The doubloon polynomial triangle

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Dedicated to George Andrews,
on the occasion of his seventieth birthday.

ABSTRACT. The doubloon polynomials are generating functions for a class of combinatorial objects called normalized doubloons by the compressed major index. They provide a refinement of the q-tangent numbers and also involve two major specializations: the Poupard triangle and the Catalan triangle.

1. Introduction

The *doubloon* (*) *polynomials* $d_{n,j}(q)$ $(n \geq 1,\ 2 \leq j \leq 2n)$ introduced in this paper serve to globalize the *Poupard triangle* [Po89] and the classical *Catalan triangle* [Sl07]. They also provide a refinement of the *q-tangent numbers*, fully studied in our previous paper [FH08]. Finally, as generating polynomials for the doubloon model, they constitute a common combinatorial set-up for the above integer triangles. They may be defined by the following recurrence:

$(D1)$ $d_{0,j}(q) = \delta_{1,j}$ (Kronecker symbol);
$(D2)$ $d_{n,j}(q) = 0$ for $n \geq 1$ and $j \leq 1$ or $j \geq 2n+1$;
$(D3)$ $d_{n,2}(q) = \sum\limits_{j} q^{j-1}\, d_{n-1,j}(q)$ for $n \geq 1$;
$(D4)$ $d_{n,j}(q) - 2\, d_{n,j-1}(q) + d_{n,j-2}(q)$

$$= -(1-q)\sum_{i=1}^{j-3} q^{n+i+1-j}\, d_{n-1,i}(q) - (1+q^{n-1})\, d_{n-1,j-2}(q) + (1-q)\sum_{i=j-1}^{2n-1} q^{i-j+1}\, d_{n-1,i}(q)$$

for $n \geq 2$ and $3 \leq j \leq 2n$.

The polynomials $d_{n,j}(q)$ $(n \geq 1,\ 2 \leq j \leq 2n)$ are easily evaluated using $(D1)$–$(D4)$ and form the *doubloon polynomial triangle*, as shown in Fig. 1.1 and 1.1′.

$$\begin{array}{ccccccc}
 & & & d_{1,2}(q) & & & \\
 & & d_{2,2}(q) & d_{2,3}(q) & d_{2,4}(q) & & \\
 & d_{3,2}(q) & d_{3,3}(q) & d_{3,4}(q) & d_{3,5}(q) & d_{3,6}(q) & \\
d_{4,2}(q) & d_{4,3}(q) & d_{4,4}(q) & d_{4,5}(q) & d_{4,6}(q) & d_{4,7}(q) & d_{4,8}(q)
\end{array}$$

Fig. 1.1. The doubloon polynomial triangle

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(*) Although the word "doubloon" originally refers to a Spanish gold coin, it is here used to designate a permutation written as a two-row matrix.

$d_{1,2}(q)=1; \qquad d_{2,2}(q)=q; \quad d_{2,3}(q)=q+1; \quad d_{2,4}(q)=1;$
$d_{3,2}(q)=2q^3+2q^2; \quad d_{3,3}(q)=2q^3+4q^2+2q; \quad d_{3,4}(q)=q^3+4q^2+4q+1;$
$d_{3,5}(q)=2q^2+4q+2; \quad d_{3,6}(q)=2q+2;$
$d_{4,2}(q)=5q^6+12q^5+12q^4+5q^3; \quad d_{4,3}(q)=5q^6+17q^5+24q^4+17q^3+5q^2;$
$d_{4,4}(q)=3q^6+15q^5+29q^4+29q^3+15q^2+3q;$
$d_{4,5}(q)=q^6+9q^5+25q^4+34q^3+25q^2+9q+1;$
$d_{4,6}(q)=3q^5+15q^4+29q^3+29q^2+15q+3;$
$d_{4,7}(q)=5q^4+17q^3+24q^2+17q+5; \quad d_{4,8}(q)=5q^3+12q^2+12q+5.$

Fig. 1.1′. The first doubloon polynomials

Notice the different symmetries of the coefficients of the polynomials $d_{n,j}(q)$, which will be fully exploited in Section 4 (Corollaries 4.3, 4.7, 4.8). Various specializations are displayed in Fig. 1.2 below, where $C_n = \frac{1}{2n+1}\binom{2n}{n}$ stands for the celebrated Catalan number and t_n for the *reduced tangent number* occurring in the Taylor expansion

$$\begin{aligned}\sqrt{2}\tan(u/\sqrt{2}) &= \sum_{n\ge 0}\frac{u^{2n+1}}{(2n+1)!}t_n\\ &= \frac{u}{1!}1+\frac{u^3}{3!}1+\frac{u^5}{5!}4+\frac{u^7}{7!}34+\frac{u^9}{9!}496+\frac{u^{11}}{11!}11056+\cdots\end{aligned}\tag{1.1}$$

The symbol Σ attached to each vertical arrow has the meaning "make the summation over j" and $d_n(q)$ is the polynomial further defined in (1.3).

$$\begin{array}{ccccc} d_{n,j}(0) & \xleftarrow{q=0} & d_{n,j}(q) & \xrightarrow{q=1} & d_{n,j}(1)\\ \downarrow{\scriptstyle\Sigma} & & \downarrow{\scriptstyle\Sigma} & & \downarrow{\scriptstyle\Sigma}\\ C_n & \xleftarrow{q=0} & d_n(q) & \xrightarrow{q=1} & t_n \end{array}$$

