pos}\,w, \quad \mathrm{Pos} \circ \mathbf{F}_3(w') = \mathrm{Pos}\,w',$$

since $\mathbf{F}_3$ maps each shuffle class onto itself. Hence,

$$\begin{aligned}\mathrm{Zero} \circ \mathrm{dw} \circ \mathbf{F}_3(w) &= \mathrm{Zero} \circ \mathbf{F}_3(w);\\ \mathrm{Pos} \circ \mathrm{dw} \circ \mathbf{F}_3(w) &= (\mathrm{DW} \circ \mathrm{Pos}\,\mathbf{F}_3(w))^{-1} = (\mathrm{DW} \circ \mathrm{Pos}\,w)^{-1};\\ \mathrm{Zero} \circ \mathbf{F}_3 \circ \mathrm{dw}(w) &= \mathrm{Zero} \circ \mathbf{F}_3(w);\\ \mathrm{Pos} \circ \mathbf{F}_3 \circ \mathrm{dw}(w) &= \mathrm{Pos} \circ \mathrm{dw}(w) = (\mathrm{DW} \circ \mathrm{Pos}\,w)^{-1}.\end{aligned}$$

The word $\mathrm{dw} \circ \mathbf{F}_3(w)$ is characterized by the pair

$$(\mathrm{Zero} \circ \mathrm{dw} \circ \mathbf{F}_3(w), \mathrm{Pos} \circ \mathrm{dw} \circ \mathbf{F}_3(w)),$$

which is equal to the pair

$$(\mathrm{Zero} \circ \mathbf{F}_3 \circ \mathrm{dw}(w), \mathrm{Pos} \circ \mathbf{F}_3 \circ \mathrm{dw}(w)),$$

which corresponds itself to the word $\mathbf{F}_3(w) \circ \mathrm{dw}(w)$. Hence,

$$\mathrm{dw} \circ \mathbf{F}_3 = \mathbf{F}_3 \circ \mathrm{dw}\,. \tag{3.10}$$

This shows that the top square in Fig. 2 is a commutative diagram, so is the bottom one. ∎

Example 3.4. Noting that $\mathrm{DW}(4\,3\,1\,2) = 3\,2\,4\,1$ we have the commutative diagram

$$\begin{array}{ccc}
7\,4\,3\,1\,5\,6\,2 & \xrightarrow{\mathrm{DW}^{\mathrm{loc}}} & 3\,5\,6\,4\,2\,7\,1 \\
{\scriptstyle \mathrm{F}_3}\Big\uparrow & & \Big\uparrow{\scriptstyle \mathrm{F}_3'} \\
1\,7\,3\,5\,2\,6\,4 & \xrightarrow{\mathrm{DW}^{\mathrm{loc}}} & 1\,3\,6\,5\,4\,7\,2
\end{array}$$

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FIX-MAHONIAN CALCULUS III; A QUADRUPLE DISTRIBUTION

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Abstract

A four-variable distribution on permutations is derived, with two dual combinatorial interpretations. The first one includes the number of fixed points "fix", the second the so-called "pix" statistic. This shows that the duality between derangements and desarrangements can be extended to the case of multivariable statistics. Several specializations are obtained, including the joint distribution of (des, exc), where "des" and "exc" stand for the number of descents and excedances, respectively.

1. Introduction

Let

$$(a;q)_n := \begin{cases} 1, & \text{if } n=0;\\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & \text{if } n\ge 1;\end{cases}$$
$$(a;q)_\infty := \prod_{n\ge 0}(1-aq^n),$$

be the traditional notation for the q-ascending factorial. For each $r\ge 0$ form the rational fraction

$$C(r;u,s,q,Y) := \frac{(1-sq)\,(u;q)_r\,(usq;q)_r}{((u;q)_r - sq(usq;q)_r)(uY;q)_{r+1}} \tag{1.1}$$

in four variables u, s, q, Y and expand it as a formal power series in u:

$$C(r;u,s,q,Y) = \sum_{n\ge 0} u^n C_n(r;s,q,Y). \tag{1.2}$$

It can be verified that each coefficient $C_n(r;s,q,Y)$ is actually a polynomial in three variables with nonnegative integral coefficients. For $r,n\ge 0$ consider the set $W_n(r)=[0,r]^n$ of all finite words of length n, whose letters

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are taken from the alphabet $[0,r]=\{0,1,\ldots,r\}$. The first purpose of this paper is to show that $C_n(r;s,q,Y)$ is the generating polynomial for $W_n(r)$ by two three-variable statistics $(\mathrm{dec},\mathrm{tot},\mathrm{single})$ and $(\mathrm{wlec},\mathrm{tot},\mathrm{wpix})$, respectively, defined by means of two classical *word factorizations*, the *Lyndon factorization* and the *H-factorization.* See Theorems 2.1 and 2.3 thereafter and their corollaries.

The second purpose of this paper is to consider the formal power series

$$(1.3)\quad \sum_{r\ge 0} t^r \frac{(1-sq)\,(u;q)_r\,(usq;q)_r}{((u;q)_r - sq(usq;q)_r)(uY;q)_{r+1}} = \sum_{r\ge 0} t^r C(r;u,s,q,Y) = \sum_{r\ge 0} t^r \sum_{n\ge 0} u^n C_n(r;s,q,Y),$$

expand it as a formal power series in u, but normalized by denominators of the form $(t;q)_{n+1}$, that is,

$$(1.4)\quad \sum_{r\ge 0} t^r \frac{(1-sq)\,(u;q)_r\,(usq;q)_r}{((u;q)_r - sq(usq;q)_r)(uY;q)_{r+1}} = \sum_{n\ge 0} A_n(s,t,q,Y)\frac{u^n}{(t;q)_{n+1}},$$

and show that each $A_n(s,t,q,Y)$ is actually the *generating polynomial* for the symmetric group $\mathfrak{S}_n$ by two four-variable statistics $(\mathrm{exc},\mathrm{des},\mathrm{maj},\mathrm{fix})$ and $(\mathrm{lec},\mathrm{ides},\mathrm{imaj},\mathrm{pix})$, respectively. The first (resp. second) statistic involves the number of fixed points "fix" (resp. the variable "pix") and is referred to as the *fix-version* (resp. the *pix-version*). Several specializations of the polynomials $A_n(s,t,q,Y)$ are then derived with their combinatorial interpretations. In particular, the *joint* distribution of the two classical Eulerian statistics "des" and "exc" is explicitly calculated.

The *fix-version* statistic on $\mathfrak{S}_n$, denoted by $(\mathrm{exc},\mathrm{des},\mathrm{maj},\mathrm{fix})$, contains the following classical integral-valued statistics: the *number of excedances* "exc," the *number of descents* "des," the *major index* "maj," the *number of fixed points* "fix," defined for each permutation $\sigma=\sigma(1)\sigma(2)\cdots\sigma(n)$ from $\mathfrak{S}_n$ by

$$\begin{aligned}
\mathrm{exc}\,\sigma &:= \#\{i : 1\le i\le n-1,\ \sigma(i)>i\};\\
\mathrm{des}\,\sigma &:= \#\{i : 1\le i\le n-1,\ \sigma(i)>\sigma(i+1)\};\\
\mathrm{maj}\,\sigma &:= \sum_i i \quad (1\le i\le n-1,\ \sigma(i)>\sigma(i+1));\\
\mathrm{fix}\,\sigma &:= \#\{i : 1\le i\le n,\ \sigma(i)=i\}.
\end{aligned}$$

As was introduced by Désarménien [5], a *desarrangement* is defined to be a word $w=x_1x_2\cdots x_n$, whose letters are *distinct* positive integers such that the inequalities $x_1>x_2>\cdots>x_{2j}$ and $x_{2j}<x_{2j+1}$ hold

for some j with $1 \le j \le n/2$ (by convention: $x_{n+1} = +\infty$). There is no desarrangement of length 1. Each desarrangement $w = x_1x_2\cdots x_n$ is called a *hook*, if $x_1 > x_2$ and either $n = 2$, or $n \ge 3$ and $x_2 < x_3 < \cdots < x_n$. As proved by Gessel [12], each permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ admits a unique factorization, called its *hook factorization*, $p\tau_1\tau_2\cdots\tau_k$, where p is an *increasing* word and each factor $\tau_1, \tau_2, \ldots, \tau_k$ is a hook. To derive the hook factorization of a permutation, it suffices to start from the right and at each step determine the right factor which is a hook, or equivalently, the shortest right factor which is a desarrangement.

The *pix-version* statistic is denoted by (lec, ides, imaj, pix). The second and third components are classical: if σ^{-1} denotes the inverse of the permutation σ, they are simply defined by

$$\operatorname{ides}\sigma := \operatorname{des}\sigma^{-1};$$
$$\operatorname{imaj}\sigma := \operatorname{maj}\sigma^{-1}.$$

The first and fourth components refer to the hook factorization $p\tau_1\tau_2\cdots\tau_k$ of σ. For each i let $\operatorname{inv}\tau_i$ denote the *number of inversions* of τ_i. Then, we define:

$$\operatorname{lec}\sigma := \sum_{1\le i\le k} \operatorname{inv}\tau_i;$$
$$\operatorname{pix}\sigma := \text{length of the factor } p.$$

For instance, the hook factorization of the following permutation of order 14 is indicated by vertical bars.

$$\sigma = 1\ 3\ 4\ 14 \mid 12\ 2\ 5\ 11\ 15 \mid 8\ 6\ 7 \mid 13\ 9\ 10$$

We have $p = 1\ 3\ 4\ 14$, so that $\operatorname{pix}\sigma = 4$. Also $\operatorname{inv}(12\ 2\ 5\ 11\ 15) = 3$, $\operatorname{inv}(8\ 6\ 7) = 2$, $\operatorname{inv}(13\ 9\ 10) = 2$, so that $\operatorname{lec}\sigma = 7$. Our main two theorems are the following.

Theorem 1.1 (The fix-version). *Let $A_n(s,t,q,Y)$ $(n \ge 0)$ be the sequence of polynomials in four variables, whose factorial generating function is given by* (1.4). *Then, the generating polynomial for $\mathfrak{S}_n$ by the four-variable statistic* (exc, des, maj, fix) *is equal to $A_n(s,t,q,Y)$. In other words,*

$$\sum_{\sigma\in\mathfrak{S}_n} s^{\operatorname{exc}\sigma} t^{\operatorname{des}\sigma} q^{\operatorname{maj}\sigma} Y^{\operatorname{fix}\sigma} = A_n(s,t,q,Y). \tag{1.5}$$

Theorem 1.2 (The pix-version). *Let $A_n(s,t,q,Y)$ $(n \ge 0)$ be the sequence of polynomials in four variables, whose factorial generating function is given by* (1.4). *Then, the generating polynomial for $\mathfrak{S}_n$ by* *the four-variable statistic* (lec, ides, imaj, pix) *is equal to* $A_n(s,t,q,Y)$. *In other words,*

$$\sum_{\sigma\in\mathfrak{S}_n} s^{\operatorname{lec}\sigma}t^{\operatorname{ides}\sigma}q^{\operatorname{imaj}\sigma}Y^{\operatorname{pix}\sigma}=A_n(s,t,q,Y). \tag{1.6}$$

The *ligne of route*, $\operatorname{Ligne}\sigma$, of a permutation $\sigma=\sigma(1)\sigma(2)\cdots\sigma(n)$ (also called *descent set*) is defined to be the *set* of all i such that $1\le i\le n-1$ and $\sigma(i)>\sigma(i+1)$. In particular, $\operatorname{des}\sigma=\#\operatorname{Ligne}\sigma$ and $\operatorname{maj}\sigma$ is the sum of all i such that $i\in\operatorname{Ligne}\sigma$. Also, let the *inverse ligne of route* of σ be defined by $\operatorname{Iligne}\sigma:=\operatorname{Ligne}\sigma^{-1}$, so that $\operatorname{ides}\sigma=\#\operatorname{Iligne}\sigma$ and $\operatorname{imaj}\sigma=\sum_i i\ (i\in\operatorname{Iligne}\sigma)$. Finally, let $\operatorname{iexc}\sigma:=\operatorname{exc}\sigma^{-1}$.

It follows from Theorem 1.1 and Theorem 1.2 that the two four-variable statistics (iexc, ides, imaj, fix) and (lec, ides, imaj, pix) are equidistributed on each symmetric group $\mathfrak{S}_n$. The third goal of this paper is to prove the following stronger result.

Theorem 1.3. *The two three-variable statistics*

(iexc, fix, Iligne) *and* (lec, pix, Iligne)

are equidistributed on each symmetric group $\mathfrak{S}_n$.

Note that the third component in each of the previous triples is a *set-valued* statistic. So far, it was known that the two pairs (fix, Iligne) and (pix, Iligne) were equidistributed, a result derived by Désarménien and Wachs [6, 7], so that Theorem 1.3 may be regarded as an extension of their result. In the following table we reproduce the nine derangements (resp. desarrangements) σ from $\mathfrak{S}_4$, which are such that $\operatorname{fix}\sigma=0$ (resp. $\operatorname{pix}\sigma=0$), together with the values of the pairs $(\operatorname{iexc}\sigma,\operatorname{Iligne}\sigma)$ (resp. $(\operatorname{lec}\sigma,\operatorname{Iligne}\sigma)$).

<table>
<tr><th>lec</th><th>Iligne</th><th>Desarrangements</th><th>Derangements</th><th>Iligne</th><th>iexc</th></tr>
<tr><td>1</td><td>1</td><td>2134</td><td>2341</td><td>1</td><td>1</td></tr>
<tr><td rowspan="6">2</td><td>1, 2</td><td>3241</td><td>3421</td><td>1, 2</td><td rowspan="6">2</td></tr>
<tr><td>1, 3</td><td>4231</td><td>2413</td><td>1, 3</td></tr>
<tr><td>2</td><td>3124
3142</td><td>3142
3412</td><td>2</td></tr>
<tr><td>1, 3</td><td>2143</td><td>2143</td><td>1, 3</td></tr>
<tr><td>2, 3</td><td>4132</td><td>4312</td><td>2, 3</td></tr>
<tr><td>1, 2, 3</td><td>4321</td><td>4321</td><td>1, 2, 3</td></tr>
<tr><td>3</td><td>3</td><td>4123</td><td>4123</td><td>3</td><td>3</td></tr>
</table>

Theorem 1.3 has been recently used by Han and Xin [14] to set up a relation between the generating polynomial for derangements by number of excedances and the corresponding polynomial for permutations with one fixed point.

In our previous papers [9, 10] we have introduced three statistics "dez," "maz" and "maf" on $\mathfrak{S}_n$. If σ is a permutation, let $i_1, i_2, \ldots, i_h$ be the increasing sequence of its fixed points. Let $D\sigma$ (resp. $Z\sigma$) be the word derived from $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ by *deleting* all the fixed points (resp. by *replacing* all those fixed points by 0). Then those three statistics are simply defined by: $\operatorname{dez}\sigma := \operatorname{des} Z\sigma$, $\operatorname{maz}\sigma := \operatorname{maj} Z\sigma$ and $\operatorname{maf}\sigma := (i_1 - 1) + (i_2 - 2) + \cdots + (i_j - h) + \operatorname{maj} D\sigma$. For instance, with $\sigma = 8\,2\,1\,3\,5\,6\,4\,9\,7$ we have $(i_1, \ldots, i_h) = (2, 5, 6)$, $Z\sigma = 8\,0\,1\,3\,0\,0\,4\,9\,7$, $D\sigma = 8\,1\,3\,4\,9\,7$ and $\operatorname{dez}\sigma = 3$, $\operatorname{maz}\sigma = 1 + 4 + 8 = 13$, $\operatorname{maf}\sigma = (2 - 1) + (5 - 2) + (6 - 3) + \operatorname{maj}(8\,1\,3\,4\,9\,7) = 13$. Theorem 1.4 in [9] and Theorem 1.1 above provide another combinatorial interpretation for $A_n(s, t, q, Y)$, namely

$$\sum_{\sigma\in\mathfrak{S}_n} s^{\operatorname{exc}\sigma} t^{\operatorname{dez}\sigma} q^{\operatorname{maz}\sigma} Y^{\operatorname{fix}\sigma} = A_n(s, t, q, Y). \tag{1.7}$$

In the sequel we need the notations for the q-multinomial coefficients

$$\begin{bmatrix} n \\ m_1, \ldots, m_k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_{m_1} \cdots (q;q)_{m_k}} \qquad (m_1 + \cdots + m_k = n);$$

and the first q-exponential

$$e_q(u) = \sum_{n\ge 0} \frac{u^n}{(q;q)_n} = \frac{1}{(u;q)_\infty}.$$

Multiply both sides of (1.4) by $1 - t$ and let $t = 1$. We obtain the factorial generating function for a sequence of polynomials $(A_n(s, 1, q, Y))$ $(n \ge 0)$ in three variables:

$$\sum_{n\ge 0} A_n(s, 1, q, Y) \frac{u^n}{(q;q)_n} = \frac{(1 - sq)e_q(Yu)}{e_q(squ) - sqe_q(u)}. \tag{1.8}$$

It follows from Theorem 1.1 that

$$\sum_{\sigma\in\mathfrak{S}_n} s^{\operatorname{exc}\sigma} q^{\operatorname{maj}\sigma} Y^{\operatorname{fix}\sigma} = A_n(s, 1, q, Y) \tag{1.9}$$

holds for every $n \ge 0$, a result stated and proved by Shareshian and Wachs [19] by means of a symmetric function argument, so that identity (1.8) with the interpretation (1.9) belongs to those two authors. Identity (1.4) can be regarded as a graded form of (1.8). The interest of the graded form also lies in the fact that it provides the joint distribution of $(\operatorname{exc}, \operatorname{des})$, as shown in (1.15) below.

Of course, Theorem 1.2 yields a second combinatorial interpretation for the polynomials $A_n(s,1,q,Y)$ in the form

$$\sum_{\sigma\in\mathfrak{S}_n} s^{\operatorname{lec}\sigma} q^{\operatorname{imaj}\sigma} Y^{\operatorname{pix}\sigma} = A_n(s,1,q,Y). \tag{1.10}$$

However we have a third combinatorial interpretation, where the statistic "imaj" is replaced by the number of inversions "inv." We state it as our fourth main theorem.

Theorem 1.4. *Let $A_n(s,1,q,Y)$ $(n\ge 0)$ be the sequence of polynomials in three variables, whose factorial generating function is given by* (1.8). *Then, the generating polynomial for $\mathfrak{S}_n$ by the three-variable statistic* $(\operatorname{lec},\operatorname{inv},\operatorname{pix})$ *is equal to $A_n(s,1,q,Y)$. In other words,*

$$\sum_{\sigma\in\mathfrak{S}_n} s^{\operatorname{lec}\sigma} q^{\operatorname{inv}\sigma} Y^{\operatorname{pix}\sigma} = A_n(s,1,q,Y). \tag{1.11}$$

Again, Theorem 1.4 in [9] and Theorem 1.1 provide a fourth combinatorial interpretation of $A_n(s,1,q,Y)$, namely

$$\sum_{\sigma\in\mathfrak{S}_n} s^{\operatorname{exc}\sigma} q^{\operatorname{maf}\sigma} Y^{\operatorname{fix}\sigma} = A_n(s,1,q,Y). \tag{1.12}$$

Note that the statistic "maf" was introduced and studied in [4].

Let $s=1$ in identity (1.4). We get:

$$\sum_{n\ge 0} A_n(1,t,q,Y)\frac{u^n}{(t;q)_{n+1}} = \sum_{r\ge 0} t^r \Bigl(1-u\sum_{i=0}^{r} q^i\Bigr)^{-1}\frac{(u;q)_{r+1}}{(uY;q)_{r+1}}, \tag{1.13}$$

so that Theorem 1.1 implies

$$\sum_{\sigma\in\mathfrak{S}_n} t^{\operatorname{des}\sigma} q^{\operatorname{maj}\sigma} Y^{\operatorname{fix}\sigma} = A_n(1,t,q,Y), \tag{1.14}$$

an identity derived by Gessel and Reutenauer [13].

Finally, by letting $q=Y:=1$ we get the generating function for polynomials in two variables $A_n(s,t,1,1)$ $(n\ge 0)$ in the form

$$\sum_{n\ge 0} A_n(s,t,1,1)\frac{u^n}{(1-t)^{n+1}} = \sum_{r\ge 0} t^r \frac{1-s}{(1-u)^{r+1}(1-us)^{-r}-s(1-u)}. \tag{1.15}$$

It then follows from Theorems 1.1 and 1.2 that

$$\sum_{\sigma\in\mathfrak{S}_n} s^{\operatorname{exc}\sigma} t^{\operatorname{des}\sigma} = \sum_{\sigma\in\mathfrak{S}_n} s^{\operatorname{lec}\sigma} t^{\operatorname{ides}\sigma} = A_n(s,t,1,1). \tag{1.16}$$

As is well-known (see, *e.g.*, [11]) "exc" and "des" are equidistributed over $\mathfrak{S}_n$, their common generating polynomial being the Eulerian polynomial $A_n(t) := A_n(t,1,1,1) = A_n(1,t,1,1)$, which satifies the identity

$$\frac{A_n(t)}{(1-t)^{n+1}} = \sum_{r\geq 0} t^r (r+1)^n, \tag{1.17}$$

easily deduced from (1.15).