Fig. 1.2. The specializations of $d_{n,j}(q)$

When $q=1$, the $(D1)$–$(D4)$ recurrence becomes

$(P1)$ $d_{0,j}(1)=\delta_{0,j}$ (Kronecker symbol);
$(P2)$ $d_{n,j}(1)=0$ for $n\ge 1$ and $j\le 1$ or $j\ge 2n+1$;
$(P3)$ $d_{n,2}(1)=\sum\limits_j d_{n-1,j}(1)$ for $n\ge 1$;
$(P4)$ $d_{n,j}(1)-2\,d_{n,j-1}(1)+d_{n,j-2}(1)=-2\,d_{n-1,j-2}(1)$
for $n\ge 2$ and $3\le j\le 2n$,

which is exactly the recurrence introduced by Christiane Poupard [Po89].

$$\begin{array}{c} 1\\ 1\quad 2\quad 1\\ 4\quad 8\quad 10\quad 8\quad 4\\ 34\ 68\ 94\ 104\ 94\ 68\ 34 \end{array}$$

Fig. 1.3. The Poupard triangle $(d_{n,j}(1))$

The integers $d_{n,j}(1)$ are easily evaluated using $(P1)$–$(P4)$ and form the *Poupard triangle*, as shown in Fig. 1.3.

When $q = 0$, relation $(D4)$ becomes:

$$d_{n,j}(0) - 2\,d_{n,j-1}(0) + d_{n,j-2}(0) = -d_{n-1,j-2}(0) + d_{n-1,j-1}(0),$$

which can be rewritten as

$$d_{n,j}(0) - d_{n,j-1}(0) - d_{n-1,j-1}(0) = d_{n,j-1}(0) - d_{n,j-2}(0) - d_{n-1,j-2}(0),$$

so that by induction

$$d_{n,j}(0) - d_{n,j-1}(0) - d_{n-1,j-1}(0) = d_{n,2}(0) = \sum_j q^{j-1}\, d_{n-1,j}\,|_{q=0} = 0$$

using $(D2)$ and $(D3)$ when $n \ge 2$. Consequently, the integers $d_{n,j}(0)$ satisfy the recurrence relation

$$(C1) \qquad d_{n,j}(0) = d_{n,j-1}(0) + d_{n-1,j-1}(0)$$

for $n \ge 2$ and $3 \le j \le 2n$ with the initial conditions

$$(C2) \qquad \begin{gathered} d_{n,n+1}(0) = 1 \quad (n \ge 1); \\ d_{n,j}(0) = 0 \quad (n \ge 1 \text{ and } j \le 1 \text{ or } j \ge 2n+1); \\ d_{n,2}(0) = d_{n,3}(0) = \cdots = d_{n,n}(0) = 0 \quad (n \ge 2). \end{gathered}$$

In view of $(C1)$ and $(C2)$ the integers $d_{n,j}(0)$ $(n \ge 1,\ n+1 \le j \le 2n)$ obey the rules of the classical *Catalan triangle* that has been studied by many authors (see the sequence A00976 in Sloane [Sl07] and its abundant bibliography). They form the *Catalan triangle* displayed in Fig. 1.4.

$$\begin{array}{c}
1 \\
0\ 1\ 1 \\
0\ 0\ 1\ 2\ \ 2 \\
0\ 0\ 0\ 1\ 3\ \ 5\ \ \ 5 \\
0\ 0\ 0\ 0\ 1\ 4\ \ 9\ \ 14\ 14 \\
0\ 0\ 0\ 0\ 0\ 1\ 5\ 14\ 28\ 42\ 42
\end{array}$$

Fig. 1.4. The Catalan triangle $(d_{n,j}(0))$

For each $n \ge 0$ let $A_n(t,q)$ be the Carlitz [Ca54, Ca75] q-analog of the Eulerian polynomial defined by the identity

$$(1.2) \qquad \frac{A_n(t,q)}{(t;q)_{n+1}} = \sum_{j\ge 0} t^j\,([j+1]_q)^n\ ,$$

where $(t;q)_{n+1} = (1-t)(1-tq)\cdots(1-tq^n)$ and $[j+1]_q = 1+q+q^2+\cdots+q^n$ are the traditional q-ascending factorials and q-analogs of the positive integers.

The polynomial $d_n(q)$ under study was introduced in [FH08]. It is defined by

$$d_n(q) = \frac{(-1)^n q^{\binom{n}{2}} A_{2n+1}(-q^{-n}, q)}{(1+q)(1+q^2)\cdots(1+q^n)}. \tag{1.3}$$

It was shown to be a *polynomial* of degree $\binom{n-1}{2}$, with *positive integral* coefficients, having the following two properties:

$$d_n(1) = t_n; \tag{1.4}$$

$$d_n(0) = C_n \quad (n \geq 0); \tag{1.5}$$

(see the bottom row in the diagram of Fig. 1.2.)

The first values of the polynomials $d_n(q)$ are: $d_0(q) = d_1(q) = 1$; $d_2(q) = 2 + 2q$; $d_3(q) = 5 + 12q + 12q^2 + 5q^3$; $d_4(q) = 14 + 56q + 110q^2 + 136q^3 + 110q^4 + 56q^5 + 14q^6$.

The main purpose of this paper is to prove the following theorem.

Theorem 1.1. *Let $(d_{n,j}(q))$ be the set of polynomials defined by $(D1)$–$(D4)$ and $d_n(q)$ be defined by (1.3). Then, the following identity holds*

$$\sum_j d_{n,j}(q) = d_n(q). \tag{1.6}$$

In other words, the diagram in Fig. 1.2 is commutative.