The polynomials $A_n(s,t,1,1)$ do not have any particular symmetries. This is perhaps the reason why their generating function has never been calculated before, to the best of the authors' knowledge. However, with $q=1$ and $Y=0$ we obtain

$$\sum_{n\geq 0} A_n(s,t,1,0) \frac{u^n}{(1-t)^{n+1}} = \sum_{r\geq 0} t^r \frac{1-s}{(1-us)^{-r} - s(1-u)^{-r}}. \tag{1.18}$$

The right-hand side is invariant under the change of variables $u \leftarrow us$, $s \leftarrow s^{-1}$, so that the polynomials $A_n(s,t,1,0)$, which are the generating polynomials for the set of all *derangements* by the pair $(\mathrm{exc}, \mathrm{des})$, satisfy $A_n(s,t,1,0) = s^n\, A_n(s^{-1},t,1,0)$. This means that $(\mathrm{exc}, \mathrm{des})$ and $(\mathrm{iexc}, \mathrm{des})$ are equidistributed on the set of all derangements. There is a stronger combinatorial result that can be derived as follows. Let **c** be the *complement of* $(n+1)$ and **r** the *reverse image*, which map each permutation $\sigma = \sigma(1)\ldots\sigma(n)$ onto $\mathbf{c}\,\sigma := (n+1-\sigma(1))(n+1-\sigma(2))\ldots(n+1-\sigma(n))$ and $\mathbf{r}\,\sigma := \sigma(n)\,\ldots\,\sigma(2)\sigma(1)$, respectively. Then

$$(\mathrm{exc}, \mathrm{fix}, \mathrm{des}, \mathrm{ides})\,\sigma = (\mathrm{iexc}, \mathrm{fix}, \mathrm{des}, \mathrm{ides})\,\mathbf{c}\,\mathbf{r}\,\sigma. \tag{1.19}$$

The paper is organized as follows. In order to prove that $C(r;u,s,q,Y)$ is the generating polynomial for $W_n(r)$ by two multivariable statistics, we show in the next section that it suffices to construct two explicit bijections ϕ^{fix} and ϕ^{pix}. The first bijection ϕ^{fix}, defined in Section 3, relates to the algebra of *Lyndon words*, first introduced by Chen, Fox and Lyndon [3], popularized in Combinatorics by Schützenberger [18] and now set in common usage in Lothaire [17]. It is based on the techniques introduced by Kim and Zeng [15]. In particular, we show that the V-cycle decomposition introduced by those two authors, which is attached to each permutation, can be extended to the case of words. This is the content of Theorem 3.4, which may be regarded as our fifth main result.

The second bijection ϕ^{pix}, constructed in Section 4, relates to the less classical H-factorization, the analog for words of the hook factorization introduced by Gessel [12].

In Section 5 we complete the proofs of Theorems 1.1 and 1.2. By combining the two bijections ϕ^{fix} and ϕ^{pix} we obtain a transformation on words serving to prove that two bivariable statistics are equidistributed on the same rearrangement class. This is done in Section 6, as well as the proof of Theorem 1.3. Finally, Theorem 1.4 is proved in Section 7 by means of a new property of the second fundamental transformation.

2. Two multivariable generating functions for words

As $1/(u;q)_r = \sum\limits_{n\ge 0} \begin{bmatrix} r+n-1 \\ n \end{bmatrix}_q u^n$ (see, *e.g.*, [2, chap. 3]), we may rewrite the fraction $C(r;u,s,q,Y) = \dfrac{(1-sq)\,(u;q)_r\,(usq;q)_r}{((u;q)_r - sq(usq;q)_r)(uY;q)_{r+1}}$ as

$$\text{(2.1)}\quad C(r;u,s,q,Y) = \Bigl(1-\sum_{n\ge 2}\begin{bmatrix} r+n-1 \\ n \end{bmatrix}_q u^n((sq)+(sq)^2+\cdots+(sq)^{n-1})\Bigr)^{-1}\frac{1}{(uY;q)_{r+1}}.$$

If $c = c_1c_2\cdots c_n$ is a word, whose letters are nonnegative integers, let $\lambda\, c := n$ be the *length* of c and $\operatorname{tot} c := c_1 + c_2 + \cdots + c_n$ the *sum* of its letters. Furthermore, NIW_n (resp. $\mathrm{NIW}_n(r)$) designates the set of all *monotonic nonincreasing* words $c = c_1c_2\cdots c_n$ of length n, whose letters are nonnegative integers (resp. nonnegative integers at most equal to r): $c_1 \ge c_2 \ge \cdots \ge c_1 \ge 0$ (resp. $r \ge c_1 \ge c_2 \ge \cdots \ge c_1 \ge 0$). Also let $\mathrm{NIW}(r)$ be the union of all $\mathrm{NIW}_n(r)$ for $n \ge 0$. It is q-routine (see, *e.g.*, [2, chap. 3]) to prove

$$\begin{bmatrix} r+n-1 \\ n \end{bmatrix}_q = \sum_{w\in \mathrm{NIW}_n(r-1)} q^{\operatorname{tot} w}.$$

The sum $\sum\limits_{n\ge 2}\begin{bmatrix} r+n-1 \\ n \end{bmatrix}_q u^n((sq)+(sq)^2+\cdots+(sq)^{n-1})$ can then be rewritten as $\sum\limits_{(w,i)} s^i q^{i+\operatorname{tot} w} u^{\lambda w}$, where the sum is over all pairs (w,i) such that $w \in \mathrm{NIW}(r-1)$, $\lambda w \ge 2$ and i is an integer satisfying $1 \le i \le \lambda w - 1$. Let $D(r)$ (resp. $D_n(r)$) denote the set of all those pairs (w,i) (resp. those pairs such that $\lambda w = n$). Therefore, equation (2.1) can also be expressed as

$$C(r;u,s,q,Y) = (1-\sum_{(w,i)\in D(r)} s^i q^{i+\operatorname{tot} w} u^{\lambda w})^{-1} \sum_{n\ge 0} u^n \sum_{w\in \mathrm{NIW}_n(r)} q^{\operatorname{tot} w} Y^{\lambda w},$$

and the coefficient $C_n(r;s,q,Y)$ of u^n defined in (1.3) as

$$\text{(2.2)}\ C_n(r;s,q,Y) = \sum s^{i_1+\cdots+i_m}\, q^{i_1+\cdots+i_m+\operatorname{tot} w_0+\operatorname{tot} w_1+\cdots+\operatorname{tot} w_m}\, Y^{\lambda w_0},$$

the sum being over all sequences $(w_0,(w_1,i_1),\ldots,(w_m,i_m))$ such that $w_0 \in \mathrm{NIW}(r)$, each of the pairs (w_1,i_1), $\ldots$, (w_m,i_m) belongs to $D(r)$, and $\lambda w_0 + \lambda w_1 + \cdots + \lambda w_m = n$. Denote the set of those sequences by $D_n^*(r)$. The next step is to construct the two bijections $\phi^{\rm fix}$ and $\phi^{\rm pix}$ of $D_n^*(r)$ onto $W_n(r)$ enabling us to calculate certain multivariable statistical distributions *on words*.

Let $l = x_1 x_2 \cdots x_n$ be a nonempty word, whose letters are nonnegative integers. Then l is said to be a *Lyndon word*, if either $n = 1$, or if $n \geq 2$ and, with respect to the lexicographic order, the inequality $x_1 x_2 \cdots x_n > x_i x_{i+1} \cdots x_n x_1 \cdots x_{i-1}$ holds for every i such that $2 \leq i \leq n$. When $n \geq 2$, we always have $x_1 \geq x_i$ for all $i = 2, \ldots, n$ and $x_i > x_{i+1}$ for at least one integer i $(1 \leq i \leq n-1)$, so that it makes sense to define the *rightmost minimal letter* of l, denoted by $\mathrm{rmin}\, l$, as the unique letter x_{i+1} satisfying the inequalities $x_i > x_{i+1}$, $x_{i+1} \leq x_{i+2} \leq \cdots \leq x_n$.

Let w, w' be two nonempty primitive words (none of them can be expressed as v^b, where v is a word and b an integer greater than or equal to 2). We write $w \preceq w'$ if and only if $w^b \leq w'^b$, with respect to the lexicographic order, when b is large enough. As shown for instance in [17, Theorem 5.1.5] each nonnempty word w, whose letters are nonnegative integers, can be written uniquely as a product $l_1 l_2 \cdots l_k$, where each l_i is a Lyndon word and $l_1 \preceq l_2 \preceq \cdots \preceq l_k$. Classically, each Lyndon word is defined to be the minimum within its class of cyclic rearrangements, so that the sequence $l_1 \preceq l_2 \preceq \cdots$ is replaced by $l_1 \geq l_2 \geq \cdots$ The modification made here is for convenience.

For instance, the factorization of the following word as a nondecreasing product of Lyndon words with respect to "$\preceq$" [in short, *Lyndon word factorization*] is indicated by vertical bars:

$$w = |\, 2 \,|\, 3\,2\,1\,1 \,|\, 3 \,|\, 5 \,|\, 6\,4\,2\,1\,3\,2\,3 \,|\, 6\,6\,3\,1\,6\,6\,2 \,|\, 6 \,|\, .$$

Now let $w = x_1 x_2 \cdots x_n$ be an *arbitrary* word. We say that a positive integer i is a *decrease* of w if $1 \leq i \leq n-1$ and $x_i \geq x_{i+1} \geq \cdots \geq x_j > x_{j+1}$ for some j such that $i \leq j \leq n-1$. In particular, i is a decrease if $x_i > x_{i+1}$. The letter x_i is said to be a *decrease value* of w. If $1 \leq i_1 < i_2 < \cdots < i_m \leq n-1$ is the increasing sequence of the decreases of w, the subword $x_{i_1} x_{i_2} \cdots x_{i_m}$ is called the *decrease value subword* of w. It will be denoted by $\mathrm{decval}\, w$. The *number of decreases* itself of w is denoted by $\mathrm{dec}(w)$. We have $\mathrm{dec}(w) = 0$ if all letters of w are equal. Also $\mathrm{dec}(w) \geq 1$ if w is a Lyndon word having at least two letters. In the previous example we have $\mathrm{decval}\, w = 3\,2\,6\,4\,2\,3\,6\,6\,3\,6\,6$, of length 11, so that $\mathrm{dec}\, w = 11$.

Let $l_1 l_2 \ldots l_k$ be the Lyndon word factorization of a word w and let $(l_{i_1}, l_{i_2}, \ldots, l_{i_h})$ $(1 \leq i_1 < i_2 < \cdots < i_h \leq k)$ be the sequence of all the *one-letter* factors in its Lyndon word factorization. Form the nonincreasing word $\mathrm{Single}\, w$ defined by $\mathrm{Single}\, w := l_{i_h} \cdots l_{i_2} l_{i_1}$ and let $\mathrm{single}\, w = h$

be the number of letters of $\operatorname{Single} w$. In the previous example we have: $\operatorname{Single} w = 6\,5\,3\,2$ and $\operatorname{single} w = 4$.

Theorem 2.1. *The map $\phi^{\mathrm{fix}} : (w_0, (w_1, i_1), \ldots, (w_m, i_m)) \mapsto w$ of $D_n^*(r)$ onto $W_n(r)$, defined in Section 3, is a bijection having the properties:*

$$\begin{aligned} & i_1 + \cdots + i_m = \operatorname{dec} w; \\ (2.3) \qquad & i_1 + \cdots + i_m + \operatorname{tot} w_0 + \operatorname{tot} w_1 + \cdots + \operatorname{tot} w_m = \operatorname{tot} w; \\ & \lambda w_0 = \operatorname{single} w. \end{aligned}$$

The next Corollary is then a consequence of (2.2) and the above theorem.

Corollary 2.2. *The sum $C_n(r; s, q, Y)$ defined in (2.2) is also equal to*

$$C_n(r; s, q, Y) = \sum_w s^{\operatorname{dec} w} q^{\operatorname{tot} w} Y^{\operatorname{single} w}, \tag{2.4}$$

where the sum is over all words $w \in W_n(r)$.

To define the second bijection $\phi^{\mathrm{pix}} : D_n^*(r) \to W_n(r)$ another class of words is in use. We call them *H-words*. They are defined as follows: let $h = x_1 x_2 \cdots x_n$ be a word of length $\lambda h \geq 2$, whose letters are nonnegative integers. Say that h is a *H-word*, if $x_1 < x_2$, and either $n = 2$, or $n \geq 3$ and $x_2 \geq x_3 \geq \cdots \geq x_n$.

Each nonnempty word w, whose letters are nonnegative integers, can be written uniquely as a product $u h_1 h_2 \cdots h_k$, where u is a monotonic *nonincreasing* word (possibly empty) and each h_i a H-word. This factorization is called the *H-factorization* of w. Unless w is monotonic nonincreasing, it ends with a H-word, so that its H-factorization is obtained by removing that H-word and determining the next rightmost H-word. Note the discrepancy between the hook factorization for *permutations* mentioned in the introduction and the present H-factorization used for *words*.

For instance, the H-factorization of the following word is indicated by vertical bars:

$$w = \mid 6\,5\,3\,2 \mid 1\,3\,2\,1 \mid 3\,6\,4 \mid 1\,2 \mid 2\,3 \mid 1\,6\,6\,3 \mid 2\,6\,6 \mid .$$

Three statistics are now defined that relate to the H-factorization $u h_1 h_2 \cdots h_k$ of each *arbitrary* word w. First, let $\operatorname{wpix}(w)$ be the length λu of u. Then, if $\mathbf{r}$ denotes the *reverse image*, which maps each word $x_1 x_2 \ldots x_n$ onto $x_n \ldots x_2 x_1$, define the statistic $\operatorname{wlec}(w)$ by

$$\operatorname{wlec}(w) := \sum_{i=1}^{k} \operatorname{rinv}(h_i),$$

where $\operatorname{rinv}(w) = \operatorname{inv}(\mathbf{r}(w))$. In the previous example, $\operatorname{wpix} w = \lambda(6532) = 4$ and $\operatorname{wlec} w = \operatorname{inv}(1231) + \operatorname{inv}(463) + \operatorname{inv}(21) + \operatorname{inv}(32) + \operatorname{inv}(3661) + \operatorname{inv}(662) = 2 + 2 + 1 + 1 + 3 + 2 = 11$.

Theorem 2.3. *The map* $\phi^{\text{pix}} : (w_0, (w_1, i_1), \ldots, (w_m, i_m)) \mapsto w$ *of* $D_n^*(r)$ *onto* $W_n(r)$*, defined in Section* 4*, is a bijection having the properties:*

$$\begin{aligned} & i_1 + \cdots + i_m = \operatorname{wlec} w; \\ (2.5) \qquad & i_1 + \cdots + i_m + \operatorname{tot} w_0 + \operatorname{tot} w_1 + \cdots + \operatorname{tot} w_m = \operatorname{tot} w; \\ & \lambda w_0 = \operatorname{wpix} w. \end{aligned}$$

Corollary 2.4. *The sum* $C_n(r; s, q, Y)$ *defined in* (2.2) *is also equal to*

$$C_n(r; s, q, Y) = \sum_w s^{\operatorname{wlec} w} q^{\operatorname{tot} w} Y^{\operatorname{wpix} w},$$

where the sum is over all words $w \in W_n(r)$*.*

3. The bijection ϕ^{fix}

The construction of the bijection ϕ^{fix} of $D_n^*(r)$ onto $W_n(r)$ proceeds in four steps and involves three subclasses of Lyndon words: the V-words, U-words and L-words. We can say that V- and U-words are the word analogs of the V- and U-cycles introduced by Kim and Zeng [15] for permutations. The present construction is directly inspired by their work.

Each word $w = x_1 x_2 \cdots x_n$ is said to be a *V-word* (resp. a *U-word*), if it is of length $n \geq 2$ and its letters satisfy the following inequalities

(3.1) $x_1 \geq x_2 \geq \cdots \geq x_i > x_{i+1}$ and $x_{i+1} \leq x_{i+2} \leq \cdots \leq x_n < x_i$,
(resp.
(3.2) $x_1 \geq x_2 \geq \cdots \geq x_i > x_{i+1}$ and $x_{i+1} \leq x_{i+2} \leq \cdots \leq x_n < x_1$)

for some i such that $1 \leq i \leq n-1$. Note that if (3.1) or (3.2) holds, then $\operatorname{dec}(w) = i$. Also $\max w$ (the maximum letter of w) $= x_1 > x_n$. For example, $v = 5\,5\,4\,\underline{1}\,1\,2$ is a V-word and $u = 8\,7\,\underline{5}\,7\,7$ is a U-word, but not a V-word. Their rightmost minimal letters have been underlined.

Now w is said to be a *L-word*, if it is a Lyndon word of length at least equal to 2 and whenever $x_1 = x_i$ for some i such that $2 \leq i \leq n$, then $x_1 = x_2 = \cdots = x_i$. For instance, $6\,6\,3\,1\,6\,6\,2$ is a Lyndon word, but not a L-word, but $6\,6\,3\,1\,2\,1$ is a L-word.

Let $V_n(r)$ (resp. $U_n(r)$, resp. $L_n(r)$, $\text{Lyndon}_n(r)$) be the set of V-words (resp. U-words, resp. L-words, resp. Lyndon words), of length n, whose letters are at most equal to r. Also, let $V(r)$ (resp. $U(r)$, resp. $L(r)$, resp. $\text{Lyndon}(r)$) be the union of the $V_n(r)$'s (resp. the $U_n(r)$'s, resp. the $L_n(r)$'s, resp. the $\text{Lyndon}_n(r)$'s) for $n \geq 2$. Clearly, $V_n(r) \subset U_n(r) \subset L_n(r) \subset$

$\mathrm{Lyndon}_n(r)$. Parallel to $D_n^*(r)$, whose definition was given in (2.2), we introduce three sets $V_n^*(r)$, $U_n^*(r)$, $L_n^*(r)$ of sequences $(w_0, w_1, \ldots, w_k)$ of words from $W(r)$ such that $w_0 \in \mathrm{NIW}(r)$, $\lambda w_0 + \lambda w_1 + \cdots + \lambda w_k = n$ and

(i) for $V_n^*(r)$ the components w_i $(1 \le i \le k)$ belong to $V(r)$;

(ii) for $U_n^*(r)$ the components w_i $(1 \le i \le k)$ belong to $U(r)$ and are such that: $\mathrm{rmin}\, w_1 < \max w_2$, $\mathrm{rmin}\, w_2 < \max w_3$, ... , $\mathrm{rmin}\, w_{k-1} < \max w_k$;

(iii) for $L_n^*(r)$ the components w_i $(1 \le i \le k)$ belong to $L(r)$ and are such that: $\max w_1 \le \max w_2 \le \cdots \le \max w_k$.

The first step consists of mapping the set $D_n(r)$ onto $V_n(r)$. This is made by means of a very simple bijection, defined as follows: let $w = x_1 x_2 \cdots x_n$ be a nonincreasing word and let (w, i) belong to $D_n(r)$, so that $n \ge 2$ and $1 \le i \le n-1$. Let

$$y_1 := x_1 + 1,\ y_2 := x_2 + 1,\ \ldots,\ y_i := x_i + 1,$$
$$y_{i+1} := x_n,\ y_{i+2} := x_{n-1},\ \ldots,\ y_n := x_{i+1};$$
$$v := y_1 y_2 \ldots y_n.$$

The following proposition is evident.

Proposition 3.1. *The mapping* $(w, i) \mapsto v$ *is a bijection of* $D_n(r)$ *onto* $V_n(r)$ *satisfying* $\mathrm{dec}(v) = i$ *and* $\mathrm{tot}\, v = \mathrm{tot}\, w + i$.

For instance, the image of $(w = 4\,4\,3\,2\,1\,1, i = 3)$ is the V-word $v = 5\,5\,4\,1\,1\,2$ under the above bijection and $\mathrm{dec}(v) = 3$.

Let $(w_0, (w_1, i_1), (w_2, i_2), \ldots, (w_k, i_k))$ belong to $D_n^*(r)$ and, using the bijection of Proposition 3.1, let $(w_1, i_1) \mapsto v_1$, $(w_2, i_2) \mapsto v_2$, ... , $(w_k, i_k) \mapsto v_k$. Then

$$(w_0, (w_1, i_1), (w_2, i_2), \ldots, (w_k, i_k)) \mapsto (w_0, v_1, v_2, \ldots, v_k) \tag{3.3}$$

is a bijection of $D_n^*(r)$ onto $V_n^*(r)$ having the property that

$$\begin{aligned} &i_1 + i_2 + \cdots + i_k = \mathrm{dec}(v_1) + \mathrm{dec}(v_2) + \cdots + \mathrm{dec}(v_k); \\ &i_1 + i_2 + \cdots + i_k + \mathrm{tot}\, w_0 + \mathrm{tot}\, w_1 + \mathrm{tot}\, w_2 + \cdots + \mathrm{tot}\, w_k \\ &\qquad = \mathrm{tot}\, w_0 + \mathrm{tot}\, v_1 + \mathrm{tot}\, v_2 + \cdots + \mathrm{tot}\, v_k. \end{aligned} \tag{3.4}$$

For instance, the sequence

$$\big(6\,5\,3\,2,\ (2\,1\,1\,1,\ 2), (5\,3\,3,\ 2), (1\,1,\ 1), (2\,2,\ 1), (5\,5\,2\,1,\ 3), (5\,5\,2,\ 2)\big)$$

from $D_{22}^*(6)$ is mapped under (3.3) onto the sequence

$$(6\,5\,3\,2,\ 3\,2\,1\,1,\ 6\,4\,3,\ 2\,1,\ 3\,2,\ 6\,6\,3\,1,\ 6\,6\,2) \in V_{22}^*(6).$$

Also $\mathrm{dec}(3\,2\,1\,1) + \mathrm{dec}(6\,4\,3) + \mathrm{dec}(2\,1) + \mathrm{dec}(3\,2) + \mathrm{dec}(6\,6\,3\,1) + \mathrm{dec}(6\,6\,2) = 2 + 2 + 1 + 1 + 3 + 2 = 11$.

The second step is to map $V_n^*(r)$ onto $U_n^*(r)$. Let $u = y_1y_2\cdots y_k \in U(r)$ and $v = z_1z_2\cdots z_l \in V(r)$. Suppose that $\operatorname{rmin} u$ is the $(i+1)$-st leftmost letter of u and $\operatorname{rmin} v$ is the $(j+1)$-st letter of v. Also assume that $\operatorname{rmin} u \geq \max v$. Then, the word $[u,v] := y_1\cdots y_i z_1\cdots z_j z_{j+1}\cdots z_l y_{i+1}\cdots y_k$ belongs to $U(r)$. Furthermore, $\operatorname{rmin}[u,v]$ is the $(i+j+1)$-st leftmost letter of $[u,v]$ and its value is z_{j+1}. We also have the inequalities: $y_i > y_{i+1}$ (by definition of $\operatorname{rmin} u$), $y_{i+1} \geq z_1$ (since $\min u \geq \max v$) and $z_j > z_l$ (since v is a V-word). These properties allow us to get back the pair (u,v) from $[u,v]$ by *successively* determining the *critical* letters z_{j+1}, z_j, z_l, y_{i+1}, z_1. The mapping $(u,v) \mapsto [u,v]$ is perfectly reversible.

For example, with $u = 8\,7\,\underline{5}\,7\,7$ and $v = \mathbf{5}\,\underline{\mathbf{2}}\,\mathbf{2}$ we have $[u,v] = 8\,7\,\mathbf{5}\,\underline{\mathbf{2}}\,\mathbf{2}\,5\,7\,7$.