Even for $q = 1$ identity (1.6), which then reads $\sum_j d_{n,j}(1) = t_n$, is not at all straightforward. It was elegantly proved by Christiane Poupard [Po89] by means of the bivariable generating function

$$Z(u,v) = 1 + \sum_{n\geq 1} \sum_{1\leq l\leq 2n-1} \frac{u^{2n-l}}{(2n-l)!} \frac{v^l}{l!} d_{n,l+1}(1). \tag{1.7}$$

She even obtained the following stronger result

$$Z(u,v) = \frac{\cos((u-v)/\sqrt{2})}{\cos((u+v)/\sqrt{2})}, \tag{1.8}$$

so that

$$\frac{\partial}{\partial u} Z(u,v)\,\Big|_{\{v=0\}} = \sum_{n\geq 1} \frac{u^{2n-1}}{(2n-1)!} d_{n,2}(1) = \sqrt{2}\tan(u/\sqrt{2}), \tag{1.9}$$

which proves $\sum_j d_{n,j}(1) = t_n$ by appealing to $(P3)$.

Finally, she obtains a combinatorial interpretation for the polynomial

$$d_n(s,1) = \sum_j d_{n,j}(1)\, s^j \tag{1.10}$$

in terms of strictly ordered binary trees, or in an equivalent manner, of André permutations, which are also alternating (see, e.g. [FS71], [FS71a], [FS73]).

For $q = 0$ identity (1.6) reads $\sum_j d_{n,j}(0) = C_n = \frac{1}{n+1}\binom{2n}{n}$. This is a consequence of the identities

$$d_{n,2n}(0) = \sum_j d_{n-1,j}(0); \tag{1.11}$$

$$d_{n,j}(0) = \binom{j-2}{n-1} - \binom{j-2}{n} = \frac{2n-j+1}{n}\binom{j-2}{n-1}; \tag{1.12}$$

so that, in particular,

$$d_{n,2n}(0) = \frac{1}{n}\binom{2n-2}{n-1} = C_{n-1}; \tag{1.13}$$

easily obtained from $(C1)$ and $(C2)$ by induction and iteration, as well as an expression for the generating function

$$\sum_{n\ge 1} u^n \sum_j d_{n,j}(0) v^j = \frac{1}{1-v-uv}\Bigl(uv^2 - \frac{v}{2}\Bigl(1-\sqrt{1-4uv^2}\Bigr)\Bigr). \tag{1.14}$$

Again, the reader is referred to the excellent commented bibliography about the sequence A009766 in Sloane's On-Line Encyclopedia of Integer Sequences [Sl07], in particular the contributions made by David Callan [Cal05] and Emeric Deutsch [De04], where identities (1.11)–(1.14) are actually derived with other initial conditions.

The proof of Theorem 1.1 will be of *combinatorial nature*. In our previous paper [FH08] we proved that each polynomial $d_n(q)$ was a polynomial with positive integral coefficients by showing that it was the generating function for the class $\mathcal{N}^0_{2n+1}$ of permutations called *normalized doubloons*, by an integral-valued statistic "cmaj" called the *compressed major index*:

$$d_n(q) = \sum_{\delta\in\mathcal{N}^0_{2n+1}} q^{\mathrm{cmaj}\,\delta}. \tag{1.15}$$

In particular, $d_n(1) = \#\mathcal{N}^0_{2n+1} = t_n$ by (1.4). Normalized doubloon and "cmaj" will be fully described in Section 2. Here we just recall that each normalized doubloon $\delta \in \mathcal{N}^0_{2n+1}$ is a two-row matrix $\delta =$

$\binom{a_0\,a_1\,\cdots\,a_n}{b_0\,b_1\,\cdots\,b_n}$, where the word $\rho(\delta) = a_0a_1\cdots a_nb_nb_{n-1}\cdots b_0$ is a permutation of $012\cdots(2n+1)$ having further properties that will be given shortly. The set of all normalized doubloons $\delta = \binom{a_0\,a_1\,\cdots\,a_n}{b_0\,b_1\,\cdots\,b_n}$ such that $b_0 = j$ is denoted by $\mathcal{N}^0_{2n+1,j}$. It will be shown that $\mathcal{N}^0_{2n+1,j}$ is an empty set for $j = 0, 1$ and $2n+1$, so that the subsets $\mathcal{N}^0_{2n+1,j}$ $(j = 2, 3, \ldots, 2n)$ form a *partition* of $\mathcal{N}^0_{2n+1}$. Consequently, the following identity holds:

$$d_n(q) = \sum_{j=2}^{2n} \sum_{\delta\in\mathcal{N}^0_{2n+1,j}} q^{\operatorname{cmaj}\delta}. \tag{1.16}$$

Accorgingly, Theorem 1.1 is a simple corollary of the next theorem.

Theorem 1.2. *Let $(d_{n,j}(q))$ be the set of polynomials in one variable q defined by $(D1)$–$(D4)$. Then $d_{n,j}(q)$ is the generating polynomial for $\mathcal{N}^0_{2n+1,j}$ by the compressed major index. In other words,*

$$d_{n,j}(q) = \sum_{\delta\in\mathcal{N}^0_{2n+1,j}} q^{\operatorname{cmaj}\delta}. \tag{1.17}$$

To prove that the polynomial $d_{n,j}(q)$, as expressed in (1.17), satisfies $(D4)$, symmetry properties must be derived (see Corollaries 4.3, 4.7 and 4.8). This requires a careful geometric study of those doubloons and how the statistic "cmaj" evolves. All this is developed in Sections 3 and 4. The proof of the recurrence is completed in Section 5 and in the final Section concluding remarks are made.

2. Doubloons

A *doubloon* of order $2n+1$ is a $2\times(n+1)$-matrix $\delta = \binom{a_0\,a_1\,\cdots\,a_n}{b_0\,b_1\,\cdots\,b_n}$ such that the word $\rho(\delta) = a_0a_1\cdots a_nb_nb_{n-1}\cdots b_0$, called the *reading* of δ, is a permutation of $012\cdots(2n+1)$. Let $\mathcal{D}_{2n+1}$ (resp. $\mathcal{D}^0_{2n+1}$) denote the set of all doubloons $\delta = \binom{a_0\,a_1\,\cdots\,a_n}{b_0\,b_1\,\cdots\,b_n}$ of order $(2n+1)$ (resp. the subset of all doubloons such that $a_0 = 0$).

Let $1 \le k \le n$; each doubloon $\delta = \binom{a_0\,a_1\,\cdots\,a_n}{b_0\,b_1\,\cdots\,b_n}$ is said to be *normalized at k*, if the following two conditions are satisfied:

$(N1)$ *exactly one* of the two integers a_k, b_k lies between a_{k-1} and b_{k-1} (we also say that δ is *interlaced at k*);

$(N2)$ either $a_{k-1} > a_k$ and $b_{k-1} > b_k$, or $a_{k-1} < b_k$ and $b_{k-1} < a_k$.

In an equivalent manner, δ is *normalized at k*, if one of the four following orderings holds:

$$\begin{aligned} &a_{k-1} < b_k < b_{k-1} < a_k;\\ &b_k < b_{k-1} < a_k < a_{k-1};\\ &b_{k-1} < a_k < a_{k-1} < b_k;\\ &a_k < a_{k-1} < b_k < b_{k-1}. \end{aligned} \tag{2.1}$$

For each pair of distinct integers (i,j) the symbol $\mathcal{N}^i_{2n+1,j}$ will denote the set of all doubloons $\delta = \binom{a_0\,a_1\,\cdots\,a_n}{b_0\,b_1\,\cdots\,b_n}$, normalized at *every* $k = 1,2,\ldots,n$, such that $a_0 = i$ and $b_0 = j$. The doubloons belonging to $\mathcal{N}^0_{2n+1,j}$ for some j are simply called *normalized.* Also $\mathcal{N}^0_{2n+1}$ designates the union of the sets $\mathcal{N}^0_{2n+1,j}$'s. For further results on permutations studied as two-row matrices see [Ha92, Ha94, FH00].

Let $\delta = \binom{a_0\,a_1\,\cdots\,a_n}{b_0\,b_1\,\cdots\,b_n}$ be a doubloon of order $(2n+1)$ and h an integer. For each $k = 0,1,\ldots,n$ let $a'_k = a_k + h$, $b'_k = b_k + h$ be expressed as *residues* $\operatorname{mod}(2n+2)$. The two-row matrix $\binom{a'_0\,a'_1\,\cdots\,a'_n}{b'_0\,b'_1\,\cdots\,b'_n}$, denoted by $\delta + h$, is still a doubloon. Let $T_h : \delta \mapsto \delta + h$.

Proposition 2.1. *The map T_h, restricted to $\mathcal{N}^i_{2n+1,j}$, is a bijection onto $\mathcal{N}^{i+h}_{2n+1,j+h}$ (superscript and subscript being taken* $\operatorname{mod}(2n+2)$*).*

Proof. The four orderings in (2.1) are *cyclic* rearrangements of each other, so that if δ is normalized at each i, the doubloon $T_h\delta = \delta + h$ has the same property. $\square$

The *number of descents*, $\operatorname{des}\delta$, (resp. the *major index*, $\operatorname{maj}\delta$) of a doubloon $\delta = \binom{a_0\,a_1\,\cdots\,a_n}{b_0\,b_1\,\cdots\,b_n}$ is defined to be the number of descents (resp. the major index) of the permutation $\rho(\delta) = a_0a_1\cdots a_nb_nb_{n-1}\cdots b_0$, so that if the descents ($a_i > a_{i+1}$, or $a_n > b_n$, or still $b_{i-1} > b_i$) occur at positions $l_1, l_2, \ldots, l_r$ in $\rho(\delta)$, then $\operatorname{des}\delta = r$ and $\operatorname{maj}\delta = l_1 + l_2 + \cdots + l_r$. The *compressed major index*, $\operatorname{cmaj}\delta$, of δ is defined by

$$\operatorname{cmaj}\delta = \operatorname{maj}\delta - (n+1)\operatorname{des}\delta + \binom{n}{2}. \tag{2.2}$$

For example, there is one normalized doubloon of order 3 ($n = 1$): $\delta = \binom{0\,3}{2\,1}$ and $\operatorname{cmaj}\delta = \operatorname{maj}(0\,3\,1\,2) - (1+1)\operatorname{des}(0\,3\,1\,2) + \binom{1}{2} = 0$, so that $d_{1,2}(q) = 1$. There are four normalized doubloons of order 5 ($n = 2$) and the partition of $\mathcal{N}^0_5$ reads:

$$\mathcal{N}^0_{5,2} = \left\{\begin{pmatrix}0\,4\,3\\2\,1\,5\end{pmatrix}\right\},\ \mathcal{N}^0_{5,3} = \left\{\begin{pmatrix}0\,5\,4\\3\,2\,1\end{pmatrix},\quad \begin{pmatrix}0\,4\,2\\3\,1\,5\end{pmatrix}\right\},\quad \mathcal{N}^0_{5,4} = \left\{\begin{pmatrix}0\,5\,3\\4\,2\,1\end{pmatrix}\right\}.$$

We have $\operatorname{cmaj}(043512) = \operatorname{maj}(043512) - 3\operatorname{des}(043512) + \binom{2}{2} = (2+4) - 3\times 2 + 1 = 1$, so that $d_{2,2}(q) = q$. Furthermore, $d_{2,3}(q) = 1 + q$ and $d_{2,4}(q) = 1$, as expected (see Fig. 1.2).