Now let $(w_0, v_1, v_2, \ldots, v_k) \in V_n^*(r)$. If $k = 1$, let $(w_0, u_1) := (w_0, v_1) \in U_n^*(r)$. If $k \geq 2$, let $(1, 2, \ldots, a)$ be the longest sequence of integers such that $\operatorname{rmin} v_1 \geq \max v_2 > \operatorname{rmin} v_2 \geq \max v_2 > \cdots \geq \max v_a > \operatorname{rmin} v_a$ and, either $a = k$, or $a \leq k-1$ and $\operatorname{rmin} v_a < \max v_{a+1}$. Let

$$u_1 := \begin{cases} v_1, & \text{if } a = 1; \\ [\cdots[\,[v_1, v_2], v_3], \,\cdots, v_a], & \text{if } a \geq 2. \end{cases} \tag{3.5}$$

We have $u_1 \in U(r)$ and $(w_0, u_1) \in U_n^*(r)$ if $a = k$. Otherwise, $\operatorname{rmin} u_1 < \max v_{a+1}$. We can then apply the procedure described in (3.5) to the sequence $(v_{a+1}, v_{a+2}, \ldots, v_k)$. When reaching v_k we obtain a sequence $(w_0, u_1, \ldots, u_h) \in U_n^*(r)$. The whole procedure is perfectly reversible. We have then the following proposition.

Proposition 3.2. *The mapping*

$$(w_0, v_1, v_2, \ldots, v_k) \mapsto (w_0, u_1, u_2, \ldots, u_h) \tag{3.6}$$

described in (3.5) *is a bijection of* $V_n^*(r)$ *onto* $U_n^*(r)$ *having the following properties:*
(i) $u_1u_2\cdots u_h$ *is a rearrangement of* $v_1v_2\cdots v_k$, *so that* $\operatorname{tot} u_1 + \operatorname{tot} u_2 + \cdots + \operatorname{tot} u_h = \operatorname{tot} v_1 + \operatorname{tot} v_2 + \cdots + \operatorname{tot} v_k$;
(ii) $\operatorname{dec}(u_1) + \operatorname{dec}(u_2) + \cdots + \operatorname{dec}(u_h) = \operatorname{dec}(v_1) + \operatorname{dec}(v_2) + \cdots + \operatorname{dec}(v_k)$.

For example, the above sequence

$$(6\,5\,3\,2,\ 3\,2\,1\,1,\ 6\,4\,3,\ 2\,1,\ 3\,2,\ 6\,6\,3\,1,\ 6\,6\,2) \in V_{22}^*(6)$$

is mapped onto the sequence

$$(6\,5\,3\,2,\ 3\,2\,1\,1,\ 6\,4\,2\,1\,3,\ 3\,2,\ 6\,6\,3\,1,\ 6\,6\,2) \in U_{22}^*(6).$$

where $[6\,4\,3,\ 2\,1] = 6\,4\,2\,1\,3$. Also $\operatorname{dec}(3\,2\,1\,1) + \operatorname{dec}(6\,4\,2\,1\,3) + \operatorname{dec}(3\,2) + \operatorname{dec}(6\,6\,3\,1) + \operatorname{dec}(6\,6\,2) = 11$.

The third step is to map $U_n^*(r)$ onto $L_n^*(r)$. Let $l = x_1x_2\cdots x_j \in L(r)$ and $u = y_1y_2\cdots y_{j'} \in U(r)$. Suppose that $\operatorname{rmin} l$ is the $(i+1)$-st leftmost letter of l and $\operatorname{rmin} u$ is the $(i'+1)$-st leftmost letter of u. Also assume that $\operatorname{rmin} l < \max u$ and $\max l > \max u$. If $x_j < y_1$, let $\langle l, u\rangle := lu$. If $x_j \ge y_1$, there is a unique integer $a \ge i+1$ such that $x_a < y_1 = \max u \le x_{a+1}$. Then, let

$$\langle l, u\rangle := x_1\cdots x_i x_{i+1}\cdots x_a y_1 \cdots y_{i'} y_{i'+1}\cdots y_{j'} x_{a+1}\cdots x_j.$$

The word $\langle l, u\rangle$ belongs to $L(r)$ and $\operatorname{rmin}\langle l, u\rangle$ is the $(a+i'+1)$-st leftmost letter of $\langle l, u\rangle$, its value being $y_{i'+1}$. Now y_1 is the rightmost letter of $\langle l, u\rangle$ to the left of $\operatorname{rmin}\langle l, u\rangle = y_{i'+1}$ such that the letter preceding it, that is x_a, satisfies $x_a < y_1$ and the letter following it, that is y_2, is such that $y_1 \ge y_2$. On the other hand, $y_{j'}$ is the unique letter in the nondecreasing factor $y_{i'+1}\cdots y_{j'} x_{a+1}\cdots x_j$ that satisfies $y_{j'} < y_1 = \max u \le x_{a+1}$. Hence the mapping $(l, u) \mapsto \langle l, u\rangle$ is completely reversible.

For example, with $l = 8\,2\,5\,\underline{3}\,3\,5\,7\,7$ and $u = \mathbf{7\,6\,\underline{1}\,2}$ we have $\langle l, u\rangle = 8\,2\,5\,3\,3\,5\,\mathbf{7\,6\,\underline{1}\,2}\,7\,7$ (the leftmost minimal letters have been underlined). The letter $\mathbf{7}$ is the rightmost letter in $\langle l, u\rangle$ greater than its predecessor 5 and greater than or equal to its successor $\mathbf{6}$. Also $\mathbf{2}$ is the unique letter in the factor $1\,\mathbf{2}\,7\,7$ that satisfies $\mathbf{2} < \max u = \mathbf{7} \le 7$.

Let $(w_0, u_1, u_2, \ldots, u_h) \in U_n^*(r)$. If $h = 1$, let $(w_0, l_1) := (w_0, u_1) \in L_n^*(r)$. If $h \ge 2$, let $(1, 2, \ldots, a)$ be the longest sequence of integers such that $\max u_1 > \max u_j$ for all $j = 2, \ldots, a$ and, and either $a = h$, or $a \le h-1$ and $\max u_1 \le \max u_{a+1}$. Let

$$l_1 := \begin{cases} u_1, & \text{if } a = 1;\\ \langle \cdots \langle\, \langle u_1, u_2\rangle, u_3\rangle, \cdots, u_a\rangle, & \text{if } a \ge 2. \end{cases} \tag{3.7}$$

We have $l_1 \in L(r)$ and $(w_0, l_1) \in L_n^*(r)$ if $a = h$. Otherwise, $\max l_1 \le \max u_{a+1}$. We then apply the procedure described in (3.7) to the sequence $(u_{a+1}, u_{a+2}, \ldots, u_h)$. When reaching u_h we obtain a sequence $(w_0, l_1, \ldots, l_m) \in L_n^*(r)$. The whole procedure is perfectly reversible. We have then the following proposition.

Proposition 3.3. *The mapping*

$$(w_0, u_1, u_2, \ldots, u_h) \mapsto (w_0, l_1, l_2, \ldots, l_m) \tag{3.8}$$

described in (3.7) *is a bijection of* $U_n^*(r)$ *onto* $L_n^*(r)$ *having the following properties:*

(i) $l_1 l_2 \cdots l_m$ *is a rearrangement of* $u_1u_2\cdots u_h$, *so that* $\operatorname{tot} l_1 + \operatorname{tot} l_2 + \cdots + \operatorname{tot} l_m = \operatorname{tot} u_1 + \operatorname{tot} u_2 + \cdots + \operatorname{tot} u_h$;

(ii) $\operatorname{dec}(l_1) + \operatorname{dec}(l_2) + \cdots + \operatorname{dec}(l_m) = \operatorname{dec}(u_1) + \operatorname{dec}(u_2) + \cdots + \operatorname{dec}(u_h)$.

For example the above sequence

$$(6\,5\,3\,2,\ 3\,2\,1\,1,\ 6\,4\,2\,1\,3,\ 3\,2,\ 6\,6\,3\,1,\ 6\,6\,2)) \in U^*_{22}(6).$$

is mapped onto the sequence

$$(6\,5\,3\,2,\ 3\,2\,1\,1,\ 6\,4\,2\,1\,3\,2\,3,\ 6\,6\,3\,1,\ 6\,6\,2) \in L^*_{22}(6),$$

where $\langle 6\,4\,2\,1\,3,\ 3\,2\rangle = 6\,4\,2\,1\,3\,2\,3$. Also $\operatorname{dec}(3\,2\,1\,1) + \operatorname{dec}(6\,4\,2\,1\,3\,2\,3) + \operatorname{dec}(6\,6\,3\,1) + \operatorname{dec}(6\,6\,2) = 11$.

The fourth step is to map $L^*_n(r)$ onto $W_n(r)$. Let $(w_0, l_1, l_2, \ldots, l_m) \in L^*_n(r)$. If w_0 is nonempty, of length b, denote by $f_1,\ f_2,\ \ldots\ ,\ f_b$ its b letters from left to right, so that $r \geq f_1 \geq f_2 \geq \cdots \geq f_b \geq 0$. If $m = 1$, let $\sigma_1 := l_1$. If $m \geq 2$, let a be the greatest integer such that $l_1 \succ l_2$, $l_1 l_2 \succ l_3,\ \ldots\ ,\ l_1 \cdots l_{a-1} \succ l_a$. If $a \leq h-1$, let $a' > a$ be the greatest integer such that $l_{a+1} \succ l_{a+2},\ l_{a+1} l_{a+2} \succ l_{a+3},\ \ldots\ ,\ l_{a+1} \cdots l_{a'-1} \succ l_{a'}$, etc. Form $\sigma_1 := l_1 l_2 \cdots l_a$, $\sigma_2 := l_{a+1} \cdots l_{a'}$, etc. The sequence $(\sigma_1, \sigma_2, \ldots)$ is a *nonincreasing* sequence of Lyndon words. Let $(\tau_1, \tau_2, \ldots, \tau_p)$ be the *nonincreasing* rearrangement of the sequence $(\sigma_1, \sigma_2, \ldots, f_1, f_2, \ldots, f_b)$ if w_0 is nonempty, and of $(\sigma_1, \sigma_2, \ldots)$ otherwise. Then, $(\tau_1, \tau_2, \ldots, \tau_p)$ is the *Lyndon word factorization* of a unique word $w \in W_n(r)$. The mapping

$$(w_0, l_1, l_2, \ldots, l_m) \mapsto w \tag{3.9}$$

is perfectly reversible. Also the verification of $\operatorname{dec}(w) = \operatorname{dec}(l_1) + \operatorname{dec}(l_2) + \cdots + \operatorname{dec}(l_m)$ is immediate.

For example, the above sequence

$$(6\,5\,3\,2,\ 3\,2\,1\,1,\ 6\,4\,2\,1\,3\,2\,3,\ 6\,6\,3\,1,\ 6\,6\,2) \in L^*_{22}(6)$$

is mapped onto the Lyndon word factorization

$$w = \mid 2 \mid 3\,2\,1\,1 \mid 3 \mid 5 \mid 6\,4\,2\,1\,3\,2\,3 \mid 6\,6\,3\,1\,6\,6\,2 \mid 6 \mid . \tag{3.10}$$

The map ϕ^{fix} is then defined as being the composition product of

$$\begin{aligned}
(w_0, (w_1, i_1), (w_2, i_2), \ldots, (w_k, i_k)) &\mapsto (w_0, v_1, v_2, \ldots, v_k) && \text{in (3.3)}\\
(w_0, v_1, v_2, \ldots, v_k) &\mapsto (w_0, u_1, u_2, \ldots, u_h) && \text{in (3.6)}\\
(w_0, u_1, u_2, \ldots, u_h) &\mapsto (w_0, l_1, l_2, \ldots, l_m) && \text{in (3.8)}\\
(w_0, l_1, l_2, \ldots, l_m) &\mapsto w. && \text{in (3.9)}
\end{aligned}$$

Therefore, ϕ^{fix} is a bijection of $D^*_n(r)$ onto $W_n(r)$ having the properties stated in Theorem 2.1. The latter theorem is then proved.

From the property of the bijection $w \mapsto (w_0, v_1, v_2, \ldots, v_k)$ of $W_n(r)$ onto $V^*_n(r)$ we deduce the following theorem, which may be regarded as a *word analog* of Theorem 2.4 in Kim-Zeng's paper [15].

Theorem 3.4 (V-word decomposition)**.** *To each word $w = x_1x_2\cdots x_n$ whose letters are nonnegative integers there corresponds a unique sequence $(w_0, v_1, v_2, \ldots, v_k)$, where w_0 is a nondecreasing word and $v_1, v_2, \ldots, v_k$ are V-words with the further property that $w_0v_1v_2\cdots v_k$ is a rearrangement of w and* decval w *is the juxtaposition product of the* decval v_i*'s:*

$$\operatorname{decval} w = (\operatorname{decval} v_1)(\operatorname{decval} v_2)\cdots(\operatorname{decval} v_k).$$

In particular,

$$\operatorname{dec} w = \operatorname{dec} v_1 + \operatorname{dec} v_2 + \cdots + \operatorname{dec} v_k.$$

For instance, the decrease values of the word w below and of the V-factors of its V-decomposition are reproduced in boldface:

$$w = 2\,\mathbf{3}\,\mathbf{2}\,1\,1\,3\,5\,\mathbf{6}\,\mathbf{4}\,\mathbf{2}\,1\,\mathbf{3}\,2\,3\,\mathbf{6}\,\mathbf{6}\,\mathbf{3}\,1\,\mathbf{6}\,\mathbf{6}\,2\,6;$$
$$(6\,5\,3\,2,\ \mathbf{3}\,\mathbf{2}\,1\,1,\ \mathbf{6}\,\mathbf{4}\,3,\ \mathbf{2}\,1,\ \mathbf{3}\,2,\ \mathbf{6}\,\mathbf{6}\,\mathbf{3}\,1,\ \mathbf{6}\,\mathbf{6}\,2).$$

4. The bijection ϕ^{pix}

The bijection $\phi^{\mathrm{pix}} : D_n^*(r) \to W_n(r)$ whose properties were stated in Theorem 3.2 is easy to construct. Let $H_n(r)$ be the set of all H-words of length n, whose letters are at most equal to r and $H(r)$ be the union of all $H_n(r)$'s for $n \geq 2$. We first map $D_n(r)$ onto $H_n(r)$ as follows. Let $w = x_1x_2\cdots x_n$ be a nonincreasing word and let (w, i) belong to $D_n(r)$, so that $n \geq 2$ and $1 \leq i \leq n-1$. Define:

$$h := x_{i+1}(x_1+1)(x_2+1)\ldots(x_i+1)x_{i+2}x_{i+3}\ldots x_n.$$

The following proposition is evident.

Proposition 4.1. *The mapping $(w, i) \mapsto h$ is a bijection of $D_n(r)$ onto $H_n(r)$ satisfying* $\operatorname{rinv}(h) = i$ *and* $\operatorname{tot} h = \operatorname{tot} w + i$*.*

For instance, the image of $(w = 4\,4\,3\,2\,2\,1, i = 3)$ is the H-word $h = 2\,5\,5\,4\,2\,1$ under the above bijection and $\operatorname{rinv}(h) = 3$.

Let $(w_0, (w_1, i_1), (w_2, i_2), \ldots, (w_k, i_k))$ belong to $D_n^*(r)$ and, using the bijection of Proposition 4.1, let $(w_1, i_1) \mapsto h_1$, $(w_2, i_2) \mapsto h_2$, … , $(w_k, i_k) \mapsto h_k$. Then $w_0h_1h_2\cdots h_k$ is the H-factorization of a word $w \in W_n(r)$. Accordingly,

$$\phi^{\mathrm{pix}} : (w_0, (w_1, i_1), (w_2, i_2), \ldots, (w_k, i_k)) \mapsto w := w_0h_1h_2\cdots h_k$$

is a bijection of $D_n^*(r)$ onto $W_n(r)$ having the properties listed in (2.5). This completes the proof of Theorem 2.3.

For instance, the sequence

$$\big(6\,5\,3\,2,\ (2\,1\,1\,1,\ 2), (5\,3\,3,\ 2), (1\,1,\ 1), (2\,2,\ 1), (5\,5\,2\,1,\ 3), (5\,5\,2,\ 2)\big)$$

from $D^*_{22}(6)$ is mapped under ϕ^{pix} onto the word

$$6\,5\,3\,2 \mid 1\,3\,2\,1 \mid 3\,6\,4 \mid 1\,2 \mid 2\,3 \mid 1\,6\,6\,3 \mid 2\,6\,6 \in W_{22}(6).$$

Also $(\mathrm{wlec}, \mathrm{tot}, \mathrm{wpix})\, w = (11, 74, 4)$.

5. From words to permutations

We are now in a position to prove Theorems 1.1 and 1.2. Suppose that identity (1.5) holds. As

$$\frac{1}{(t;q)_{n+1}} = \sum_{j\ge 0} t^j \sum_{w\in \mathrm{NIW}_n(j)} q^{\mathrm{tot}\, w},$$

the right-hand side of (1.4) can then be written as

$$\sum_{r\ge 0} t^r \sum_{n\ge 0} B_n^{\mathrm{fix}}(r; s, q, Y) u^n,$$

where

$$B_n^{\mathrm{fix}}(r; s, q, Y) := \sum_{(\sigma,c)} s^{\mathrm{exc}\,\sigma}\, q^{\mathrm{maj}\,\sigma + \mathrm{tot}\, c}\, Y^{\mathrm{fix}\,\sigma}, \tag{5.1}$$

the sum being over all pairs (σ, c) such that $\sigma \in \mathfrak{S}_n$, $\mathrm{des}\,\sigma \le r$ and $c \in \mathrm{NIW}_n(r - \mathrm{des}\,\sigma)$. Denote the set of all those pairs by $\mathfrak{S}_n(r, \mathrm{des})$.

In the same manner, let $\mathfrak{S}_n(r, \mathrm{ides})$ denote the set of all pairs (σ, c) such that $\sigma \in \mathfrak{S}_n$, $\mathrm{ides}\,\sigma \le r$ and $c \in \mathrm{NIW}_n(r - \mathrm{ides}\,\sigma)$ and let

$$B_n^{\mathrm{pix}}(r; s, q, Y) := \sum_{(\sigma,c)} s^{\mathrm{lec}\,\sigma}\, q^{\mathrm{imaj}\,\sigma + \mathrm{tot}\, c}\, Y^{\mathrm{pix}\,\sigma}, \tag{5.2}$$

where the sum is over all $(\sigma, c) \in \mathfrak{S}_n(r, \mathrm{ides})$. If (1.6) holds, the right-hand side of (1.4) is equal to $\sum\limits_{r\ge 0} t^r \sum\limits_{n\ge 0} B_n^{\mathrm{pix}}(r; s, q, Y) u^n$.

Accordingly, for proving identity (1.5) (resp. (1.6)) it suffices to show that $C_n(r; s, q, Y) = B_n^{\mathrm{fix}}(r; s, q, Y)$ (resp. $C_n(r; s, q, Y) = B_n^{\mathrm{pix}}(r; s, q, Y)$) holds for all pairs (r, n). Referring to Corollaries 2.2 and 2.4 it suffices to construct a bijection

$$\psi^{\mathrm{fix}} : w \mapsto (\sigma, c)$$

of $W_n(r)$ onto $\mathfrak{S}_n(r,\mathrm{des})$ having the following properties

$$\begin{aligned} \operatorname{dec} w &= \operatorname{exc} \sigma; \\ \operatorname{tot} w &= \operatorname{maj} \sigma + \operatorname{tot} c; \\ \operatorname{single} w &= \operatorname{fix} \sigma; \end{aligned} \tag{5.3}$$

and a bijection

$$\psi^{\mathrm{pix}} : w \mapsto (\sigma, c)$$

of $W_n(r)$ onto $\mathfrak{S}_n(r,\mathrm{ides})$ having the following properties

$$\begin{aligned} \operatorname{wlec} w &= \operatorname{lec} \sigma; \\ \operatorname{tot} w &= \operatorname{imaj} \sigma + \operatorname{tot} c; \\ \operatorname{wpix} w &= \operatorname{pix} \sigma. \end{aligned} \tag{5.4}$$

The construction of ψ^{fix} is achieved by adapting a classical bijection used by Gessel-Reutenauer [13] and Désarménien-Wachs [6, 7]. Start with the Lyndon word factorization $(\tau_1, \tau_2, \ldots, \tau_p)$ of a word $w \in W_n(r)$. If x is a letter of the factor $\tau_i = y_1 \cdots y_{j-1} x y_{j+1} \cdots y_h$, form the cyclic rearrangement $\operatorname{cyc}(x) := x y_{j+1} \cdots y_h y_1 \cdots y_{j-1}$. If x, y are two letters of w, we say that x *precedes* y, if $\operatorname{cyc} x \succ \operatorname{cyc} y$, or if $\operatorname{cyc} x = \operatorname{cyc} y$ and the letter x is to the right of the letter y in the word w. Accordingly, to each letter x of w there corresponds a unique integer $p(x)$, which is the number of letters *preceding* x plus one.

When replacing each letter x in the Lyndon word factorization of w by $p(x)$, we obtain a *cycle decomposition* of a permutation σ. Furthermore, the cycles start with their minima and when reading the word from left to right the cycle minima are in *decreasing* order.