3. Operations on doubloons

In this section we study the actions of several operators on the statistic "cmaj." First, we recall the action of the dihedral group on the traditional statistics "des" and "maj," in particular characterize the images of each permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ under the *reversal* $\mathbf{r}$ (resp. *complement* $\mathbf{c}$) that maps σ onto its mirror image $\mathbf{r}\,\sigma = \sigma(n)\cdots\sigma(2)\sigma(1)$ (resp. onto its complement $\mathbf{c}\,\sigma = (n+1-\sigma(1))(n+1-\sigma(2))\cdots(n+1-\sigma(n))$). The next properties are well-known (see, e.g., [FS78]): if σ belongs to $\mathfrak{S}_n$, then

(3.1) $\quad \operatorname{des}\mathbf{r}\,\sigma = n-1-\operatorname{des}\sigma; \quad \operatorname{maj}\mathbf{r}\,\sigma = \operatorname{maj}\sigma - n\operatorname{des}\sigma + \binom{n}{2};$

(3.2) $\quad \operatorname{des}\mathbf{c}\,\sigma = n-1-\operatorname{des}\sigma; \quad \operatorname{maj}\mathbf{c}\,\sigma = \binom{n}{2} - \operatorname{maj}\sigma;$

(3.3) $\quad \operatorname{des}\mathbf{r}\,\mathbf{c}\,\sigma = \operatorname{des}\sigma; \quad \operatorname{maj}\mathbf{r}\,\mathbf{c}\,\sigma = n\operatorname{des}\sigma - \operatorname{maj}\sigma.$

Now let i, j be the two integers defined by $\sigma(1) = n-j$, $\sigma(n) = n-i$ and σ' be the permutation mapping k onto

$$\sigma'(k) = \begin{cases} \sigma(k)+i, & \text{if } \sigma(k)+i \le n; \\ \sigma(k)+i-n, & \text{if } \sigma(k)+i > n. \end{cases}$$

The operation $\sigma \mapsto \sigma'$ and Property (3.5) below already appear in [Ha92b] for the study of the Z-statistic.

Lemma 3.1. *We have:*

$$\operatorname{des}\sigma - \operatorname{des}\sigma' = \begin{cases} 0, & \textit{if } i<j; \\ 1, & \textit{if } i>j; \end{cases} \tag{3.4}$$

$$\operatorname{maj}\sigma - \operatorname{maj}\sigma' = i. \tag{3.5}$$

Proof. As $\sigma(n) = n-i$, we can factorize the word $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ as a product $p_0q_1p_1\cdots q_rp_r$ $(r\ge 1)$, where the letters of all p_k's (resp. all q_k's) are smaller than or equal to (resp. greater than) $n-i$. Also, let $p'_0q'_1p'_1\cdots q'_rp'_r$ be the corresponding factorization of σ' such that $\lambda p'_0 = \lambda p_0$, $\lambda q'_1 = \lambda q_1$, ... ($\lambda$ being the word length). In particular, p_r (and p'_r) is never empty since $\sigma(n) = n-i$ (and $\sigma'(n) = n$), but p_0 (resp. and p'_0) is nonempty if and only if $\sigma(1)+i = n-j+i \le n$, that is, if $i<j$.

As the rightmost letter of each factor q_k is larger than the leftmost letter of p_k, we have

$$\operatorname{des}\sigma = \operatorname{des}p_0 + \operatorname{des}q_1 + 1 + \operatorname{des}p_1 + \cdots + \operatorname{des}q_r + 1 + \operatorname{des}p_r.$$

In σ' the rightmost letter of each p'_k is greater than the leftmost letter of q'_{k+1}, so that, if $i<j$,

$$\begin{aligned} \operatorname{des}\sigma' &= \operatorname{des}p'_0 + 1 + \operatorname{des}q'_1 + \cdots + \operatorname{des}p'_{r-1} + 1 + \operatorname{des}q'_r + \operatorname{des}p'_r \\ &= \operatorname{des}p_0 + \operatorname{des}q_1 + 1 + \cdots + \operatorname{des}p_{r-1} + \operatorname{des}q_r + 1 + \operatorname{des}p_r \\ &= \operatorname{des}\sigma, \end{aligned}$$

while, if $i>j$, the factor p'_0 is empty and

$$\begin{aligned} \operatorname{des}\sigma' &= \operatorname{des}q'_1 + \cdots + \operatorname{des}p'_{r-1} + 1 + \operatorname{des}q'_r + \operatorname{des}p'_r \\ &= \operatorname{des}\sigma - 1. \end{aligned}$$

However, in both cases we have:

$$\begin{aligned}\operatorname{maj}\sigma-\operatorname{maj}\sigma' &= \lambda(p_0q_1)+\lambda(p_0q_1p_1q_2)+\cdots+\lambda(p_0q_1\cdots p_{r-1}q_r)\\ &\qquad -\left(\lambda(p'_0)+\lambda(p'_0q'_1p'_1)+\cdots+\lambda(p'_0q'_1\cdots p'_{r-1}\right)\\ &= \lambda(q_1)+\lambda(q_2)+\cdots+\lambda(q_r)\\ &= \#\{k:\sigma(k)>n-i\}=i. \quad \square\end{aligned}$$

Lemma 3.2. *Let* $\delta=\binom{a_0\,a_1\cdots a_n}{b_0\,b_1\cdots b_n}$ *be a doubloon such that* $a_0=i$ *and* $b_0=j$. *Also, let* $\delta'=T_{-i}(\delta)=\delta-i=\binom{0\quad a_1-i\cdots a_n-i}{j-i\,b_1-i\cdots b_n-i}$. *Then*

$$\operatorname{cmaj}\delta-\operatorname{cmaj}\delta'=\begin{cases}-i, & \text{if } i<j;\\ n+1-i, & \text{if } i>j.\end{cases}\tag{3.6}$$

Proof. Add 1 to each letter of the reading $\rho(\delta)=a_0a_1\cdots a_nb_nb_{n-1}\cdots b_0$ of δ and do the same for the reading $\rho(\delta')$ of δ'. We obtain two permutations σ, σ' of $12\cdots(2n+2)$ of the form:

$$\begin{aligned}\sigma&=(i+1)\ \sigma(2)\ \cdots\ \sigma(2n+1)\ (j+1);\\ \sigma'&=1\ \sigma'(2)\ \cdots\ \sigma'(2n+1)\ (j-i+1).\end{aligned}$$

When the transformation $\mathbf{r}\,\mathbf{c}$ is applied to each of them, we get

$$\begin{aligned}\mathbf{r}\,\mathbf{c}\,\sigma&=(2n+2-j)\ \cdots\ (2n+2-i);\\ \mathbf{r}\,\mathbf{c}\,\sigma'&=(2n+2-j+i)\ \cdots\ (2n+2).\end{aligned}$$

It follows from Lemma 3.1 (with $(2n+2)$ replacing n, $\mathbf{r}\,\mathbf{c}\,\sigma$ instead of σ and $\mathbf{r}\,\mathbf{c}\,\sigma'$ instead of σ') that

$$\operatorname{des}\mathbf{r}\,\mathbf{c}\,\sigma-\operatorname{des}\mathbf{r}\,\mathbf{c}\,\sigma'=\begin{cases}0, & \text{if } i<j;\\ 1, & \text{if } i>j;\end{cases}$$

$$\operatorname{maj}\mathbf{r}\,\mathbf{c}\,\sigma-\operatorname{maj}\mathbf{r}\,\mathbf{c}\,\sigma'=i,$$

so that by (3.3)

$$\operatorname{des}\sigma-\operatorname{des}\sigma'=\begin{cases}0, & \text{if } i<j;\\ 1, & \text{if } i>j;\end{cases}$$

$$\begin{aligned}\operatorname{maj}\sigma-\operatorname{maj}\sigma'&=(2n+2)(\operatorname{des}\sigma-\operatorname{des}\sigma')-(\operatorname{maj}\mathbf{r}\,\mathbf{c}\,\sigma-\operatorname{maj}\mathbf{r}\,\mathbf{c}\,\sigma')\\ &=(2n+2)(\operatorname{des}\sigma-\operatorname{des}\sigma')-i.\end{aligned}$$

As $\operatorname{cmaj}\delta=\operatorname{maj}\sigma-(n+1)\operatorname{des}\sigma+\binom{n}{2}$, we get

$$\begin{aligned}\operatorname{cmaj}\delta-\operatorname{cmaj}\delta'&=(\operatorname{maj}\sigma-\operatorname{maj}\sigma')-(n+1)(\operatorname{des}\sigma-\operatorname{des}\sigma')\\ &=(n+1)(\operatorname{des}\sigma-\operatorname{des}\sigma')-i\\ &=\begin{cases}-i, & \text{if } i<j;\\ n+1-i, & \text{if } i>j.\end{cases}\quad\square\end{aligned}$$

Lemma 3.3. *Let* $\delta = \binom{0\ a_1 \cdots a_n}{b_0\ b_1 \cdots b_n} \mapsto \delta' = \binom{a'_0\ a'_1 \cdots a'_n}{b'_0\ b'_1 \cdots b'_n}$ *be the transformation mapping* $\delta \in \mathcal{D}^0_{2n+1}$ *onto the doubloon* δ' *obtained from* δ *by replacing each entry* a_k *(resp.* b_k*) by* $a'_k = (2n+1) - a_k$ *(resp.* $b'_k = (2n+1) - b_k$*). Then* $\operatorname{cmaj} \delta = n(n-1) - \operatorname{cmaj} \delta'$.

Proof. Again, add 1 to each element of $\rho(\delta)$ and of $\rho(\delta')$ to obtain two permutations σ and σ' of $12 \cdots (2n+2)$. We have: $\sigma' = \mathbf{c}\,\sigma$. By (3.2) $\operatorname{des} \sigma + \operatorname{des} \sigma' = 2n+1$ and $\operatorname{maj} \sigma + \operatorname{maj} \sigma' = \binom{2n+2}{2} = (n+1)(2n+1)$. Hence, $\operatorname{cmaj} \delta + \operatorname{cmaj} \delta' = \operatorname{maj} \sigma + \operatorname{maj} \sigma' - (n+1)(\operatorname{des} \sigma + \operatorname{des} \sigma') + n(n-1) = (n+1)(2n+1) - (n+1)(2n+1) + n(n-1) = n(n-1)$. $\square$

Let $\Gamma : \binom{0\ a_1 \cdots a_n}{b_0\ b_1 \cdots b_n} \mapsto \binom{0 \quad 2n+2-a_1 \cdots 2n+2-a_n}{2n+2-b_0\ 2n+2-b_1 \cdots 2n+2-b_n}$ [$\Gamma : \delta \mapsto \delta''$ in short] be the transformation mapping each doubloon $\delta = \binom{0\ a_1 \cdots a_n}{b_0\ b_1 \cdots b_n} \in \mathcal{D}^0_{2n+1}$ onto the doubloon $\delta'' = \binom{0\ a''_1 \cdots a''_n}{b''_0\ b''_1 \cdots b''_n}$ obtained from δ by replacing each entry a_k (resp. b_k) by the residue $a''_k = (2n+2) - a_k$ (resp. $b''_k = (2n+2) - b_k$).