When this replacement is applied to the Lyndon word factorization displayed in (3.10), we obtain:

$$\begin{array}{rl} w = & 2 \mid 3\ 2\ 1\ 1 \mid 3 \mid 5 \mid 6\ 4\ 2\ 1\ 3\ 2\ 3 \mid 6\ 6\ 3\ 1\ 6\ 6\ 2 \mid 6 \\ \sigma = & \mathbf{16} \mid 12\ 18\ 22\ 21 \mid \mathbf{10} \mid \mathbf{7} \mid 4\ 8\ 17\ 20\ 11\ 15\ 9 \mid 2\ 5\ 13\ 19\ 3\ 6\ 14 \mid \mathbf{1} \end{array}$$

Let $\overline{c}_i := p^{-1}(i)$ for $i = 1, 2, \ldots, n$. As the permutation σ is expressed as the product of its disjoint cycles, we can form the three-row matrix

$$\begin{array}{rcccc} \mathrm{Id} = & 1 & 2 & \cdots & n \\ \sigma\ = & \sigma(1) & \sigma(2) & \cdots & \sigma(n) \\ \overline{c}\ = & \overline{c}_1 & \overline{c}_2 & \cdots & \overline{c}_n \end{array}$$

The essential feature is that the word $\overline{c}$ just defined is the monotonic *nonincreasing* rearrangement of w and it has the property that

$$\sigma(i) > \sigma(i+1) \Rightarrow \overline{c}_i > \overline{c}_{i+1}. \tag{5.5}$$

See [13, 7] for a detailed proof. The rest of the proof is routine. Let $z = z_1 z_2 \cdots z_n$ be the word defined by

$$z_i := \#\{j : i \le j \le n-1,\, \sigma(j) > \sigma(j+1)\}.$$

In other words, z_i is the number of descents of σ within the right factor $\sigma(i)\sigma(i+1)\cdots\sigma(n)$. In particular, $z_1 = \operatorname{des}\sigma$. Because of (5.5) the word $c = c_1 c_2 \cdots c_n$ defined by $c_i := \overline{c}_i - z_i$ for $i = 1, 2, \ldots, n$ belongs to $\mathrm{NIW}(r - \operatorname{des}\sigma)$ and $\operatorname{des}\sigma \le r$.

Finally, the verification of the three properties (5.3) is straightforward. Thus, we have constructed the desired bijection $\psi^{\mathrm{fix}} : w \mapsto (\sigma, c)$, as the reverse construction requires no further development.

With the above example we have:

$$\begin{array}{rcl}
\mathrm{Id} &=& 1\ 2\ 3\ 4\ 5\ \ 6\ \ 7\ \ 8\ \ 9\ 10\ 11\ 12\ 13\ 14\ 15\ 16\ 17\ 18\ 19\ 20\ 21\ 22 \\
\sigma &=& \mathbf{1}\ \underline{5}\ \underline{6}\ \underline{8}\ \underline{13}\ \underline{14}\ \mathbf{7}\ \underline{17}\ 4\ \mathbf{10}\ \underline{15}\ \underline{18}\ \underline{19}\ 2\ 9\ \mathbf{16}\ \underline{20}\ \underline{22}\ 3\ 11\ 12\ 21 \\
\overline{c} &=& 6\ 6\ 6\ 6\ 6\ 6\ 5\ 4\ 3\ 3\ 3\ 3\ 3\ 2\ 2\ 2\ 2\ 2\ 1\ 1\ 1\ 1 \\
z &=& 4\ 4\ 4\ 4\ 4\ 4\ 3\ 3\ 2\ 2\ 2\ 2\ 2\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0 \\
c &=& 2\ 2\ 2\ 2\ 2\ 2\ 2\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1
\end{array}$$

The excedances of σ have been underlined ($\operatorname{exc}\sigma = 11$). As $\operatorname{tot} z = \operatorname{maj}\sigma$, we have $74 = \operatorname{tot} w = \operatorname{maj}\sigma + \operatorname{tot} c = 45 + 29$. The fixed points are written in boldface ($\operatorname{fix}\sigma = 4$).

The bijection $\psi^{\mathrm{pix}} : w \mapsto (\sigma, c)$ of $W_n(r)$ onto $\mathfrak{S}_n(r, \mathrm{ides})$ is constructed by means of the classical *standardisation* of words. Read w from left to right and label 1, 2, … all the maximal letters. If there are m such letters, restart the reading from left to right and label $m+1$, $m+2$, … the second greatest letters. Pursue this reading method until reaching the minimal letters. Call $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ the permutation derived by reading those labels from left to right.

The permutation σ and the word w have the same hook-factorization *type*. This means that if $ah_1h_2\ldots h_s$ (resp. $bp_1p_2\ldots p_k$) is the hook-factorization of σ (resp. H-factorization of w), then $k = s$ and $\lambda a = \lambda b$. For each $1 \le i \le k$ we have $\lambda h_i = \lambda p_i$ and $\operatorname{inv}(h_i) = \operatorname{rinv}(p_i)$. Hence $\operatorname{wlec} w = \operatorname{lec}\sigma$ and $\operatorname{wpix} w = \operatorname{pix}\sigma$.

Now define the word $z = z_1 z_2 \ldots z_n$ as follows. If $\sigma(j) = n$ is the maximal letter, then $z_j := 0$; if $\sigma(j) = \sigma(k) - 1$ and $j < k$, then $z_j := z_k$; if $\sigma(j) = \sigma(k) - 1$ and $j > k$, then $z_j := z_k + 1$. We can verify that $\operatorname{imaj}\sigma = \operatorname{tot} z$. With $w = x_1 x_2 \cdots x_n$ define the word $d = d_1 d_2 \cdots d_n$ by $d_i := x_i - z_i$ $(1 \le i \le n)$. As $z_j = z_k + 1 \Rightarrow x_j \ge x_k + 1$, the letters of d are all nonnegative. The final word c is just defined to be the monotonic nonincreasing rearrangement of d. Finally, properties (5.4) are easily verified.

For defining the reverse of ψ^{pix} we just have to remember that the following inequality holds: $\sigma(j) < \sigma(k) \Rightarrow d_j \geq d_k$. This achieves the proofs of Theorems 1.1 and 1.2. For example,

$\mathrm{Id} =$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
$w =$	6	5	3	2	1	3	2	1	3	6	4	1	2	2	3	1	6	6	3	2	6	6
$\sigma =$	1	7	9	14	19	10	15	20	11	2	8	21	16	17	12	22	3	4	13	18	5	6
$z =$	4	3	2	1	0	2	1	0	2	4	3	0	1	1	2	0	4	4	2	1	4	4
$d =$	2	2	1	1	1	1	1	1	1	2	1	1	1	1	1	1	2	2	1	1	2	2
$c =$	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

6. A bijection on words and the proof of Theorem 1.3

Consider the two bijections ϕ^{fix} and ϕ^{pix} that have been constructed in Sections 3 and 4 and consider the bijection F defined by the following diagram:

$$\begin{tikzcd}
 & W_n(r) \arrow[dd, "F"] \\
D_n^*(r) \arrow[ur, "\phi^{\text{fix}}"] \arrow[dr, "\phi^{\text{pix}}"'] & \\
 & W_n(r)
\end{tikzcd}$$

Fig. 1. Definition of F

On the other hand, go back to the definition of the bijection $(w,i) \mapsto v$ (resp. $(w,i) \mapsto h$) given in Proposition 3.1 (resp. in Proposition 4.1). If $w = x_1x_2\cdots x_n$, then *both* v and h are *rearrangements* of the word $(x_1+1)(x_2+1)\cdots(x_i+1)x_{i+1}\cdots x_n$. Now consider the two bijections

$$\phi^{\text{fix}} : (w_0, (w_1, i_1), \ldots, (w_m, i_m)) \mapsto w;$$
$$\phi^{\text{pix}} : (w_0, (w_1, i_1), \ldots, (w_m, i_m)) \mapsto w'.$$

It then follows from Proposition 3.2, Proposition 3.3 and (3.9), on the one hand, and from the very definition of ϕ^{pix}, on the other hand, that the words w and w' are *rearrangements of each other.* Finally, Theorems 2.1 and 2.3 imply the following result.

Theorem 6.1. *The transformation F defined by the diagram of Fig.* 1 *maps each word whose letters are nonnegative integers on another word $F(w)$ and has the following properties:*

(i) $F(w)$ is a rearrangement of w and the restriction of F to each rearrangement class is a bijection of that class onto itself;

(ii) $(\mathrm{dec}, \mathrm{single})\, w = (\mathrm{wlec}, \mathrm{wpix})\, F(w)$.

Let $\mathbf{c}$ be the *complement to* $(n+1)$ that maps each permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ onto $\mathbf{c}\,\sigma := (n+1-\sigma(1))(n+1-\sigma(2))\cdots(n+1-\sigma(n))$.

When restricted to the symmetric group $\mathfrak{S}_n$ the mapping $F\circ\mathbf{c}$ maps $\mathfrak{S}_n$ onto itself and has the property

$$(\mathrm{des},\mathrm{single})\,\sigma=(\mathrm{lec},\mathrm{pix})\,(F\circ\mathbf{c})(\sigma).$$

Note that "dec" was replaced by "des", as all the decreases in a permutation are descents. Finally, the so-called first fundamental transformation (see [11]) $\sigma\mapsto\hat\sigma$ maps $\mathfrak{S}_n$ onto itself and is such that

$$(\mathrm{exc},\mathrm{fix})\,\sigma=(\mathrm{des},\mathrm{single})\,\hat\sigma.$$

Hence

$$(\mathrm{exc},\mathrm{fix})\,\sigma=(\mathrm{lec},\mathrm{pix})\,(F\circ\mathbf{c})(\hat\sigma).$$

As announced in the introduction we have a stronger result stated in Theorem 1.3. Its proof is as follows.

Proof of Theorem 1.3. For each composition $J=j_1j_2\cdots j_m$ (word with positive letters) define the set $L(J)$ and the monotonic nonincreasing word $c(J)$ by

$$\begin{aligned}L(J)&:=\{j_m,j_m+j_{m-1},\ldots,j_m+j_{m-1}+\cdots+j_2+j_1\};\\ c(J)&:=m^{j_m}(m-1)^{j_{m-1}}\cdots 2^{j_2}1^{j_1}.\end{aligned}$$

For example, with $J=455116$ we have $L(J)=\{6,7,8,13,18,22\}$ and $c(J)=6666665433333222221111$.

Fix a composition J of n (i.e., $\operatorname{tot}J=n$) and let $\mathfrak{S}^J$ be the set of all permutations σ of order n such that $\operatorname{Iligne}\sigma\subset L(J)$. Using the bijection ψ^{pix} given in Section 5, define a bijection $w_1\mapsto\sigma_1$ between the set R_J of all rearrangements of $c(J)$ and $\mathfrak{S}^J$ by

$$(\sigma_1,*)=\psi^{\mathrm{pix}}(w_1).$$

For defining the reverse $\sigma_1\mapsto w_1$ we only have to take the multiplicity of $w_1\in R_J$ into account. This is well-defined because $\operatorname{Iligne}\sigma_1\subset L(J)$. For example, take the same example used in Section 5 for ψ^{pix}:

$$\begin{array}{rcllllllllllllllllllllll}
\mathrm{Id}&=&1&2&3&4&5&6&7&8&9&10&11&12&13&14&15&16&17&18&19&20&21&22\\
w_1&=&6&5&3&2&1&3&2&1&3&6&4&1&2&2&3&1&6&6&3&2&6&6\\
\sigma_1&=&1&7&9&14&19&10&15&20&11&2&8&21&16&17&12&22&3&4&13&18&5&6
\end{array}$$

Then $\operatorname{Iligne}\sigma_1=\{6,8,13,18\}\subset \mathrm{L}(J)$ and the basic properties of this bijection are

$$\operatorname{wlec}w_1=\operatorname{lec}\sigma_1,\quad \operatorname{wpix}w_1=\operatorname{pix}\sigma_1.$$

On the other hand the bijection $\psi^{\rm fix}$ given in Section 5 defines a bijection $w_2 \mapsto \sigma_2$ between R_J and $\mathfrak{S}^J$ by

$$(\sigma_2^{-1}, *) = \psi^{\rm fix}(w_2).$$

Again, for the reverse $\sigma_2 \mapsto w_2$ the multiplicity of $w_2 \in R_J$ is to be taken into account. This is also well-defined, since $\operatorname{Iligne}\sigma_2 \subset L(J)$. With the example used in Section 5 for $\psi^{\rm fix}$ we have:

$$\begin{array}{rl} \mathrm{Id} & = 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15\ 16\ 17\ 18\ 19\ 20\ 21\ 22 \\ w_2 & = 2\ 3\ 2\ 1\ 1\ 3\ 5\ 6\ 4\ 2\ 1\ 3\ 2\ 3\ 6\ 6\ 3\ 1\ 6\ 6\ 2\ 6 \\ \sigma_2^{-1} & = \mathbf{1}\ 5\ 6\ 8\ 13\ 14\ \mathbf{7}\ 17\ 4\ \mathbf{10}\ 15\ 18\ 19\ 2\ 9\ \mathbf{16}\ 20\ 22\ 3\ 11\ 12\ 21 \\ \sigma_2 & = \mathbf{1}\ 14\ 19\ 9\ 2\ 3\ \mathbf{7}\ 4\ 15\ \mathbf{10}\ 20\ 21\ 5\ 6\ 11\ \mathbf{16}\ 8\ 12\ 13\ 17\ 22\ 18 \end{array}$$

Also $\operatorname{Iligne}\sigma_2 = \operatorname{Ligne}\sigma_2^{-1} = \{6, 8, 13, 18\} \subset \mathrm{L}(J)$.

The basic properties of this bijection are

$$\operatorname{dec} w_2 = \operatorname{iexc}\sigma_2, \quad \operatorname{single} w_2 = \operatorname{fix}\sigma_2.$$

We can use those two bijections and the bijection F defined in Fig. 1 to form the chain

$$\sigma \mapsto w_1 \overset{F}{\mapsto} w_2 \mapsto \sigma_2,$$

and therefore obtain a bijection $\sigma_1 \mapsto \sigma_2$ of $\mathfrak{S}^J$ onto itself having the following properties

$$\operatorname{iexc}\sigma_2 = \operatorname{lec}\sigma_1, \quad \operatorname{fix}\sigma_2 = \operatorname{pix}\sigma_1.$$

In other words, the pairs (iexc, fix) and (lec, pix) are equidistributed on $\{\sigma \in \mathfrak{S}_n, \operatorname{Iligne}\sigma \subset J\}$ for all compositions J of n. By the inclusion-exclusion principle those pairs are also equidistributed on each set $\{\sigma \in \mathfrak{S}_n, \operatorname{Iligne}\sigma = J\}$. Hence the triples (iexc, fix, Iligne) and (lec, pix, Iligne) are equidistributed on $\mathfrak{S}_n$. $\square$

7. Proof of Theorem 1.4

If $\mathbf{m} = (m_1, m_2, \ldots, m_n)$ is a sequence of n nonnegative integers, the rearrangement class of the nondecreasing word $1^{m_1}2^{m_2}\ldots r^{m_n}$, that is, the class of all the words than can be derived from $1^{m_1}2^{m_2}\ldots r^{m_n}$ by permutation of the letters, is denoted by $R_{\mathbf{m}}$. The definitions of "des," "maj" and "inv" used so far for permutations are also valid for words. The *second fundamental transformation*, as it was called later on (see [8], [17], §10.6 or [16], ex. 5.1.1.19) denoted by Φ, maps each word w on another word $\Phi(w)$ and has the following properties:

(a) $\operatorname{maj} w = \operatorname{inv}\Phi(w)$;

(b) $\Phi(w)$ is a rearrangement of w, and the restriction of Φ to each rearrangement class $R_{\mathbf{m}}$ is a bijection of $R_{\mathbf{m}}$ onto itself.

Further properties were further proved by Foata, Schützenberger [11] and Björner, Wachs [1], in particular, when the transformation is restricted to act on rearrangement classes $R_{\mathbf{m}}$ such that $m_1 = \cdots = m_n = 1$, that is, on symmetric groups $\mathfrak{S}_n$.

Ligne and inverse ligne of route have been defined in the Introduction. As was proved in [11], the transformation Φ preserves the inverse ligne of route, so that the pairs (Iligne, maj) and (Iligne, inv) are equidistributed on $\mathfrak{S}_n$, a result that we express as

$$(\mathrm{Iligne}, \mathrm{maj}) \simeq (\mathrm{Iligne}, \mathrm{inv}); \tag{7.1}$$

or as

$$(\mathrm{Ligne}, \mathrm{imaj}) \simeq (\mathrm{Ligne}, \mathrm{inv}). \tag{7.2}$$

The refinement of (7.2) we now derive (see Proposition 7.1 and Theorem 7.2 below) is based on the properties of a new statistic called LAC.

For each permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ and each integer i such that $1 \le i \le n$ define $\ell_i := 0$ if $\sigma(i) < \sigma(i+1)$ and $\ell_i := k$ if $\sigma(i)$ is greater than all the letters $\sigma(i+1)$, $\sigma(i+2)$, ..., $\sigma(i+k)$, but $\sigma(i) < \sigma(i+k+1)$. [By convention, $\sigma(n+1) = +\infty$.]

Definition. The statistic $\mathrm{LAC}\,\sigma$ attached to each permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ is defined to be the word $\mathrm{LAC}\,\sigma = \ell_1\ell_2\ldots\ell_n$.

Example. We have

id	=	1	2	3	4	5	6	7	8	9	10	11	12
σ	=	3	4	8	1	9	2	5	10	12	7	6	11
$\mathrm{LAC}\,\sigma$	=	0	0	1	0	2	0	0	0	3	1	0	0

In the above table $\ell_5 = 2$ because $\sigma = \ldots \mathbf{9}\ 2\ 5\ \mathbf{10} \cdots$ and $\ell_9 = 3$ because $\sigma = \ldots \mathbf{12}\ 7\ 6\ 11$.

Proposition 7.1. *Let $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ be a permutation and let $\mathrm{LAC}\,\sigma = \ell_1\ell_2\ldots\ell_n$. Then $i \in \mathrm{Ligne}\,\sigma$ if and only if $\ell_i \ge 1$.*

Theorem 7.2. *We have*

$$(\mathrm{LAC}, \mathrm{imaj}) \simeq (\mathrm{LAC}, \mathrm{inv}). \tag{7.3}$$

Proof. Define $\mathrm{ILAC}\,\sigma := \mathrm{LAC}\,\sigma^{-1}$. Since Φ maps "maj" to "inv," property (7.3) will be proved if we show that Φ preserves "ILAC", that is,

$$\mathrm{ILAC}\,\Phi(\sigma) = \mathrm{ILAC}\,\sigma. \tag{7.4}$$

A direct description of ILAC σ can be given as follows. Let ILAC $\sigma = f_1 f_2 \dots f_n$. Then $f_i = j$ if and only if within the word $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ the integer j is such that the letters of σ equal to $i+1, i+2, \dots, i+j$ are on the left of the letter equal to i and either $(i+j+1)$ is on the right of i, or $i+j=n$.

As can be seen in ([17], chap. 10), the second fundamental transformation Φ is defined by induction: $\Phi(x) = x$ for each letter x and $\Phi(wx) = \gamma_x(\Phi(w))x$ for each word w and each letter x, where γ_x is a well-defined bijection. See the above reference for an explicit description of γ_x. Identity (7.4) is then a simple consequence of the following property of γ_x (we omit its proof): *Let w be a word and x a letter. If u is a subword of w such that all letters of u are smaller (resp. greater) than x, then u is also a subword of $\gamma_x(w)$.* $\square$

Proposition 7.3. *Let σ and τ be two permutations of order n. If* LAC $\sigma =$ LAC τ, *then*

(i) $\mathrm{Ligne}\,\sigma = \mathrm{Ligne}\,\tau$;
(ii) $(\mathrm{des}, \mathrm{maj})\sigma = (\mathrm{des}, \mathrm{maj})\tau$;
(iii) $\mathrm{pix}\,\sigma = \mathrm{pix}\,\tau$;
(iv) $\mathrm{lec}\,\sigma = \mathrm{lec}\,\tau$.

Proof. (i) follows from Proposition 7.1. (ii) follows from (i). By (i) we see that σ and τ have the same hook-factorization *type*. That means that if $ah_1h_2\dots h_s$ (resp. $bp_1p_2\dots p_k$) is the hook-factorization of σ (resp. of τ), then $k = s$ and $\lambda\, a = \lambda\, b$, $\lambda\, h_i = \lambda\, p_i$ for $1 \le i \le k$. Hence (iii) holds. For proving (iv) it suffices to prove that $\mathrm{inv}(h_i) = \mathrm{inv}(p_i)$ for $1 \le i \le k$. This is true since LAC $\sigma =$ LAC τ by hypothesis. $\square$

It follows from Proposition 7.3 that

$$(\mathrm{lec}, \mathrm{imaj}, \mathrm{pix}) \simeq (\mathrm{lec}, \mathrm{inv}, \mathrm{pix}) \tag{7.5}$$

and this is all we need to prove Theorem 1.4.

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DECREASES AND DESCENTS IN WORDS

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ABSTRACT. The generating function for words by a multivariable statistic involving decrease, increase, descent and rise values is explicitly calculated by using the MacMahon Master Theorem and the properties of the first fundamental transformation on words. Applications to statistical study of the symmetric group are also given.

1. Introduction

In our recent papers [2, 3, 4, 5, 6, 7, 8, 9] that involve the calculations of factorial generating functions for the symmetric and hyperoctahedral groups, we have obtained several results on statistical distributions on *words*. In this paper we take up again the study of those word statistics in a more general context. They do not necessarily have counterparts on permutations, but are essential in this word calculus. The MacMahon Master Theorem [14, p. 97-98] and the first fundamental transformation on words [13, chap. 10] will be our basic tools.

The *number of decreases* is a crucial statistic; as such it is at the origin of our word studies. The definition of *decrease* slightly differs from the definition of the classical *descent*. Let $w = x_1x_2\cdots x_n$ be an *arbitrary* word, whose letters are nonnegative integers. Recall that a positive integer i is said to be a *descent* (or *descent place*) of w if $1 \le i \le n-1$ and $x_i > x_{i+1}$. We say that i is a *decrease* of w if $1 \le i \le n-1$ and $x_i = x_{i+1} = \cdots = x_j > x_{j+1}$ for some j such that $i \le j \le n-1$. The letter x_i is said to be a *decrease value* (resp. *descent value*) of w. The set of all decreases (resp. descents) is denoted by $\mathrm{DEC}(w)$ (resp. $\mathrm{DES}(w)$). Each descent is a decrease, but not conversely. This means that $\mathrm{DES}(w) \subset \mathrm{DEC}(w)$. However, $\mathrm{DES}(w) = \mathrm{DEC}(w)$ when w is a word *without repetitions*.

In the present paper our intention is to go back to the study of the number of decreases, this time associated with several other word statistics, and derive the Ur-result that should have been at the origin of several of our statistical distribution studies. This Ur-result is stated in Theorem 1.1, but has two equivalent forms, as written in Theorems 1.2 and 1.3.