Remark. If δ is interlaced at each k (condition $(N1)$ holds at each k), the same property holds for $\Gamma(\delta)$. However, if δ is normalized at each k, the property is *not* preserved under the transformation Γ.

Lemma 3.4. *For each* $\delta \in \mathcal{D}^0_{2n+1}$ *we have:* $\operatorname{cmaj} \delta + \operatorname{cmaj} \Gamma(\delta) = n^2$.

Proof. First, transform δ into the doubloon δ' defined in Lemma 3.3, so that $\delta' = \binom{2n+1 \quad a'_1 \cdots a'_n}{2n+1-j\ b'_1 \cdots b'_n}$ and $\operatorname{cmaj} \delta = n(n-1) - \operatorname{cmaj} \delta'$. Next, apply Lemma 3.2 to δ' with $i = 2n+1$, so that the new doubloon is of the form $\delta'' = \binom{0 \quad a''_1 \cdots a''_n}{2n+2-j\ b''_1 \cdots b''_n}$. Hence, $\operatorname{cmaj} \delta' - \operatorname{cmaj} \delta'' = (n+1) - (2n+1) = -n$ and $\operatorname{cmaj} \delta = n(n-1) - \operatorname{cmaj} \delta'' + n = n^2 - \operatorname{cmaj} \delta''$. $\square$

4. Further operations on doubloons

In our previous paper [FHa08] we also introduced a class of transformations ϕ_i $(0 \le i \le n)$ on $\mathcal{D}_{2n+1}$, called *micro flips*, which permute the entries in a given column. By definition,

$$\phi_i : \begin{pmatrix} a_0 \cdots a_{i-1}\ a_i\ a_{i+1} \cdots a_n \\ b_0 \cdots b_{i-1}\ b_i\ b_{i+1} \cdots b_n \end{pmatrix} \mapsto \begin{pmatrix} a_0 \cdots a_{i-1}\ b_i\ a_{i+1} \cdots a_n \\ b_0 \cdots b_{i-1}\ a_i\ b_{i+1} \cdots b_n \end{pmatrix} \ (0 \le i \le n).$$

Next, the *macro flips* Φ_i are defined by $\Phi_i = \phi_i \phi_{i+1} \cdots \phi_n$ $(1 \le i \le n)$. Note that both ϕ_i and Φ_i are *involutions* of $\mathcal{D}_{2n+1}$. In particular,

$$\Phi_1 \Phi_2 \cdots \Phi_i = \prod_{\substack{j \text{ odd}, \\ j \le i}} \phi_j. \tag{4.1}$$

By means of the transformation Γ (see Lemma 3.4 above) and the involutions Φ_i's we now construct a bijection of $\mathcal{N}^0_{2n+1,j}$ onto $\mathcal{N}^0_{2n+1,2n+2-j}$.

Lemma 4.1. *Let* $\delta = \binom{0\ a_1 \cdots a_n}{b_0\ b_1 \cdots b_n}$ *be a normalized doubloon and let* $\Gamma(\delta) = \delta'' = \binom{0\ a_1'' \cdots a_n''}{b_0''\ b_1'' \cdots b_n''}$. *Then, for each* $i = 1, 2, \ldots, n$ *the doubloon* $\Phi_1\Phi_2 \cdots \Phi_i(\delta'')$ *is normalized at each integer* $1, 2, \ldots, i$. *In particular,* $\Phi_1\Phi_2 \cdots \Phi_n(\delta'')$ *is normalized.*

Proof. Using (4.1) we have:

$$\delta^* = \Phi_1\Phi_2 \cdots \Phi_i(\delta'') = \begin{cases} \binom{0\ b_1''\ a_2'' \cdots a_{i-1}''\ b_i''}{b_0''\ a_1''\ b_2'' \cdots b_{i-1}''\ a_i''}, & \text{if } i \text{ odd;} \\ \binom{0\ b_1''\ a_2'' \cdots b_{i-1}''\ a_i''}{b_0''\ a_1''\ b_2'' \cdots a_{i-1}''\ b_i''}, & \text{if } i \text{ even.} \end{cases}$$

The doubloon δ'' is not normalized (see the Remark before Lemma 3.4), but is interlaced at each i, so that the relations $a_{j-1}'' < a_j''$, $b_{j-1}'' < b_j''$ or $a_{j-1}'' > a_j''$, $b_{j-1}'' > b_j''$ hold for all j. This shows that δ^* is normalized whenever j is odd or even. ∎

For the next Proposition we need the following property proved in our previous paper [FHa08, Theorem 3.5]:

(4.2) *Let* $\delta = \binom{0\ a_1 \cdots a_n}{b_0\ b_1 \cdots b_n}$ *be an interlaced doubloon, normalized at* i. *Then* $\operatorname{cmaj} \Phi_i(\delta) - \operatorname{cmaj} \delta = n - i + 1 \quad (1 \le i \le n)$.