In parallel with the notion of decrease, we say that that a positive integer i is an *increase* (resp. a *rise*) of w if $1 \le i \le n$ and $x_i = x_{i+1} =$

$\cdots = x_j < x_{j+1}$ for some j such that $i \le j \le n$ (resp. if $1 \le i \le n$ and $x_i < x_{i+1}$). By convention, $x_{n+1} = +\infty$. The letter x_i is said to be *an increase value* (resp. *a rise value*) of w. Thus, the rightmost letter x_n is always a rise value. Again, the set of all increases (resp. rises) is denoted by $\mathrm{INC}(w)$ (resp. $\mathrm{RISE}(w)$). Each rise is an increase, but not conversely. This means that $\mathrm{RISE}(w) \subset \mathrm{INC}(w)$.

Furthermore, a position i $(1 \le i \le n)$ is said to be a *record* if $x_j \le x_i$ for all j such that $1 \le j \le i-1$. The letter x_i is said to be a *record value.* The set of all records of w is denoted by $\mathrm{REC}(w)$.

Introduce six sequences of commuting variables (X_i), (Y_i), (Z_i), (T_i), (Y_i'), (T_i') $(i = 0, 1, 2, \ldots)$ and for each word $w = x_1x_2\ldots x_n$ from $[0,r]^*$ define the *weight* $\psi(w)$ of $w = x_1x_2\ldots x_n$ to be

$$\begin{aligned}(1.1)\quad \psi(w) := &\prod_{i\in\mathrm{DES}} X_{x_i} \prod_{i\in\mathrm{RISE}\setminus\mathrm{REC}} Y_{x_i} \prod_{i\in\mathrm{DEC}\setminus\mathrm{DES}} Z_{x_i}\\ &\times \prod_{i\in(\mathrm{INC}\setminus\mathrm{RISE})\setminus\mathrm{REC}} T_{x_i} \prod_{i\in\mathrm{RISE}\cap\mathrm{REC}} Y'_{x_i} \prod_{i\in(\mathrm{INC}\setminus\mathrm{RISE})\cap\mathrm{REC}} T'_{x_i},\end{aligned}$$

where the argument "(w)" has not been written for typographic reasons. For example, $i \in \mathrm{RISE}\setminus\mathrm{REC}$ stands for $i \in \mathrm{RISE}(w)\setminus\mathrm{REC}(w)$.

Example. For the word $w = 3\,2\,4\,4\,5\,5\,5\,3\,1\,1\,1\,4\,1\,3\,5$ the sets DES, DEC, INC, RISE, REC of w are indicated by bullets.

$$\begin{array}{lcccccccccccccccc}
w & = & 3 & 2 & 4 & 4 & 5 & 5 & 5 & 3 & 1 & 1 & 1 & 4 & 1 & 3 & 5\\
\mathrm{DES} & = & \bullet & & & & & & \bullet & \bullet & & & & \bullet & & & \\
\mathrm{DEC} & = & \bullet & & & & \bullet & \bullet & \bullet & \bullet & & & & \bullet & & & \\
\mathrm{RISE} & = & & \bullet & & \bullet & & & & & & & \bullet & & \bullet & \bullet & \bullet\\
\mathrm{INC} & = & & \bullet & \bullet & \bullet & & & & & \bullet & \bullet & \bullet & & \bullet & \bullet & \bullet\\
\mathrm{REC} & = & \bullet & & \bullet & \bullet & \bullet & \bullet & \bullet & & & & & & & & \bullet
\end{array}$$

We have $\psi(w) = X_3Y_2T_4'Y_4'Z_5Z_5X_5X_3T_1T_1Y_1X_4Y_1Y_3Y_5'$.

Now let C be the $(r+1)\times(r+1)$ matrix

$$(1.2)\qquad C = \begin{pmatrix}
0 & \dfrac{X_1}{1-Z_1} & \dfrac{X_2}{1-Z_2} & \cdots & \dfrac{X_{r-1}}{1-Z_{r-1}} & \dfrac{X_r}{1-Z_r}\\
\dfrac{Y_0}{1-T_0} & 0 & \dfrac{X_2}{1-Z_2} & \cdots & \dfrac{X_{r-1}}{1-Z_{r-1}} & \dfrac{X_r}{1-Z_r}\\
\dfrac{Y_0}{1-T_0} & \dfrac{Y_1}{1-T_1} & 0 & \cdots & \dfrac{X_{r-1}}{1-Z_{r-1}} & \dfrac{X_r}{1-Z_r}\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
\dfrac{Y_0}{1-T_0} & \dfrac{Y_1}{1-T_1} & \dfrac{Y_2}{1-T_2} & \cdots & 0 & \dfrac{X_r}{1-Z_r}\\
\dfrac{Y_0}{1-T_0} & \dfrac{Y_1}{1-T_1} & \dfrac{Y_2}{1-T_2} & \cdots & \dfrac{Y_{r-1}}{1-T_{r-1}} & 0
\end{pmatrix}.$$

Theorem 1.1. *The generating function for the set* $[0,r]^*$ *by the weight* ψ *is given by*

$$\sum_{w\in[0,r]^*}\psi(w)=\frac{\prod_{0\le j\le r}\Bigl(1+\dfrac{Y_j'}{1-T_j'}\Bigr)}{\det(I-C)},\tag{1.3}$$

where I *is the identity matrix of order* $(r+1)$.

Of course, the expression $1/\det(I-C)$ is too redolent of the MacMahon Master Theorem [14, p. 97-98] for not having it play a crucial role in the proof of (1.3). It does indeed. However we further need the properties of the first fundamental transformation, as it is developed in Cartier-Foata [1] and also in Lothaire [13, chap. 10]. As mentioned earlier, Theorem 1.1 must be regarded as our Ur-result. Its proof is given in Section 2. Its two equivalent forms come next.

Theorem 1.2. *We also have:*

$$\sum_{w\in[0,r]^*}\psi(w)=\frac{\dfrac{\prod\limits_{0\le j\le r}\dfrac{1-Z_j}{1-Z_j+X_j}}{\prod\limits_{0\le j\le r}\dfrac{1-T_j'}{1-T_j'+Y_j'}}}{1-\sum\limits_{0\le l\le r}\dfrac{\prod\limits_{0\le j\le l-1}\dfrac{1-Z_j}{1-Z_j+X_j}}{\prod\limits_{0\le j\le l-1}\dfrac{1-T_j}{1-T_j+Y_j}}\,\dfrac{X_l}{1-Z_l+X_l}}.\tag{1.4}$$

By definition each letter equal to 0 cannot be a decrease value. Consequently, the weight $\psi(w)$ of each word w must not contain the variables X_0, Z_0. There is then another expression for the right-hand side of (1.4) which does not involve the variables X_0, Z_0. To obtain it we factor out $(1-Z_0)/(1-Z_0+X_0)$ from both numerator and denominator of the right-hand side, as done in the next theorem.

Theorem 1.3. *We also have:*

$$\sum_{w\in[0,r]^*}\psi(w)=\frac{\dfrac{\prod\limits_{1\le j\le r}\dfrac{1-Z_j}{1-Z_j+X_j}}{\prod\limits_{0\le j\le r}\dfrac{1-T_j'}{1-T_j'+Y_j'}}}{1-\sum\limits_{1\le l\le r}\dfrac{\prod\limits_{1\le j\le l-1}\dfrac{1-Z_j}{1-Z_j+X_j}}{\prod\limits_{0\le j\le l-1}\dfrac{1-T_j}{1-T_j+Y_j}}\,\dfrac{X_l}{1-Z_l+X_l}}.\tag{1.5}$$

Theorem 1.2 (and therefore Theorem 1.3) is proved in Section 3 by using a determinantal manipulation and summing the weights $\psi(w)$ according to their so-called *keys*. An alternate proof of Theorem 1.2, which is not reproduced in this paper, is based on the word-analog of the Kim-Zeng transformation [11] and follows the pattern developed in our previous paper [9]. Specializations of those two theorems for deriving generating functions *on words* only appear in Section 6. There is however a specialization of (1.5) that deserves a special development and is now presented.

Let γ be the homomorphism defined by the following substitutions of variables:

$$\gamma := \{X_j \leftarrow s\,Y_{j-1}, \quad Z_j \leftarrow s\,Y_{j-1}, \quad T_j \leftarrow Y_j, \quad T'_j \leftarrow Y'_j\}.$$

For each word $w = x_1x_2\cdots x_n \in [0,r]^*$ we then have:

$$\gamma\,\psi(w) = \prod_{x_i\in\mathrm{INC}\cap\mathrm{REC}} Y'_{x_i} \times \prod_{x_i\in\mathrm{DEC}} sY_{x_i-1} \times \prod_{x_i\in\mathrm{INC}\setminus\mathrm{REC}} Y_{x_i}. \tag{1.6}$$

Applying γ to (1.5) we get:

$$\sum_{w\in[0,r]^*} \gamma\,\psi(w) = \frac{\dfrac{\prod\limits_{1\le j\le r}(1-sY_{j-1})}{\prod\limits_{0\le j\le r}(1-Y'_j)}}{1-\sum\limits_{1\le l\le r}\dfrac{\prod\limits_{1\le j\le l-1}(1-sY_{j-1})}{\prod\limits_{0\le j\le l-1}(1-Y_j)}sY_{l-1}}. \tag{1.7}$$

The above right-hand side can be further simplified as stated in the following theorem, whose proof is given in Section 4.

Theorem 1.4. *We have:*

$$\sum_{w\in[0,r]^*} \gamma\,\psi(w) = \frac{(1-s)\prod\limits_{0\le j\le r-1}(1-Y_j)\prod\limits_{0\le j\le r-1}(1-s\,Y_j)}{\prod\limits_{0\le j\le r}(1-Y'_j)\Big(\prod\limits_{0\le j\le r-1}(1-Y_j)-s\prod\limits_{0\le j\le r-1}(1-s\,Y_j)\Big)}. \tag{1.8}$$

For each word $w = x_1x_2\cdots x_n$ let $\mathrm{inrec}\,w$ denote the *number* of letters of w, which are increase *and* record values. Also let $\mathrm{dec}\,w$ be the number of decreases in w and $\operatorname{tot} w = x_1 + x_2 + \cdots + x_n$ be the *sum* of the letters of w. Also make the substitution $Y'_j \leftarrow RY_j$, where R is a new variable and let $\psi_R := \gamma\,\psi\,|_{Y'_j \leftarrow R\,Y_j}$, so that (1.6) becomes:

$$\psi_R(w) = R^{\operatorname{inrec} w} s^{\operatorname{dec} w} \prod_{x_i \in \mathrm{DEC}} Y_{x_i - 1} \times \prod_{x_i \in \mathrm{INC}} Y_{x_i}. \tag{1.9}$$

On the other hand, let

$$H(Y) := \prod_{i \geq 0} (1 - Y_i)^{-1}; \tag{1.10}$$

$$H_r(Y) := \prod_{0 \leq i \leq r-1} (1 - Y_i)^{-1} \quad (r \geq 0). \tag{1.11}$$

Using the homomorphism ψ_R identity (1.8) may be rewritten as:

$$\sum_{w \in [0,r]^*} \psi_R(w) = \frac{(1-s)H_{r+1}(RY)}{H_r(sY) - sH_r(Y)}. \tag{1.12}$$

The left-hand side of (1.12) can be further expressed as a series over the symmetric groups $\mathfrak{S}_n$ $(n = 0, 1, \ldots)$. If $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ is a permutation of $12\cdots n$, let $z = z_1 z_2 \cdots z_n$ be the word defined by:

$$z_i := \#\{j : i \leq j \leq n-1,\, \sigma(j) > \sigma(j+1)\} \quad (1 \leq i \leq n), \tag{1.13}$$

so that z_1 is the *number of descents*, $\operatorname{des}\sigma$, and $\operatorname{tot} z = z_1 + z_2 + \cdots + z_n$ is the *major index*, $\operatorname{maj}\sigma$, of σ. Also, let $\operatorname{exc}\sigma := \{i : 1 \leq i \leq n-1, \sigma(i) > i\}$ be the *number of excedances* and $\operatorname{fix}\sigma$ be the *number of fixed points* of σ. Finally, let NIW_n (resp. $\mathrm{NIW}_n(k)$) denote the set of words $c = c_1 c_2 \cdots c_n$, of length n, whose letters are integers satisfying $c_1 \geq c_1 \geq \cdots \geq c_n \geq 0$ (resp. $k \geq c_1 \geq c_1 \geq \cdots \geq c_n \geq 0$). With each pair $(\sigma, c) \in \mathfrak{S}_n \times \mathrm{NIW}_n$ we associate the monomial

$$Y_{(\sigma,c)} := \prod_{j < \sigma(j)} Y_{c_j + z_j - 1} \times \prod_{j \geq \sigma(j)} Y_{c_j + z_j}. \tag{1.14}$$

Theorem 1.5. *We have:*

$$\sum_{n \geq 0} \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \operatorname{des}\sigma \leq r}} R^{\operatorname{fix}\sigma} s^{\operatorname{exc}\sigma} \sum_{c \in \mathrm{NIW}_n(r - \operatorname{des}\sigma)} Y_{(\sigma,c)} = \frac{(1-s)H_{r+1}(RY)}{H_r(sY) - sH_r(Y)} \quad (r \geq 0). \tag{1.15}$$

When r tends to infinity in (1.15), we get the identity

$$\sum_{n\geq 0}\sum_{\sigma\in\mathfrak{S}_n} R^{\operatorname{fix}\sigma} s^{\operatorname{exc}\sigma} \sum_{c\in \mathrm{NIW}_n} Y_{(\sigma,c)} = \frac{(1-s)H(RY)}{H(sY)-sH(Y)}, \tag{1.16}$$

derived by Shareshian and Wachs [15, Theorem 2.1] using a quasi-symmetric function approach. However, starting from (1.15), we can obtain a *graded form* of (1.16) as follows. Let

$$Y(\sigma;t) := \sum_{k\geq 0} t^k \sum_{c\in \mathrm{NIW}_n(k)} Y_{(\sigma,c)}. \tag{1.17}$$

Theorem 1.6. *The graded form of* (1.16) *reads:*

$$\sum_{n\geq 0}\sum_{\sigma\in\mathfrak{S}_n} R^{\operatorname{fix}\sigma} s^{\operatorname{exc}\sigma} t^{\operatorname{des}\sigma} Y(\sigma;t) = \sum_{r\geq 0} t^r \frac{(1-s)H_{r+1}(RY)}{H_r(sY)-sH_r(Y)}. \tag{1.18}$$

Section 5 starts with redescribing the Gessel-Reutenauer standardization [10] and showing how it is used to prove Theorem 1.5. The graded form (1.18) is deduced from Theorem 1.5 by a standard series manipulation. We end the paper by giving some specializations of our Ur-theorem and also by showing that the distribution of $(\operatorname{fix},\operatorname{exc},\operatorname{des},\operatorname{maj})$ over the symmetric groups that was found earlier in [9] using the word-analog of the Kim-Zeng transformation [11], can also be deduced from our Ur-result in two different manners.

2. Proof of Theorem 1.1

A word $w = y_1y_2\cdots y_n \in [0,r]^*$ having no equal letters in succession is called an *h-derangement* (*horizontal derangement*). The set of all h-derangement words in $[0,r]^*$ is denoted by $[0,r]^*_h$. Let α be the substitution of variables defined by

$$\alpha := \{Z_i \leftarrow 0,\quad T_i \leftarrow 0,\quad T'_i \leftarrow 0\}.$$

Then

$$\alpha\,\psi(w) = \psi(w) = \prod_{i\in \mathrm{DES}} X_{x_i} \prod_{i\in \mathrm{RISE}\setminus\mathrm{REC}} Y_{x_i} \prod_{i\in \mathrm{RISE}\cap\mathrm{REC}} Y'_{x_i},$$

if w is an h-derangement and $\alpha\,\psi(w) = 0$ otherwise. The following specialization of Theorem 1.1 is obtained by taking the image of identity (1.3) under α.

Theorem 2.1. *We have*

$$\sum_{w\in[0,r]_h^*} \psi(w) = \frac{\prod_{0\le j\le r}\Bigl(1+Y'_j\Bigr)}{\det(I-\alpha\,C)}. \tag{2.1}$$

Even though Theorem 2.1 is a special case of Theorem 1.1, it can still be used as a lemma to prove Theorem 1.1. We proceed as follows.

Proof of Theorem 1.1. Let w be a word from $[0,r]^*$. The *key* of w is defined to be the h-derangement k derived from w by erasing all letters x_i such that $x_i = x_{i+1}$. For instance, the key of $w = 324455531114135$ is the h-derangement $k = 3245314135$.

Let β be the substitution of variables defined by

$$\beta := \{X_i \leftarrow X_i/(1-Z_i),\quad Y_i \leftarrow Y_i/(1-T_i),\quad Y'_i \leftarrow Y'_i/(1-T'_i)\}.$$

Then, the generating function for the set of all w whose key is k by the weight ψ is given by

$$\sum_{w,\,\mathrm{key}(w)=k} \psi(w) = \beta\,\psi(k).$$

Since $\beta\,\alpha\,C = C$ we have

$$\begin{aligned}\sum_{w\in[0,r]^*} \psi(w) &= \sum_{k\in[0,r]_h^*}\ \sum_{\mathrm{key}(w)=k} \psi(w) \ = \sum_{k\in[0,r]_h^*} \beta\,\psi(k)\\ &= \beta\Bigl(\sum_{k\in[0,r]_h^*} \psi(k)\Bigr) \ = \ \beta\,\frac{\prod_{0\le j\le r}\Bigl(1+Y'_j\Bigr)}{\det(I-\alpha\,C)} \qquad \text{[by (2.1)]}\\ &= \frac{\prod_{0\le j\le r}\Bigl(1+\beta\,Y'_j\Bigr)}{\det(I-\beta\,\alpha\,C)} = \frac{\prod_{0\le j\le r}\Bigl(1+\dfrac{Y'_j}{1-T'_j}\Bigr)}{\det(I-C)}. \quad \square\end{aligned}$$

Proof of Theorem 2.1. A letter x_i which is a record and also a rise value is called an *riserec* value. For each h-derangement $k = x_1x_2\cdots x_n$ let w_0 be the nondecreasing word composed of all the riserec values of k and let k_0 be the word obtained from k by erasing all the riserec values of k. Since the letters of w_0 can be uniquely inserted into the word k_0 for reconstructing k, the map

$$k \mapsto (w_0, k_0) \tag{2.2}$$

is a bijection of the set of all h-derangements onto the the set of all pairs (w_0, k_0) such that w_0 is a nondecreasing word and k_0 is an h-derangement without any riserec value. Moreover, if a letter x_j of k is a record value and therefore becomes a letter, say, $x_{0,i}$ of w_0, then $x_{0,i}$ is a rise of w_0 if and only if x_j is a rise of k. Therefore

$$\psi(k) = \psi(w_0)\psi(k_0). \tag{2.3}$$

Next apply the first fundamental transformation to k_0 (see [13, § 10.5]), say, $u = \mathbf{F}_1(k_0)$. Let us recall how $\mathbf{F}_1$ is defined by means of an example. Start with the word $k_0 = 5\,3\,6\,5\,3\,2\,4\,6\,1\,2\,4\,3\,1$ and cut it before each record value to get $k_0 = 5\,3 \mid 6\,5\,3\,2\,4 \mid 6\,1\,2\,4\,3\,1 =: w_1 \mid w_2 \mid w_3$. In each compartment move the leftmost letter to the end to obtain the cyclic shifts $\delta w_1 = 3\,5$, $\delta w_2 = 5\,3\,2\,4\,6$, $\delta w_3 = 1\,2\,4\,3\,1\,6$ and form the two-row matrix $\begin{pmatrix} \delta w_1\, \delta w_2\, \delta w_3 \\ w_1 \;\; w_2 \;\; w_3 \end{pmatrix} = \begin{pmatrix} 3\,5\,5\,3\,2\,4\,6\,1\,2\,4\,3\,1\,6 \\ 5\,3\,6\,5\,3\,2\,4\,6\,1\,2\,4\,3\,1 \end{pmatrix}$. Finally, rearrange the vertical biletters of that two-row matrix in such a way that the entries on the top row are in nondecreasing order, assuming that two biletters $\binom{a}{a'}$, $\binom{b}{b'}$ can commute only when $a \neq b$. We then obtain the two-row matrix $\mathbf{F}_1(k_0) := \begin{pmatrix} \overline{u} \\ u \end{pmatrix} = \begin{pmatrix} 1\,1\,2\,2\,3\,3\,3\,4\,4\,5\,5\,6\,6 \\ 6\,3\,3\,1\,5\,5\,4\,2\,2\,3\,6\,4\,1 \end{pmatrix}$.

We can characterize the word $u = y_1 y_2 \cdots y_m$ when we start with an h-derangement k_0. Let $\overline{u} = z_1 z_2 \cdots z_m$ be the nondecreasing rearrangement of u (and of k_0). By construction $y_i \neq z_i$ for $1 \leq i \leq m$. Such a word u is called a *v-derangement* (vertical derangement). Denote the set of all v-derangements in $[0, r]^*$ by $[0, r]^*_v$. Then $\mathbf{F}_1$ provides a bijection of the set of all h-derangements onto the set of all pairs (w_0, u) such that w_0 is a nondecreasing word and u is a v-derangement:

$$k \mapsto (w_0, k_0) \mapsto (w_0, u) \quad \text{where} \quad \mathbf{F}_1(k_0) = \begin{pmatrix} \overline{u} \\ u \end{pmatrix}.$$

Example. $k = \mathbf{2}53\mathbf{5}653246124316\mathbf{6}$...

$$k = \mathbf{2}53\mathbf{5}65324612431\mathbf{6} \mapsto (w_0 = \mathbf{256}, k_0 = 5365324612431)$$
$$\mapsto (w_0 = \mathbf{256}, u = 6331554223641),$$

since
$$\mathbf{F}_1(k_0) = \begin{pmatrix} \overline{u} \\ u \end{pmatrix} = \begin{pmatrix} 1\,1\,2\,2\,3\,3\,3\,4\,4\,5\,5\,6\,6 \\ 6\,3\,3\,1\,5\,5\,4\,2\,2\,3\,6\,4\,1 \end{pmatrix}.$$

Let Φ be the homomorphism generated by

$$\Phi\begin{pmatrix} i \\ j \end{pmatrix} := \begin{cases} X_j, & \text{if } j > i; \\ Y_j, & \text{if } j < i. \end{cases}$$

By the property [13, chap. 10] of the first fundamental transformation we have

$$\psi(k) = \psi(w_0)\psi(k_0) = \psi(w_0)\Phi\binom{\overline{u}}{u}. \tag{2.4}$$

Let $\mathrm{ND}(r)$ be the set of all *non-decreasing* words from $[0,r]^*$. It follows from the properties of the above bijections that

$$\sum_{w\in[0,r]^*_h} \psi(w) = \sum_{w_0\in \mathrm{ND}(r)} \psi(w_0) \sum_{u\in[0,r]^*_v} \Phi\binom{\overline{u}}{u};$$

$$\sum_{w_0\in \mathrm{ND}(r)} \psi(w_0) = \prod_{0\le j\le r} (1+Y'_j).$$

There remains to prove the identity:

$$\sum_{u\in[0,r]^*_v} \Phi\binom{\overline{u}}{u} = \frac{1}{\det(I-\alpha\, C)}. \tag{2.5}$$

The proof is based on the celebrated MacMahon Master Theorem, using the noncommutative version developed in ([1], chap. 4). Also see ([13], chap. 10). Consider the matrix

$$C'' = \begin{pmatrix} 0 & \binom{0}{1} & \binom{0}{2} & \cdots & \binom{0}{r-1} & \binom{0}{r} \\ \binom{1}{0} & 0 & \binom{1}{2} & \cdots & \binom{1}{r-1} & \binom{1}{r} \\ \binom{2}{0} & \binom{2}{1} & 0 & \cdots & \binom{2}{r-1} & \binom{2}{r} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \binom{r-1}{0} & \binom{r-1}{1} & \binom{r-1}{2} & \cdots & 0 & \binom{r-1}{r} \\ \binom{r}{0} & \binom{r}{1} & \binom{r}{2} & \cdots & \binom{r}{r-1} & 0 \end{pmatrix}.$$

As shown in the references just mentioned, there holds the identity

$$\frac{1}{\det(I-C'')} = \sum_{u\in[0,r]^*_v} \binom{\overline{u}}{u}. \tag{2.6}$$

Applying Φ to both sides of (2.6) yields (2.5). $\square$

3. Proof of Theorem 1.2

The proof of Theorem 1.2 is similar to the proof of Theorem 1.1 given in the previous section. First we prove the following specialization of Theorem 1.2.

Theorem 3.1. *We have:*

$$\sum_{w\in[0,r]_h^*}\psi(w)=\frac{\displaystyle\prod_{0\le j\le r}\frac{1+Y_j'}{1+X_j}}{1-\displaystyle\sum_{0\le l\le r}\prod_{0\le j\le l-1}\frac{1+Y_j}{1+X_j}\frac{X_l}{1+X_l}}. \tag{3.1}$$

Proof. Denote the left-hand side of (3.1) by LHS. From Theorem 2.1 we have

$$\mathrm{LHS}=\frac{\prod_{0\le j\le r}(1+Y_j')}{D},$$

where

$$D=\begin{vmatrix} 1 & -X_1 & -X_2 & \cdots & -X_{r-1} & -X_r\\ -Y_0 & 1 & -X_2 & \cdots & -X_{r-1} & -X_r\\ -Y_0 & -Y_1 & 1 & \cdots & -X_{r-1} & -X_r\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ -Y_0 & -Y_1 & -Y_2 & \cdots & 1 & -X_r\\ -Y_0 & -Y_1 & -Y_2 & \cdots & -Y_{r-1} & 1 \end{vmatrix}.$$

In the above determinant subtract the r-th row from the $(r+1)$-st one; then the $(r-1)$-st from the r-th row; ..., the first row from the second. We obtain:

$$D=\begin{vmatrix} 1 & -X_1 & -X_2 & \cdots & -X_{r-1} & -X_r\\ -1-Y_0 & 1+X_1 & 0 & \cdots & 0 & 0\\ 0 & -1-Y_1 & 1+X_2 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1+X_{r-1} & 0\\ 0 & 0 & 0 & \cdots & -1-Y_{r-1} & 1+X_r \end{vmatrix}.$$

Now expand the determinant by the cofactors of the *first row.* We get:

$$D=\prod_{1\le j\le r}(1+X_j)-\sum_{1\le l\le r}\Bigl(\prod_{0\le j\le l-1}(1+Y_j)\prod_{l+1\le j\le r}(1+X_j)\Bigr)X_l.$$

We further have:

$$\begin{aligned} D=&\prod_{1\le j\le r}(1+X_j)+X_0\prod_{1\le j\le r}(1+X_j)\\ &\qquad-\sum_{0\le l\le r}\Bigl(\prod_{0\le j\le l-1}(1+Y_j)\prod_{l\le j\le r}(1+X_j)\Bigr)\frac{X_l}{1+X_l}. \end{aligned}$$

Hence,

$$\text{(3.2)} \qquad \text{LHS} = \frac{\displaystyle\prod_{0\le j\le r} \frac{(1+Y'_j)}{(1+X_j)}}{\displaystyle 1-\sum_{0\le l\le r}\prod_{0\le j\le l-1}\frac{(1+Y_j)}{(1+X_j)}\frac{X_l}{1+X_l}}. \quad \square$$

Proof of Theorem 1.2. As in the proof of Theorem 1.1 we may write:

$$\sum_{w\in[0,r]^*}\psi(w) = \sum_{k\in[0,r]^*_h}\sum_{\text{key}(w)=k}\psi(w) = \sum_{k\in[0,r]^*_h}\beta\,\psi(k) = \beta(\text{LHS}).$$

Using (3.2) it is immediate to verify that $\beta(\text{LHS})$ is equal to the right-hand side of (1.4). $\square$

4. Proof of Theorem 1.4

First, we may check that

$$\begin{aligned}
&\prod_{0\le j\le r}(1-sY_j) - \prod_{0\le j\le r}(1-Y_j)\\
=&\sum_{0\le l\le r}\prod_{0\le j\le l}(1-sY_j)\prod_{l+1\le j\le r}(1-Y_j) - \sum_{0\le l\le r}\prod_{0\le j\le l-1}(1-sY_j)\prod_{l\le j\le r}(1-Y_j)\\
=&(1-s)\sum_{0\le l\le r}Y_l\prod_{0\le j\le l-1}(1-sY_j)\prod_{l+1\le j\le r}(1-Y_j).
\end{aligned}$$

Proof of Theorem 1.4. Using (1.7) we have:

$$\begin{aligned}
&\frac{1}{1-s\displaystyle\sum_{0\le l\le r-1}Y_l\prod_{0\le j\le l-1}(1-sY_j)/\prod_{0\le j\le l}(1-Y_j)}\\
&=\frac{\displaystyle\prod_{0\le j\le r-1}(1-Y_j)}{\displaystyle\prod_{0\le j\le r-1}(1-Y_j) - s\sum_{0\le l\le r-1}Y_l\prod_{0\le j\le l-1}(1-sY_j)\prod_{l+1\le j\le r-1}(1-Y_j)}\\
&=\frac{\displaystyle\prod_{0\le j\le r-1}(1-Y_j)}{\displaystyle\prod_{0\le j\le r-1}(1-Y_j) - s\Big(\prod_{0\le j\le r-1}(1-sY_j) - \prod_{0\le j\le r-1}(1-Y_j)\Big)/(1-s)}\\
&=\frac{\displaystyle(1-s)\prod_{0\le j\le r-1}(1-Y_j)}{\displaystyle\prod_{0\le j\le r-1}(1-Y_j) - s\prod_{0\le j\le r-1}(1-sY_j)}. \quad \square
\end{aligned}$$

5. Proofs of Theorems 1.5 and 1.6

An updated version of the Gessel-Reutenauer standardization [10] is fully described in our previous paper [9], Section 5. The standardization consists of mapping each word w from $[0,r]^*$ of length n onto a pair (σ, c), where $\sigma \in \mathfrak{S}_n$ and $c = c_1c_2\cdots c_n$ is a word of length n, whose letters are nonnegative integers having the property: $r - \operatorname{des}\sigma \ge c_1 \ge c_2 \ge \cdots \ge c_n \ge 0$, the symbol $\operatorname{des}\sigma$ being the *number of descents* of σ. We recall the construction of the inverse $(\sigma, c) \mapsto w$ by means of an example.

$$
\begin{array}{rcccccccccccccccccccccccc}
\mathrm{Id} & = & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 \\
\to \sigma & = & \mathbf{1} & \underline{5} & \underline{6} & \underline{8} & \underline{13} & \underline{14} & \mathbf{7} & \underline{17} & 4 & \mathbf{10} & \underline{15} & \underline{18} & \underline{19} & 2 & 9 & \mathbf{16} & \underline{20} & \underline{22} & 3 & 11 & 12 & 21 \\
z & = & 4 & 4 & 4 & 4 & 4 & 4 & 3 & 3 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
\to c & = & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\overline{c} & = & 6 & 6 & 6 & 6 & 6 & 6 & 5 & 4 & 3 & 3 & 3 & 3 & 3 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\
\sigma & = & (\mathbf{16}) & (12 & 18 & 22 & 21) & (\mathbf{10}) & (\mathbf{7}) & (4 & 8 & 17 & 20 & 11 & 15 & 9) & (2 & 5 & 13 & 19 & 3 & 6 & 14) & (\mathbf{1}) \\
\check{\sigma} & = & \mathbf{16} & 12 & 18 & 22 & 21 & \mathbf{10} & \mathbf{7} & 4 & 8 & 17 & 20 & 11 & 15 & 9 & 2 & 5 & 13 & 19 & 3 & 6 & 14 & \mathbf{1} \\
\mapsto w & = & 2\mid & \underline{3} & \underline{2} & 1 & 1\mid & 3\mid & 5\mid & \underline{6} & \underline{4} & \underline{2} & 1 & \underline{3} & 2 & 3\mid & \underline{6} & \underline{6} & \underline{3} & 1 & \underline{6} & \underline{6} & 2\mid & 6
\end{array}
$$

In the example $n = 22$. The second row contains the values $\sigma(i)$ $(i = 1, 2, \ldots, n)$ of the *starting* permutation σ. The *excedances* $\sigma(i) > i$ are underlined, while the *fixed points* $\sigma(i) = i$ are written in boldface. The third row is the vector $z = z_1z_2\cdots z_n$ defined by (1.13) so that $z_1 = \operatorname{des}\sigma$ and $\operatorname{tot} z = z_1 + z_2 + \cdots + z_n$ is the *major index* of σ denoted by $\operatorname{maj}\sigma$, as already mentioned in the Introduction. The fourth row is the *starting* nonincreasing word $c = c_1c_2\cdots c_n$. The fifth row $\overline{c} = \overline{c}_1\overline{c}_2\cdots\overline{c}_n$ is the word defined by

$$\overline{c}_i := z_i + c_i \quad (1 \le i \le n).$$

In the sixth row the permutation σ is represented as the product of its disjoint cycles. Each cycle starts with its *minimum* element and those minimum elements are in *decreasing* order when reading the whole word from left to right. When removing the parentheses in the sixth row we obtain the seventh row denoted by $\check{\sigma} = \check{\sigma}(1)\check{\sigma}(2)\cdots\check{\sigma}(n)$. The bottom row is the word $w = x_1x_2\cdots x_n$ corresponding to the pair (σ, c) defined by

$$x_i := \overline{c}_{\check{\sigma}(i)} \quad (1 \le i \le n).$$

For instance, $\check{\sigma}(9) = 8$ and $\overline{c}(8) = 4$. Hence $x_9 = 4$. The decrease values of w have been underlined.

It can be verified that all the above steps are reversible and that $w \mapsto (\sigma, c)$ is a bijection of the set of all words from $[0,r]^*$ of length n onto the set of pairs (σ, c) such that $\sigma \in \mathfrak{S}_n$, $\operatorname{des}\sigma \le r$ and $c = c_1c_2\cdots c_n$

is a word satisfying $r - \operatorname{des}\sigma \ge c_1 \ge c_2 \ge \cdots \ge c_n \ge 0$. Furthermore, x_i is a decrease value of w if and only if $\check\sigma(i) < \check\sigma(i+1)$, if and only if $\check\sigma(i) < \sigma(\check\sigma(i))$. Also x_i is an increase and record value of w if and only if $\check\sigma(i)$ is a fixed point of σ. Hence,

$$\begin{aligned}
\psi_R(w) &= R^{\operatorname{inrec} w} s^{\operatorname{dec} w} \prod_{x_i \in \mathrm{DEC}} Y_{x_i - 1} \prod_{x_i \in \mathrm{INC}} Y_{x_i} \\
&= R^{\operatorname{fix}\sigma} s^{\operatorname{exc}\sigma} \prod_{\check\sigma(i) < \sigma(\check\sigma(i))} Y_{\overline{c}_{\check\sigma(i)} - 1} \prod_{\check\sigma(i) \ge \sigma(\check\sigma(i))} Y_{\overline{c}_{\check\sigma(i)}} \\
&= R^{\operatorname{fix}\sigma} s^{\operatorname{exc}\sigma} \prod_{j < \sigma(j)} Y_{\overline{c}_j - 1} \prod_{j \ge \sigma(j)} Y_{\overline{c}_j} \\
&= R^{\operatorname{fix}\sigma} s^{\operatorname{exc}\sigma} \prod_{j < \sigma(j)} Y_{c_j + z_j - 1} \prod_{j \ge \sigma(j)} Y_{c_j + z_j} \\
&= R^{\operatorname{fix}\sigma} s^{\operatorname{exc}\sigma} Y_{(\sigma,c)}.
\end{aligned}$$

Consequently,

$$\sum_{w \in [0,r]^*} \psi_R(w) = \sum_{n \ge 0} \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \operatorname{des}\sigma \le r}} R^{\operatorname{fix}\sigma} s^{\operatorname{exc}\sigma} \sum_{c \in \mathrm{NIW}_n(r - \operatorname{des}\sigma)} Y_{(\sigma,c)}. \tag{5.1}$$

This achieves the proof of Theorem 1.5 by taking identity (1.12) into account. $\square$

For the proof of Theorem 1.6 we multiply both sides of (1.15) by t^r and sum over $r \ge 0$. We obtain:

$$\begin{aligned}
&\sum_{r \ge 0} t^r \frac{(1-s) H_{r+1}(RY)}{H_r(sY) - s H_r(Y)} \\
&\quad = \sum_{r \ge 0} t^r \sum_{n \ge 0} \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \operatorname{des}\sigma \le r}} R^{\operatorname{fix}\sigma} s^{\operatorname{exc}\sigma} \sum_{c \in \mathrm{NIW}_n(r - \operatorname{des}\sigma)} Y_{(\sigma,c)} \\
&\quad = \sum_{n \ge 0} \sum_{r \ge 0} t^r \sum_{0 \le j \le r} \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \operatorname{des}\sigma = r - j}} R^{\operatorname{fix}\sigma} s^{\operatorname{exc}\sigma} \sum_{c \in \mathrm{NIW}_n(j)} Y_{(\sigma,c)} \\
&\quad = \sum_{n \ge 0} \sum_{j \ge 0} t^j \sum_{r \ge j} t^{r-j} \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \operatorname{des}\sigma = r - j}} R^{\operatorname{fix}\sigma} s^{\operatorname{exc}\sigma} \sum_{c \in \mathrm{NIW}_n(j)} Y_{(\sigma,c)} \\
&\quad = \sum_{n \ge 0} \sum_{j \ge 0} t^j \sum_{k \ge 0} t^k \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \operatorname{des}\sigma = k}} R^{\operatorname{fix}\sigma} s^{\operatorname{exc}\sigma} \sum_{c \in \mathrm{NIW}_n(j)} Y_{(\sigma,c)}
\end{aligned}$$

$$= \sum_{n\geq 0}\sum_{\sigma\in\mathfrak{S}_n} R^{\operatorname{fix}\sigma} s^{\operatorname{exc}\sigma} t^{\operatorname{des}\sigma} \sum_{j\geq 0} t^j \sum_{c\in\mathrm{NIW}_n(j)} Y_{(\sigma,c)}$$

$$= \sum_{n\geq 0}\sum_{\sigma\in\mathfrak{S}_n} R^{\operatorname{fix}\sigma} s^{\operatorname{exc}\sigma} t^{\operatorname{des}\sigma} Y(\sigma;t). \quad \square$$

6. Specializations and q-Calculus

In Theorem 1.4, replace Y_j, Y'_j $(j\geq 0)$ by u. We get the identity

$$\sum_{w\in[0,r]^*} s^{\operatorname{dec} w} u^{\lambda w} = \frac{1-s}{(1-u)^{r+1}(1-us)^{-r} - s(1-u)}, \tag{6.1}$$

where λw denotes the length of w.

Now replace X_j $(j\geq 0)$ by us and the other variables by u in Theorem 1.2. We then recover the *classical* generating function for words by number of descents:

$$\sum_{w\in[0,r]^*} s^{\operatorname{des} w} u^{\lambda w} = \frac{1-s}{(1-u+us)^{r+1} - s}. \tag{6.2}$$

As was proved in our paper [9] (formula (1.15)), when multiplying (6.1) (and not (6.2)) by t^r and summing over $r\geq 0$ we get the generating function for the pair $(\operatorname{exc},\operatorname{des})$ over the symmetric groups:

$$\sum_{r\geq 0} t^r \sum_{w\in[0,r]^*} s^{\operatorname{dec} w} u^{\lambda w} = \sum_{n\geq 0} \frac{u^n}{(1-t)^{n+1}} \sum_{\sigma\in\mathfrak{S}_n} s^{\operatorname{exc}\sigma} t^{\operatorname{des}\sigma}. \tag{6.3}$$

Now, recall the traditional notation of the q-ascending factorial

$$(a;q)_n = \begin{cases} 1, & \text{if } n=0; \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & \text{if } n\geq 1. \end{cases}$$

Theorem 6.1. *The factorial generating function for the distributions of the vector* $(\operatorname{fix},\operatorname{exc},\operatorname{des},\operatorname{maj})$ *over the symmetric groups* $\mathfrak{S}_n$ *is given by*

$$\sum_{n\geq 0} \frac{u^n}{(t;q)_{n+1}} \sum_{\sigma\in\mathfrak{S}_n} R^{\operatorname{fix}\sigma} s^{\operatorname{exc}\sigma} t^{\operatorname{des}\sigma} q^{\operatorname{maj}\sigma} \tag{6.4}$$
$$= \sum_{r\geq 0} t^r \frac{1}{(uR;q)_{r+1}} \frac{(1-sq)\,(u;q)_r\,(usq;q)_r}{((u;q)_r - sq(usq;q)_r)}.$$

The theorem was proved in our previous paper [9] by means of the word-analog of the Kim-Zeng transformation [11] and the Gessel-Reutenauer standardization. Here, it is a simple consequence of Theorem 1.6 by applying the homomorphism ϕ generated by $\phi(Y_j) := uq^j$ $(j \ge 0)$ and $\phi(s) := sq$ to both sides of (1.18).

We proceed as follows. First, $\phi H_r(Y) = \prod_{0\le j\le r-1}(1-uq^j)^{-1} = 1/(u;q)_r$ and

$$\phi\frac{(1-s)H_{r+1}(RY)}{H_r(sY)-sH_r(Y)} = \frac{(1-sq)/(uR;q)_{r+1}}{1/(usq;q)_r - sq/(u;q)_r}$$
$$= \frac{(1-sq)(u;q)_r\,(usq;q)_r}{(uR;q)_{r+1}\,\big(((u;q)_r - sq(usq;q)_r\big)}.$$

Then, for $(\sigma,c)\in\mathfrak{S}_n\times \mathrm{NIW}_n$

$$\phi Y_{(\sigma,c)} = u^n q^{\mathrm{tot}\,c+\mathrm{tot}\,z-\mathrm{exc}\,\sigma} = q^{\mathrm{maj}\,\sigma-\mathrm{exc}\,\sigma}u^n q^{\mathrm{tot}\,c};$$

$$\phi Y(\sigma;t) = q^{\mathrm{maj}\,\sigma-\mathrm{exc}\,\sigma}u^n\sum_{j\ge 0}t^j\sum_{c\in\mathrm{NIW}_n(j)}q^{\mathrm{tot}\,c} = q^{\mathrm{maj}\,\sigma-\mathrm{exc}\,\sigma}\frac{u^n}{(t;q)_{n+1}};$$

so that

$$\phi\big(R\,\mathrm{fix}^\sigma\, s^{\mathrm{exc}\,\sigma}t^{\mathrm{des}\,\sigma}Y(\sigma;t)\big) = R^{\mathrm{fix}\,\sigma}(sq)^{\mathrm{exc}\,\sigma}t^{\mathrm{des}\,\sigma}q^{\mathrm{maj}\,\sigma-\mathrm{exc}\,\sigma}\frac{u^n}{(t;q)_{n+1}}.$$

Hence, the image of identity (1.18) under ϕ gives back (6.4). $\square$

There is still another proof, which we now describe as follows. In identity (1.4) make the substitutions $X_j\leftarrow usq^j$, $Y_j\leftarrow uq^j$, $Z_j\leftarrow usq^j$, $T_j\leftarrow uq^j$, $Y_j'\leftarrow uRq^j$, $T_j'\leftarrow uRq^j$. The weight $\psi(w)$ becomes $s^{\mathrm{dec}\,w}R^{\mathrm{inrec}\,w}u^{\lambda w}q^{\mathrm{tot}\,w}$ and (1.4) yields the identity

$$\text{(6.5)}\qquad \sum_{w\in[0,r]^*}s^{\mathrm{dec}\,w}R^{\mathrm{inrec}\,w}u^{\lambda w}q^{\mathrm{tot}\,w} = \frac{\dfrac{(us;q)_{r+1}}{(uR;q)_{r+1}}}{1-\displaystyle\sum_{0\le l\le r}\frac{(us;q)_l}{(u;q)_l}usq^l}.$$

Now, use the q-telescoping argument provided by Krattenthaler [12]:

$$\frac{(us;q)_l}{(u;q)_l}usq^l = \frac{sq}{1-sq}\Big(\frac{(us;q)_{l+1}}{(u;q)_l}-\frac{(us;q)_l}{(u;q)_{l-1}}\Big)\quad(1\le l\le r).$$

We obtain:

$$\text{(6.6)}\ \sum_{w\in[0,r]^*}s^{\mathrm{dec}\,w}R^{\mathrm{inrec}\,w}u^{\lambda w}q^{\mathrm{tot}\,w} = \frac{1}{(uR;q)_{r+1}}\frac{(1-sq)\,(u;q)_r\,(usq;q)_r}{((u;q)_r - sq(usq;q)_r)}.$$

The summation can also be made over the symmetric groups by using the Gessel-Reutenauer standardization $w\mapsto(\sigma,c)$. This time only the following properties are needed:

$$\mathrm{dec}\,w = \mathrm{exc}\,\sigma;\quad \mathrm{tot}\,w = \mathrm{maj}\,\sigma+\mathrm{tot}\,c;\quad \mathrm{inrec}\,w = \mathrm{fix}\,\sigma.$$

Multiply (6.6) by t^r and sum over $r \geq 0$. We get:

$$\sum_{r\geq 0} t^r \sum_{n\geq 0} \sum_{\substack{\sigma\in\mathfrak{S}_n \\ \operatorname{des}\sigma\leq r}} \sum_{c\in \mathrm{NIW}_n(r-\operatorname{des}\sigma)} s^{\operatorname{exc}\sigma} R^{\operatorname{fix}\sigma} u^n q^{\operatorname{maj}\sigma+\operatorname{tot}c}$$
$$= \sum_{r\geq 0} t^r \frac{1}{(uR;q)_{r+1}} \frac{(1-sq)\,(u;q)_r\,(usq;q)_r}{((u;q)_r - sq(usq;q)_r)}.$$

Following the same pattern as in the proof of Theorem 1.6 we again derive identity (6.4). □

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PERMUTATIONS WITH EXTREMAL NUMBER OF FIXED POINTS

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Abstract. We extend Stanley's work on alternating permutations with extremal number of fixed points in two directions: first, alternating permutations are replaced by permutations with a prescribed descent set; second, instead of simply counting permutations we study their generating polynomials by number of excedances. Several techniques are used: Désarménien's desarrangement combinatorics, Gessel's hook-factorization and the analytical properties of two new permutation statistics "DEZ" and "lec". Explicit formulas for the maximal case are derived by using symmetric function tools.

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Key words. Alternating permutations, derangements, desarrangements, descent set

1. Introduction

Let $J = \{j_1, j_2, \ldots, j_r\}_<$ be a set of integers arranged increasingly and let $\mathfrak{S}_J$ denote the set of all permutations on J. For each permutation $\sigma = \sigma(j_1)\sigma(j_2)\cdots\sigma(j_r) \in \mathfrak{S}_J$ define the *number of excedances*, the *number of fixed points* and the *descent set* of σ to be

$$\begin{aligned} \operatorname{fix}\sigma &= |\{i : 1 \le i \le r, \sigma(j_i) = j_i\}|, \\ \operatorname{exc}\sigma &= |\{i : 1 \le i \le r, \sigma(j_i) > j_i\}|, \\ \operatorname{DES}\sigma &= \{i : 1 \le i \le r-1, \sigma(j_i) > \sigma(j_{i+1})\}, \end{aligned} \tag{1}$$

respectively. A permutation without fixed point is called a *derangement*. When $J = [n] := \{1, 2, \ldots, n\}$, we recover the classical definitions. The set $\mathfrak{S}_{[n]}$ is abbreviated by $\mathfrak{S}_n$, and for $\sigma \in \mathfrak{S}_n$ we write σ_i for $\sigma(i)$. Our main results are the following Theorems 1 and 2.

Theorem 1. *Let J be a subset of $[n-1]$.*
(i) *If $\sigma \in \mathfrak{S}_n$ and $\operatorname{DES}\sigma = J$, then*

$$\operatorname{fix}\sigma \le n - |J|.$$

(ii) *Let $F_n(J)$ be the set of all permutations σ of order n such that $\operatorname{DES}\sigma = J$ and $\operatorname{fix}\sigma = n - |J|$. Furthermore, let $G(J)$ be the set of all derangements τ on J such that $\tau(i) > \tau(i+1)$ whenever i and $i+1$ belong to J. Then*

$$\sum_{\sigma \in F_n(J)} s^{\operatorname{exc}\sigma} = \sum_{\tau \in G(J)} s^{\operatorname{exc}\tau}.$$

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Example. For $n=8$ and $J=\{1,2,3,6\}$, there are two permutations in $F_n(J)$, both having two excedances: 74315628 and 74325618. On the other hand, there are two derangements in $G(J)$, both having two excedances: 6321 and 6312.

Theorem 2. *Let $D_0^J(n)$ be the set of all derangements σ on $[n]$ such that* $\operatorname{DES}\sigma = J$, *and let $D_1^J(n)$ be the set of all permutations σ on $[n]$ such that* $\operatorname{DES}\sigma = J$ *with exactly one fixed point. If J is a proper subset of $[n-1]$, then there is a polynomial $Q_n^J(s)$ with positive integral coefficients such that*

$$\sum_{\sigma\in D_0^J(n)} s^{\operatorname{exc}\sigma} - \sum_{\sigma\in D_1^J(n)} s^{\operatorname{exc}\sigma} = (s-1)Q_n^J(s).$$

Example. For $n=6$ and $J=\{1,3,4,5\}$ there are six derangements in $D_0^J(n)$:

$$216543, 316542, 416532, 436521, 546321, 645321;$$

and there are six permutations in $D_1^J(n)$:

$$326541, 426531, 516432, 536421, 615432, 635421.$$

The numbers of excedances are respectively 3, 3, 3, 4, 3, 3 and 3, 3, 2, 3, 2, 3, so that

$$\sum_{\sigma\in D_0^J(n)} s^{\operatorname{exc}\sigma} - \sum_{\sigma\in D_1^J(n)} s^{\operatorname{exc}\sigma} = (5s^3+s^4)-(4s^3+2s^2) = (s-1)(s^3+2s^2).$$

Theorem 1 extends Stanley's work on alternating permutations (that we explain next) with maximal number of fixed points, and Theorem 2 extends the corresponding minimal case. The extensions are in two directions: first, alternating permutations are replaced by permutations with a prescribed descent set; second, instead of simply counting permutations we study their generating polynomials by number of excedances.

A permutation $\pi=\pi_1\pi_2\cdots\pi_n\in\mathfrak{S}_n$ is said to be *alternating* (resp. *reverse alternating*) if $\pi_1>\pi_2<\pi_3>\pi_4<\dots$ (resp. if $\pi_1<\pi_2>\pi_3<\pi_4>\dots$); or equivalently, if $\operatorname{DES}\pi$ is $\{1,3,5,\dots\}\cap[n-1]$ (resp. $\{2,4,6,\dots\}\cap[n-1]$). Therefore, results on permutations with a prescribed descent set apply to alternating permutations. Let $D_k(n)$ be the set of permutations in $\mathfrak{S}_n$ with exactly k fixed points. Then $D_0(n)$ is the set of derangements of order n. Write $d_k(n)$ (resp. $d_k^*(n)$) for the number of alternating (resp. reverse alternating) permutations in $D_k(n)$. The next two corollaries are immediate consequences of Theorems 1 and 2.

Corollary 3 ([14], Conjecture 6.3)**.** *Let D_n denote the number of derangements. Then, for $n\ge 2$ we have*

$$d_n(2n)=d_{n+1}(2n+1)=d_{n+1}^*(2n+1)=d_{n+2}^*(2n+2)=D_n.$$

Corollary 4 ([14], Corollary 6.2)**.** *For $n\ge 2$ we have $d_0(n)=d_1(n)$ and $d_0^*(n)=d_1^*(n)$.*

Stanley enumerated $D_k(n)$ and came up with Corollaries 3 and 4 on alternating permutations with extremal number of fixed points. He then asked for combinatorial proofs of them. This is the motivation of the paper. The results in Corollary 3, conjectured by Stanley, was recently proved by Chapman and Williams [16] in two ways, one directly and the other using the newly developed concept of permutation tableaux [15]. In Section 3 we give a direct proof of a generalized form of Corollary 3. Corollary 4 is actually a special case of a more general result due to Gessel and Reutenauer, which itself can be derived from Theorem 2 by setting $s = 1$, as stated in the next corollary.

Corollary 5 ([10], Theorem 8.3)**.** *Let J be a proper subset of $[n-1]$. Then, the number of derangements in $\mathfrak{S}_n$ with descent set J is equal to the number of permutations in $\mathfrak{S}_n$ with exactly one fixed point and descent set J.*

The paper is organized as follows. In Section 2 we give the proof of Theorem 1 that contains the results for the maximal case. Section 3 includes a direct proof of an extension of Corollary 3. Section 4 introduces the necessary part of Gessel and Reutenauer's work for enumerating the maximal case. Section 5 is devoted to the proof of Theorem 2 dealing with the minimal case. We conclude the paper by making several remarks of analytic nature (see Section 6). In particular, Corollary 19, proved combinatorially, should deserve an analytic proof. Several techniques are used: Désarménien's desarrangement combinatorics [1], Gessel's hook-factorization [9] and the analytical properties of two new permutation statistics "DEZ" and "lec" [5, 6].

2. Permutations with maximal number of fixed points

Our task in this section is to prove Theorem 1. The proof relies on the properties of the new statistic "DEZ" introduced by Foata and Han [5]. For a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathfrak{S}_n$ let $\sigma^0 = \sigma_1^0\sigma_2^0\cdots\sigma_n^0$ be the word derived from σ by replacing each fixed point $\sigma_i = i$ by 0. The set-valued statistic "DEZ" is defined by

$$\operatorname{DEZ}\sigma = \operatorname{DES}\sigma^0 := \{i : 1 \le i \le n-1, \sigma_i^0 > \sigma_{i+1}^0\}.$$

For example, if $\sigma = 8\,2\,1\,3\,5\,6\,4\,9\,7$, then $\operatorname{DES}\sigma = \{1,2,6,8\}$, $\sigma^0 = 8\,0\,1\,3\,0\,0\,4\,9\,7$ and $\operatorname{DEZ}\sigma = \operatorname{DES}\sigma^0 = \{1,4,8\}$. The basic property of the statistic "DEZ" is given in the following proposition.

Proposition 6 ([5], Theorem 1.4)**.** *The two three-variable statistics* $(\operatorname{fix}, \operatorname{exc}, \operatorname{DEZ})$ *and* $(\operatorname{fix}, \operatorname{exc}, \operatorname{DES})$ *are equi-distributed on the symmetric group $\mathfrak{S}_n$.*

More precisely, Proposition 6 asserts that there is a bijection $\Phi : \mathfrak{S}_n \mapsto \mathfrak{S}_n$ such that

$$\operatorname{fix}\pi = \operatorname{fix}\Phi(\pi), \quad \operatorname{exc}\pi = \operatorname{exc}\Phi(\pi), \quad \operatorname{DES}\pi = \operatorname{DEZ}\Phi(\pi), \text{ for all } \pi \in \mathfrak{S}_n.$$

By Proposition 6 Theorem 1 is equivalent to the following Theorem 1′, where the statistic "DES" has been replaced by "DEZ".

Theorem 1′. *Let J be a subset of $[n-1]$.*
(i) *If $\sigma \in \mathfrak{S}_n$ and $\operatorname{DEZ}\sigma = J$, then*

$$\operatorname{fix}\sigma \le n - |J|.$$

(ii) *Let $F_n'(J)$ be the set of all permutations σ of order n such that $\operatorname{DEZ}\sigma = J$ and $\operatorname{fix}\sigma = n - |J|$. Furthermore, let $G(J)$ be the set of all derangements τ on J such that*

$\tau(i) > \tau(i+1)$ *whenever* i *and* $i+1$ *belong to* J. *Then*

$$\sum_{\sigma \in F_n'(J)} s^{\operatorname{exc} \sigma} = \sum_{\tau \in G(J)} s^{\operatorname{exc} \tau}.$$

Proof of Theorem 1′. Let σ be a permutation such that $\operatorname{DEZ} \sigma = J$ and let $i \in J$. Then $\sigma_i^0 > \sigma_{i+1}^0 \geq 0$, so that i is not a fixed point of σ. It follows that σ has at least $|J|$ non-fixed points. This proves (i).

Now, consider the case where σ has exactly $n - |J|$ fixed points. Then J is the set of all the non-fixed points of σ. By removing the fixed points from σ we obtain a derangement τ on J. If $i, i+1 \in J$, then $\tau(i) = \sigma(i) > \sigma(i+1) = \tau(i+1)$. It follows that $\tau \in G(J)$. On the other hand, take any derangement $\tau \in G(J)$ and let σ be the permutation defined by

$$\sigma(i) = \begin{cases} \tau(i), & \text{if } i \in J, \\ i, & \text{if } i \notin J. \end{cases}$$

Then $\operatorname{DEZ} \sigma = J$. It is easy to see that $\sigma \in F_n'(J)$ and $\operatorname{exc} \sigma = \operatorname{exc} \tau$. This proves the second part of Theorem 1′. $\square$

Example. Suppose $n = 8$ and $J = \{1, 2, 3, 6\}$. Let us search for the permutations $\sigma \in \mathfrak{S}_8$ such that $\operatorname{fix} \sigma = 8 - |J| = 4$ and $\operatorname{DEZ} \sigma = J$. There are two derangements τ in $G(J)$, namely, 6321 and 6312, both having two excedances, so that the two corresponding elements σ in $F_n'(J)$ are 63245178 and 63145278, both having two excedances.

Remarks. (i) For permutations with descent set J it is easy to show that the maximum number of fixed points is $n - |J|$, except when J consists of an odd number of consecutive integers. In the latter exceptional case the only decreasing permutation has exactly one fixed point and therefore is not a derangement.

(ii) The first part of Theorem 1 can also be proved directly by using the fact that in any consecutive decreasing subsequence of π, say $\pi_i > \pi_{i+1} > \cdots > \pi_{i+k}$, there is at most one fixed point in $\{i, i+1, \ldots, i+k\}$. However the "DEZ" statistic is an essential tool in the proof of the second part.

3. An extension of Corollary 3

Stanley's conjectured result in Corollary 3 was first proved by Williams [16] using the newly developed concept of permutation tableaux. A direct proof without using permutation tableaux was later included in her updated version with Chapman. Our direct proof was independently derived just after Williams' first proof. It has the advantage of automatically showing the following extension (Proposition 7). We only give the generalized form for $d_n(2n) = D_n$, since the other cases are similar. All of the three proofs are bijective, and the bijections are all equivalent. Note that Proposition 7 is still a corollary of Theorem 1.

Proposition 7. *The number of alternating permutations in* $\mathfrak{S}_{2n}$ *with* n *fixed points and* k *excedances is equal to the number of derangements in* $\mathfrak{S}_n$ *with* k *excedances.*

Let π be an alternating permutation. Then, each doubleton $\{\pi_{2i-1}, \pi_{2i}\}$ contains at most one fixed point. This proves the following lemma.

Lemma 8. *Each alternating permutation $\pi \in \mathfrak{S}_n$ has at most $\lceil n/2 \rceil$ fixed points. When this maximum is reached, either $2i-1$, or $2i$ is a fixed point of π $(2 \le 2i \le n+1)$.*

When the underlying set of the permutation π is not necessarily $[n]$, we use $\pi(i)$ instead of π_i for convenience. An integer i is called an *excedance*, a *fixed point*, or a *subcedance* of π if $\pi(i) > i$, $\pi(i) = i$, or $\pi(i) < i$, respectively.

Proof of Proposition 7. Let $\pi \in \mathfrak{S}_{2n}$ be alternating and have exactly n fixed points. It follows from Lemma 8 that for each i we have the following property: either $2i-1$ is a fixed point and $2i$ a subcedance, or $2i-1$ is an excedance and $2i$ a fixed point. Conversely, if the property holds, the permutation π is necessarily alternating, because $\pi(2i) \le 2i < 2i+1 \le \pi(2i+1)$; $\pi(2i-1) \ge 2i-1 \ge \pi(2i)-1$. Those inequalities imply that $\pi(2i-1) > \pi(2i)$, since $2i-1$ and $2i$ cannot be both fixed points.

By removing all fixed points of π we obtain a derangement σ on an n-subset of $[2n]$. The standardization of σ, which consists of replacing the i-th smallest element of σ by i, yields a derangement τ on $[n]$. We claim that the map $\varphi : \pi \mapsto \tau$ is the desired bijection. Since the standardization preserves excedances, subcedances and fixed points, it maps one element of $\{\pi(2i-1), \pi(2i)\}$ to $\tau(i)$. It follows that $\tau(i) > i$ if and only if $2i-1$ is an excedance and $2i$ is a fixed point of π, and that $\tau(i) < i$ if and only if $2i-1$ is a fixed point and $2i$ is a subcedance of π. Thus, the set of all fixed points of π can be constructed from τ. The map φ is then reversible.

The proposition then follows since the bijection preserves the number of excedances. $\square$

Example. Let $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & \bar{2} & 6 & \bar{4} & \bar{5} & 1 & 10 & \bar{8} & \bar{9} & 7 \end{pmatrix}$. Removing all the fixed points gives $\sigma = \begin{pmatrix} 1 & 3 & 6 & 7 & 10 \\ 3 & 6 & 1 & 10 & 7 \end{pmatrix}$, standardized to $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$. Conversely, τ has excedances at positions $1, 2, 4$ and subcedances at positions $3, 5$. This implies that $2, 4, 8$ and $5, 9$ are fixed points of π and hence we can construct π. Furthermore, we have $\operatorname{exc} \pi = \operatorname{exc} \sigma = \operatorname{exc} \tau = 3$.

4. Enumeration for the maximal case

In this section we will use Theorem 1 to enumerate the number of permutations with a prescribed descent set and having the maximal number of fixed points. Every descent set $J \subseteq [n-1]$ can be partitioned into blocks of consecutive integers, such that numbers from different blocks differ by at least 2. Let $J^b = (a_1, a_2, \ldots, a_k)$ denote the sequence of the size of the blocks. For instance, if $J = \{1, 2, 3, 6\}$, then $1, 2, 3$ form a block and 6 itself forms another block. Hence $J^b = (3, 1)$. Let M_J denote the number of derangements in $\mathfrak{S}_n$ with descent set J having $n - |J|$ fixed points. By Theorem 1 the number M_J depends only on J^b. Thus, we can denote M_J by $M(a_1, a_2, \ldots, a_k)$.

Theorem 9. *The number $M(a_1, \ldots, a_k)$ is the coefficient of $x_1^{a_1} \cdots x_k^{a_k}$ in the expansion of*

$$\frac{1}{(1+x_1)(1+x_2)\cdots(1+x_k)(1-x_1-x_2-\cdots-x_k)}.$$

An immediate consequence of Theorem 9 is the following Corollary 10, which says that $M(a_1, a_2, \ldots, a_k)$ is symmetric in the a_i's.

Corollary 10. *For each permutation $\tau \in \mathfrak{S}_k$ we have*

$$M(a_1, a_2, \ldots, a_k) = M(a_{\tau_1}, a_{\tau_2}, \ldots, a_{\tau_k}).$$

For example, $M(3,1)$ counts two derangements 4312 and 4321; $M(1,3)$ counts two derangements 3421 and 4321. This symmetry seems not easy to prove directly. Using Theorem 9 an explicit formula for $M(a_1, a_2, \ldots, a_k)$ can be obtained when $k = 1, 2$. We have $M(a) = 1$ if a is even, and $M(a) = 0$ if a is odd; also

$$M(a,b) = \sum_{j=2}^{a+b} \sum_{i=0}^{j} (-1)^j \binom{a+b-j}{a-i}.$$

To prove Theorem 9 we need some notions from [10], where Gessel and Reutenauer represented the number of permutations with given cycle structure and descent set by the scalar product of two special characters of the symmetric group introduced by Foulkes [7, 8]. Their results were also key ingredients in [14] for the enumeration of alternating permutations by number of fixed points. In what follows, we assume the basic knowledge of symmetric functions (see, e.g., [11, 12, 13]). The scalar product $\langle\ ,\ \rangle$ of two symmetric functions is a bilinear form defined for all partitions λ and μ by

$$\langle m_\lambda, h_\mu \rangle = \langle h_\mu, m_\lambda \rangle = \delta_{\lambda\mu}, \tag{2}$$

where m_λ is the monomial symmetric function, h_μ is the complete symmetric function, and δ is the usual Kronecker symbol. Moreover, if ω is the homomorphism defined by $\omega e_i = h_i$ and $\omega h_i = e_i$, where e_i is the elementary symmetric function, then for any symmetric functions f and g we have

$$\langle f, g \rangle = \langle \omega f, \omega g \rangle. \tag{3}$$

Associate the function

$$S_J = \sum_{\mathrm{DES}\, w = J} x_{w_1} x_{w_2} \cdots x_{w_n} \tag{4}$$

with each subset $J \subseteq [n-1]$, where the sum ranges over all words on positive integers with descent set J. We claim that S_J is a symmetric function whose shape is a border strip (see [13, p. 345]). In particular, $S_{[n-1]}$ is equal to e_n, the elementary symmetric function of order n. On the other hand, every partition λ of n has an associate symmetric function L_λ related to a Lie representation. The definition of L_λ is omitted here (see

[10]); just remember that the symmetric function corresponding to derangements of order n is given by

$$\mathcal{D}_n = \sum_{\lambda} L_\lambda = \sum_{j=0}^{n} (-1)^j e_j h_1^{n-j}, \tag{5}$$

where the sum ranges over all partitions λ having no part equal to 1 [10, Theorem 8.1]. We need the following result from [10] for our enumeration.

Proposition 11 (Gessel-Reutenauer)**.** *The number of permutations having descent set J and cycle structure λ is equal to the scalar product of the symmetric functions S_J and L_λ.*

Proof of Theorem 9. For each fixed integer sequence $(a_1, a_2, \ldots, a_k)$ let $s_i = a_1 + a_2 + \cdots + a_i$ for $i = 1, \ldots, k$ and $\ell = s_k$. Then $M(a_1, a_2, \ldots, a_k)$ is the number of *derangements* $\pi \in \mathfrak{S}_\ell$ such that s_i with $i = 1, 2, \ldots, k-1$ may or may not be a descent of π, and such that all the other numbers in $[\ell - 1]$ are descents of π. There is then a set T of 2^{k-1} descent sets J to consider, depending on whether each s_i is a descent or not (for $i = 1, \ldots, k-1$). By Proposition 11 and linearity we have

$$M(a_1, a_2, \ldots, a_k) = \langle \sum_{J \in T} S_J, \mathcal{D}_\ell \rangle. \tag{6}$$

From (4) it follows that

$$\sum_{J \in T} S_J = \sum_{\mathrm{DES}\, w \in T} x_{w_1} x_{w_2} \cdots x_{w_n} = \sum_{[\ell-1] \setminus \{s_1, s_2, \ldots, s_{k-1}\} \subseteq \mathrm{DES}\, w} x_{w_1} x_{w_2} \cdots x_{w_n}.$$

Each word w occurring in the latter sum is the juxtaposition product $w = u^{(1)} u^{(2)} \cdots u^{(k)}$, where each $u^{(i)}$ is a decreasing word of length a_i $(i = 1, 2, \ldots, k)$. Hence $\sum_{J \in T} S_J = e_{a_1} e_{a_2} \cdots e_{a_k}$. In (6) replace $\sum_{J \in T} S_J$ by $e_{a_1} e_{a_2} \cdots e_{a_k}$ and $\mathcal{D}_\ell$ by the second expression in (5). We obtain

$$M(a_1, a_2, \ldots, a_k) = \langle e_{a_1} e_{a_2} \cdots e_{a_k}, \sum_{j=0}^{n} (-1)^j e_j h_1^{n-j} \rangle.$$

The image under ω yields

$$\begin{aligned} M(a_1, a_2, \ldots, a_k) &= \langle \omega e_{a_1} \cdots e_{a_k}, \omega \sum_{j=0}^{\ell} (-1)^j e_j h_1^{\ell-j} \rangle \\ &= \langle h_{a_1} \cdots h_{a_k}, \sum_{j=0}^{\ell} (-1)^j h_j e_1^{\ell-j} \rangle. \end{aligned}$$

Notice that $\sum_{j=0}^{\ell} (-1)^j h_j e_1^{\ell-j}$ is the coefficient of u^ℓ in

$$\Big(\sum_j h_j (-u)^j\Big)\Big(\sum_i e_1^i u^i\Big) = \frac{1}{(1 + x_1 u)(1 + x_2 u) \cdots (1 + x_k u)(1 - (x_1 + x_2 + \cdots + x_k) u)}.$$

It follows from (2) that $M(a_1, \ldots, a_k)$ is the coefficient of $x_1^{a_1} \cdots x_k^{a_k} u^\ell$ in the expansion of the above fraction. $\square$

5. Permutations with 0 or 1 fixed points

Our objective in this section is to prove Theorem 2. We will establish a chain of equivalent or stronger statements, leading to the final easy one. Further notations are needed. Let $w = w_1 w_2 \cdots w_n$ be a word on the letters $1, 2, \ldots, m$, each letter appearing at least once. The set-statistic $\mathrm{IDES}\, w$ is defined to be the set of all i such that the rightmost i appears to the right of the rightmost $i+1$ in w. Note that if π is a permutation on $[n]$, then $\mathrm{IDES}\, \pi = \mathrm{DES}\, \pi^{-1}$. For every proper subset J of $[n-1]$ let $\mathfrak{S}_n^J$ be the set of permutations $\sigma \in \mathfrak{S}_n$ with $\mathrm{IDES}\, \sigma = J$. Note the difference with the notation of $D_k^J(n)$ for $k = 0, 1$. We will see that it is easier to deal with IDES than with DES directly.

A word $w = w_1 w_2 \cdots w_n$ is said to be a *desarrangement* if $w_1 > w_2 > \cdots > w_{2k}$ and $w_{2k} \le w_{2k+1}$ for some $k \ge 1$. By convention, $w_{n+1} = \infty$. We may also say that the *leftmost trough* of w occurs at an *even* position [6]. This notion was introduced, for permutations, by Désarménien [1] and elegantly used in a subsequent paper [2]. A further refinement is due to Gessel [9]. A desarrangement $w = w_1 w_2 \cdots w_n$ is called a *hook*, if $n \ge 2$ and $w_1 > w_2 \le w_3 \le \cdots \le w_n$. Every nonempty word w on the letters $1, 2, 3, \ldots$ can be written uniquely as a product $u h_1 h_2 \cdots h_k$, where u is a *weakly increasing* word (possibly empty) and each h_i is a hook. This factorization is called the *hook-factorization* of w [6]. For permutations it was already introduced by Gessel [9]. For instance, the hook-factorization of the following word is indicated by vertical bars:

$$w = \mid 1\,2\,4\,5 \mid 6\,4\,5\,6 \mid 4\,1\,3 \mid 6\,5 \mid 5\,4 \mid 6\,1\,1\,4 \mid 5\,1\,1 \mid .$$

Let $u h_1 h_2 \cdots h_k$ be the hook factorization of the word w. The statistic $\mathrm{pix}\, w$ is defined to be the length of u, and the statistic $\mathrm{lec}\, w$ is defined, in terms of inversion statistics "inv", by the sum [6]

$$\mathrm{lec}\, w := \sum_{i=1}^{k} \mathrm{inv}(h_i).$$

In the previous example, $\mathrm{pix}\, w = |1245| = 4$ and $\mathrm{lec}\, w = \mathrm{inv}(6456) + \mathrm{inv}(413) + \mathrm{inv}(65) + \mathrm{inv}(54) + \mathrm{inv}(6114) + \mathrm{inv}(511) = 2 + 2 + 1 + 1 + 3 + 2 = 11$.

For each permutation σ let $\mathrm{iexc}\, \sigma = \mathrm{exc}\, \sigma^{-1}$. The next proposition was proved in Foata and Han [6].

Proposition 12. *The two three-variable statistics* $(\mathrm{iexc}, \mathrm{fix}, \mathrm{IDES})$ *and* $(\mathrm{lec}, \mathrm{pix}, \mathrm{IDES})$ *are equi-distributed on the symmetric group* $\mathfrak{S}_n$.

Let $K_0^J(n)$ denote the set of all desarrangements in $\mathfrak{S}_n^J$, and $K_1^J(n)$ the set of all permutations in $\mathfrak{S}_n^J$ with exactly one pixed point. Since the map $\sigma \to \sigma^{-1}$ preserves the number of fixed points, Theorem 2 is equivalent to asserting that

$$\sum_{\substack{\sigma \in D_0(n) \\ \mathrm{IDES}(\sigma) = J}} - \sum_{\substack{\sigma \in D_1(n) \\ \mathrm{IDES}(\sigma) = J}} = (s-1) Q_n^J(s).$$

Then by Proposition 12 this is equivalent to the following Theorem 2^a.

Theorem 2^a. *We have*

$$\sum_{\sigma\in K_0^J(n)} s^{\operatorname{lec}\sigma} - \sum_{\sigma\in K_1^J(n)} s^{\operatorname{lec}\sigma} = (s-1)Q_n^J(s),$$

where $Q_n^J(s)$ is a polynomial with positive integral coefficients.

The following lemma enables us to prove a stronger result.

Lemma 13. *Let $w = w_1w_2\cdots w_n$ be a desarrangement such that* $\operatorname{IDES} w \neq \{1,2,\ldots,n-1\}$ *and let $w' = w_nw_1w_2\cdots w_{n-1}$. Then, either* $\operatorname{lec} w' = \operatorname{lec} w$, *or* $\operatorname{lec} w' = \operatorname{lec} w - 1$.

Proof. Several cases are to be considered. Say that w belongs to type A if $\operatorname{lec}(w') = \operatorname{lec}(w)$, and say that w belongs to type B if $\operatorname{lec}(w') = \operatorname{lec}(w) - 1$.

Since w is a desarrangement, we may assume $w_1 > w_2 > \cdots > w_{2k} \le w_{2k+1}$ for some k. It follows that w' has one pixed point. Let $h_1\cdots h_k$ be the hook-factorization of w. Then the hook-factorization of w' must have the form $w_n|h'_1\cdots h'_\ell$. Thus, when computing $\operatorname{lec}(w')$, we can simply omit w_n. This fact will be used when checking the various cases. The reader is invited to look at Figures 1–3, where the letters b, c, x, y, z play a critical role.

(1) If the rightmost hook h_k has at least three elements, as shown in Figure 1, then $b \le c$ belongs to type A and $b > c$ belongs to type B. This is because the only possible change for "lec" must come from an inversion containing c. Furthermore, (b, c) forms an inversion for type B and does not form an inversion for type A.

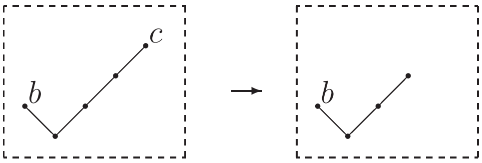

FIGURE 1. Transformation for case 1.

(2) Suppose the rightmost hook h_k has two elements $b > c$.
   (a) If there is a hook xy followed by several decreasing hooks of length 2 with $y \le z$, as shown in Figure 2, then $x \le z$ belongs to type B and $x > z$ belongs to type A.

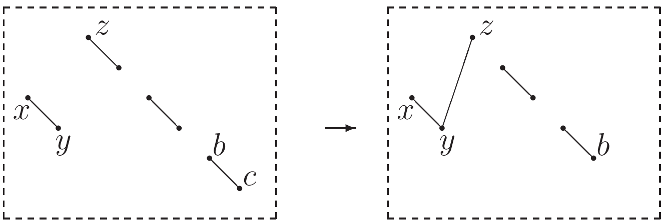

FIGURE 2. Transformation for case 2a.

   (b) If there is a hook of length at least 3, followed by several decreasing hooks of length 2, then (see Figure 3)

(i) $x > y$ belongs to type B and $x \le y$ belongs to type A in case $y > z$;
(ii) $x \le z$ belongs to type B and $x > z$ belongs to type A in case $y \le z$.

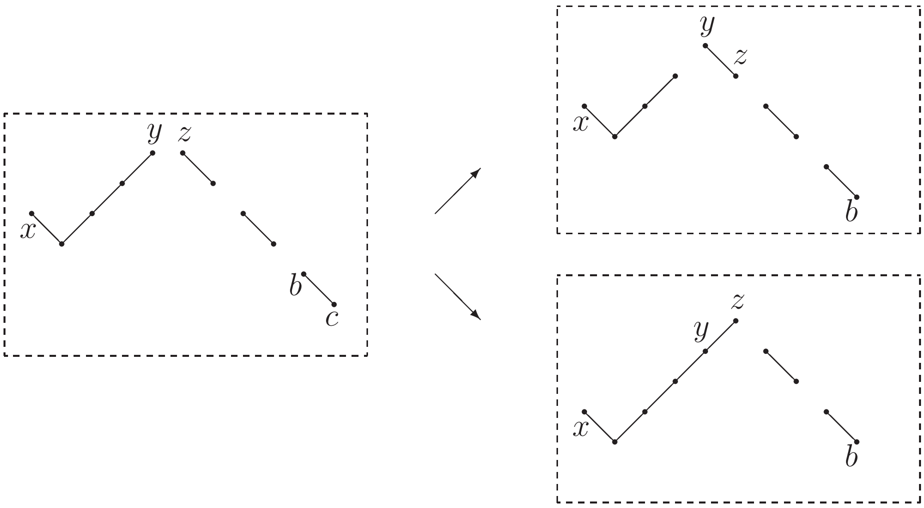

FIGURE 3. Transformations for case 2b.

This achieves the proof of the lemma. $\square$

With the notations of Lemma 13 we say that a desarrangement w is in class A_0 if $\operatorname{lec} w' = \operatorname{lec} w$ and in class B_0 if $\operatorname{lec} w' = \operatorname{lec} w - 1$. A word $w = w_1 w_2 w_3 \cdots w_n$ is said to be in class A_1 (resp. in class B_1) if the word $w_2 w_3 \cdots w_n w_1$ is in class A_0 (resp. in class B_0). Notice that a word in class A_1 or B_1 has exactly one pixed point. Then, Theorem 2^a is a consequence of the following theorem.

Theorem 2^b. *We have*

$$\sum_{\sigma \in \mathfrak{S}_n^J \cap A_0} s^{\operatorname{lec}\sigma} = \sum_{\sigma \in \mathfrak{S}_n^J \cap A_1} s^{\operatorname{lec}\sigma} \quad \textit{and} \quad \sum_{\sigma \in \mathfrak{S}_n^J \cap B_0} s^{\operatorname{lec}\sigma} = s \sum_{\sigma \in \mathfrak{S}_n^J \cap B_1} s^{\operatorname{lec}\sigma}.$$

Let $\mathfrak{S}_n^{\subseteq J}$ be the set of all permutations σ of order n such that $\operatorname{IDES}\sigma \subseteq J$. By the inclusion-exclusion principle, Theorem 2^b is equivalent to the following theorem.

Theorem 2^c. *We have*

$$\sum_{\sigma \in \mathfrak{S}_n^{\subseteq J} \cap A_0} s^{\operatorname{lec}\sigma} = \sum_{\sigma \in \mathfrak{S}_n^{\subseteq J} \cap A_1} s^{\operatorname{lec}\sigma} \quad \textit{and} \quad \sum_{\sigma \in \mathfrak{S}_n^{\subseteq J} \cap B_0} s^{\operatorname{lec}\sigma} = s \sum_{\sigma \in \mathfrak{S}_n^{\subseteq J} \cap B_1} s^{\operatorname{lec}\sigma}.$$

If $J = \{j_1, j_2, \ldots, j_{r-1}\} \subseteq [n-1]$, define a composition $\mathbf{m} = (m_1, m_2, \ldots, m_r)$ by $m_1 = j_1, m_2 = j_2 - j_1, \ldots, m_{r-1} = j_{r-1} - j_{r-2}, m_r = n - j_{r-1}$. Let $R(\mathbf{m})$ be the set of all rearrangements of $1^{m_1} 2^{m_2} \cdots r^{m_r}$. We construct a bijection ϕ from $R(\mathbf{m})$ to $\mathfrak{S}_n^{\subseteq J}$ by means of the classical *standardization* of words. Let $w \in R(\mathbf{m})$ be a word. From left to right label the letters 1 in w by $1, 2, \ldots, m_1$, then label the letters 2 in w by

$m_1+1, m_1+2, \ldots, m_1+m_2$, and so on. Then the standardization of w, denoted by $\sigma = \phi(w)$, is the permutation obtained by reading those labels from left to right. It is easy to see that ϕ is reversible and $\mathrm{IDES}\,\sigma \subseteq J$ if and only if $w \in R(\mathbf{m})$ (see [3, 6]). Moreover, the permutation σ and the word w have the same hook-factorization *type*. This means that if $ah_1h_2\ldots h_s$ (resp. $bp_1p_2\ldots p_k$) is the hook-factorization of σ (resp. hook-factorization of w), then $k = s$ and $|a| = |b|$. For each $1 \le i \le k$ we have $|h_i| = |p_i|$ and $\mathrm{inv}(h_i) = \mathrm{inv}(p_i)$. Hence $\mathrm{lec}\, w = \mathrm{lec}\,\sigma$ and $\mathrm{pix}\, w = \mathrm{pix}\,\sigma$. Furthermore, σ is in class A_0, A_1, B_0 or B_1 if and only if w is in the same class. Theorem 2^c is equivalent to the next theorem, whose proof follows from the definition of the classes A_0, A_1, B_0, B_1 and Lemma 13.

Theorem 2^d. *We have*

$$\sum_{\sigma\in R(\mathbf{m})\cap A_0} s^{\mathrm{lec}\,\sigma} = \sum_{\sigma\in R(\mathbf{m})\cap A_1} s^{\mathrm{lec}\,\sigma} \quad \textit{and} \quad \sum_{\sigma\in R(\mathbf{m})\cap B_0} s^{\mathrm{lec}\,\sigma} = s \sum_{\sigma\in R(\mathbf{m})\cap B_1} s^{\mathrm{lec}\,\sigma}.$$

The following variation of Theorem 2 follows from Theorem 2^b, but cannot be derived from Theorem 2 directly.

Theorem 14. *We have*

$$\sum_{\sigma\in D_0^J(n)} s^{\mathrm{iexc}\,\sigma} - \sum_{\sigma\in D_1^J(n)} s^{\mathrm{iexc}\,\sigma} = (s-1)Q_n^J(s)$$

for some polynomial $Q_n^J(s)$ with positive integral coefficients.

6. Further remarks

A combinatorial proof of Corollary 5 can be made by using the methods developed in the preceding section. However this proof does not need the concept of "hook" and the statistic "lec". We only list the equivalent statements, leaving the details to the reader.

Theorem 15. *Let J be a proper subset of $[n-1]$. The following statements are equivalent to Corollary 5:*

1. *The number of derangements in $\mathfrak{S}_n^J$ is equal to the number of permutations in $\mathfrak{S}_n^J$ with exactly one fixed point.*
2. *The number of desarrangements in $\mathfrak{S}_n^J$ is equal to the number of permutations in $\mathfrak{S}_n^J$ with exactly one pixed point.*
3. *The number of desarrangements in $\mathfrak{S}_n^{\subseteq J}$ is equal to the number of permutations in $\mathfrak{S}_n^{\subseteq J}$ with exactly one pixed point.*
4. *The number of desarrangements in $R(\mathbf{m})$ is equal to the number of words in $R(\mathbf{m})$ with exactly one pixed point.*

We remark that the equivalence of (1) and (2) also follows from a result of Désarménien and Wachs [2, 3]: the two bi-variable statistics (fix, IDES) and (pix, IDES) are equidistributed on the symmetric group $\mathfrak{S}_n$.

The statistics "des" and "maj" are determined by "DES": $\mathrm{des}\,\pi = \#\,\mathrm{DES}\,\pi$ and $\mathrm{maj}\,\pi = \sum_{i\in\mathrm{DES}\,\pi} i$ for $\pi \in \mathfrak{S}_n$. By using Theorem 2 for each proper subset J of $[n-1]$ and by checking the case $J = [n-1]$ directly, we have the following result.

Theorem 16. *There is a polynomial $Q_n(s,t,q)$ with positive integral coefficients such that*

$$\sum_{\sigma\in D_0(n)} s^{\operatorname{exc}\sigma}t^{\operatorname{des}\sigma}q^{\operatorname{maj}\sigma} - \sum_{\sigma\in D_1(n)} s^{\operatorname{exc}\sigma}t^{\operatorname{des}\sigma}q^{\operatorname{maj}\sigma} = (s-1)Q_n(s,t,q) + r_n(s,t,q)$$

where $r_{2k}(s,t,q) = s^k t^{2k-1} q^{k(2k-1)}$ for $k \ge 1$ and $r_{2k+1}(s,t,q) = -s^k t^{2k} q^{k(2k+1)}$ for $k \ge 0$.

A related result is the following, where we use the standard notation for q-series:

$$(z;q)_m = (1-z)(1-zq)\cdots(1-zq^{m-1}).$$

Proposition 17 ([6], Theorem 1.1). *Let $(A_n(s,t,q,Y))_{n\ge 0}$ be the sequence of polynomials in four variables, whose factorial generating function is given by*

$$\sum_{r\ge 0} t^r \frac{(1-sq)\,(u;q)_r\,(usq;q)_r}{((u;q)_r - sq(usq;q)_r)(uY;q)_{r+1}} = \sum_{n\ge 0} A_n(s,t,q,Y)\frac{u^n}{(t;q)_{n+1}}.$$

Then $A_n(s,t,q,Y)$ is the generating polynomial for $\mathfrak{S}_n$ according to the four-variable statistic (exc, des, maj, fix)*. In other words,*

$$A_n(s,t,q,Y) = \sum_{\sigma\in\mathfrak{S}_n} s^{\operatorname{exc}\sigma}t^{\operatorname{des}\sigma}q^{\operatorname{maj}\sigma}Y^{\operatorname{fix}\sigma}.$$

Since $\sum_{\sigma\in D_0(n)} s^{\operatorname{exc}\sigma}t^{\operatorname{des}\sigma}q^{\operatorname{maj}\sigma}$ is simply $A_n(s,t,q,0)$ and $\sum_{\sigma\in D_1(n)} s^{\operatorname{exc}\sigma}t^{\operatorname{des}\sigma}q^{\operatorname{maj}\sigma}$ is equal to the coefficient of Y in $A_n(s,t,q,Y)$, Theorem 16 and Proposition 17 imply the following theorem.

Theorem 18. *There is a sequence of polynomials $(Q_n(s,t,q))_{n\ge 0}$ with positive integral coefficients such that*

$$\sum_{r\ge 0} t^r \left(1 - u\frac{1-q^{r+1}}{1-q}\right) \frac{(1-sq)\,(u;q)_r\,(usq;q)_r}{((u;q)_r - sq(usq;q)_r)} - \frac{1}{1-t}$$
$$= (s-1)\sum_{n\ge 1} Q_n(s,t,q)\frac{u^n}{(t;q)_{n+1}} + r(s,t,q),$$

where

$$r(s,t,q) = \sum_{k\ge 1} s^k t^{2k-1} q^{k(2k-1)} \frac{u^{2k}}{(t;q)_{2k+1}} - \sum_{k\ge 0} s^k t^{2k} q^{k(2k+1)} \frac{u^{2k+1}}{(t;q)_{2k+2}}.$$

In the case of $t=1$ and $q=1$ the above theorem yields the following corollary.

Corollary 19. *For each $n \ge 0$ let $Q_n(s)$ be the coefficient of $u^n/n!$ in the Taylor expansion of*

$$H(s) = \frac{u-1}{se^{us} - s^2 e^u} - \frac{1}{2s\sqrt{s}}\Big(\frac{e^{u\sqrt{s}}}{\sqrt{s}+1} + \frac{e^{-u\sqrt{s}}}{\sqrt{s}-1}\Big),$$

that is

$$H(s) = \frac{u^3}{3!} + (s+3)\frac{u^4}{4!} + (s^2+17s+4)\frac{u^5}{5!} + (s^3+46s^2+80s+5)\frac{u^6}{6!} + \cdots + Q_n(s)\frac{u^n}{n!} + \cdots$$

Then, the coefficients $Q_n(s)$ are polynomials in s with positive integral coefficients.

It is easy to show that $Q_{2n-1}(1) = D_{2n-1}/2$ and $Q_{2n}(1) = (D_{2n} - 1)/2$ for $n \geq 2$. By Formula (6.19) in [4] we have

$$Q_n(1) = \sum_{2 \leq 2k \leq n-1} k \times n(n-1)(n-2) \cdots (2k+2).$$

Since $Q_n(1)$ counts the number of desarrangements of type B, Corollary 19 implies that the number of desarrangements of type A equals the number of desarrangements of type B, when excluding the decreasing desarrangement of even length. It would be interesting to have a direct (analytic) proof of Corollary 19 which would not use the combinatorial set-up of this paper.

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