Proposition 4.2. *The transformation* $\Phi_1\Phi_2 \cdots \Phi_n\Gamma : \delta \mapsto \delta^*$ *is a bijection of* $\mathcal{N}^0_{2n+1,j}$ *onto* $\mathcal{N}^0_{2n+1,2n+2-j}$ *having the property*

$$\operatorname{cmaj} \delta + \operatorname{cmaj} \delta^* = \binom{n}{2}. \tag{4.3}$$

Proof. Let $\delta'' = \Gamma(\delta)$. As $\Phi_1\Phi_2 \cdots \Phi_i(\delta'')$ is normalized at i, we have

$$\operatorname{cmaj} \Phi_i\ \Phi_1\Phi_2 \cdots \Phi_i(\delta'') - \operatorname{cmaj} \Phi_1\Phi_2 \cdots \Phi_i(\delta'') = n - i + 1$$

by (4.2). Summing over all $i = 1, 2, \ldots, n$ we get

$$\sum_{i=1}^{n} (\operatorname{cmaj} \Phi_1\Phi_2 \cdots \Phi_{i-1}(\delta'') - \operatorname{cmaj} \Phi_1\Phi_2 \cdots \Phi_i(\delta'')) = \sum_{i=1}^{n} (n - i + 1)$$

$$\operatorname{cmaj} \delta'' - \operatorname{cmaj} \Phi_1\Phi_2 \cdots \Phi_n(\delta'') = n^2 - \tbinom{n}{2}.$$

Using Lemma 3.4 we conclude that $(n^2 - \operatorname{cmaj} \delta) - \operatorname{cmaj} \delta^* = n^2 - \binom{n}{2}$ and then $\operatorname{cmaj} \delta + \operatorname{cmaj} \delta^* = \binom{n}{2}$. ∎

Example. For $n = 2$ we have $\Phi_1\Phi_2 = \phi_1$. The four normalized doubloons of order 5 ($n = 2$) displayed at the end of Section 2 are mapped under $\Phi_1\Phi_2\Gamma = \phi_1\Gamma$ as shown in Fig. 4.1. Next to each doubloon appears the value of its "cmaj".

$$
\begin{array}{cccc}
\binom{0\,4\,3}{2\,1\,5},1 & \binom{0\,5\,4}{3\,2\,1},0 & \binom{0\,4\,2}{3\,1\,5},1 & \binom{0\,5\,3}{4\,2\,1},0 \\
\big\downarrow \Gamma & \big\downarrow \Gamma & \big\downarrow \Gamma & \big\downarrow \Gamma \\
\binom{0\,2\,3}{4\,5\,1},3 & \binom{0\,1\,2}{3\,4\,5},4 & \binom{0\,2\,4}{3\,5\,1},3 & \binom{0\,1\,3}{2\,4\,5},4 \\
\big\downarrow \phi_1 & \big\downarrow \phi_1 & \big\downarrow \phi_1 & \big\downarrow \phi_1 \\
\binom{0\,5\,3}{4\,2\,1},0 & \binom{0\,4\,2}{3\,1\,5},1 & \binom{0\,5\,4}{3\,2\,1},0 & \binom{0\,4\,3}{2\,1\,5},1
\end{array}
$$

Fig. 4.1. The transformation $\Phi_1\Phi_2\cdots\Phi_n\Gamma$

Let $d_{n,j}(q) = \sum\limits_{\delta\in\mathcal{N}^0_{2n+1,j}} q^{\mathrm{cmaj}\,\delta}$. The final purpose is to show that $d_{n,j}(q)$ satisfies relations $(D1)-(D4)$. This will be done in Section 5. In the rest of this section we state some symmetry properties of the $d_{n,j}(q)$'s.

Corollary 4.3. *For $2\le j\le 2n$ we have:*

$$
d_{n,j}(q) = q^{\binom{n}{2}}\, d_{n,2n+2-j}(q^{-1}). \tag{4.4}
$$

Proof. This follows from the previous proposition:

$$
d_{n,j}(q) = \sum_{\delta\in\mathcal{N}^0_{2n+1,j}} q^{\mathrm{cmaj}\,\delta} = \sum_{\delta^*\in\mathcal{N}^0_{2n+1,2n+1-j}} q^{\binom{n}{2}-\mathrm{cmaj}\,\delta^*} = q^{\binom{n}{2}}\, d_{n,2n+2-j}(q^{-1}). \qquad \square
$$

In the next lemma we study the action of the sole transposition ϕ_0 that permutes the leftmost entries of each doubloon.

Lemma 4.4. *For each normalized doubloon δ*

$$
\mathrm{cmaj}\,\delta - \mathrm{cmaj}\,\phi_0(\delta) = -n. \tag{4.5}
$$

Proof. Let $\rho(\delta) = 0a_1\cdots a_nb_n\cdots b_1b_0$ be the reading of δ, so that $\rho(\phi_0(\delta)) = b_0a_1\cdots a_nb_n\cdots b_10$. As δ is normalized (and, in particular, interlaced), we have: $0<b_1<b_0<a_1$. Thus, $\rho(\delta)$ starts and ends with a rise ($0<a_1$ and $b_1<a_0$), while $\rho(\phi_0(\delta))$ starts with a rise $b_0<a_1$ and ends with a descent $b_1>0$. Hence, $\mathrm{des}\,\rho(\delta) = \mathrm{des}\,\rho(\phi_0(\delta))-1$ and $\mathrm{maj}\,\rho(\delta) = \mathrm{maj}\,\rho(\phi_0(\delta)) - (2n+1)$ and $\mathrm{cmaj}\,\delta - \mathrm{cmaj}\,\rho(\phi_0(\delta)) = -(2n+1)-(n+1)-1 = -n$ $\square$

The involution $\phi_0\Phi_1 = \Phi_1\phi_0 = \phi_0\phi_1\cdots\phi_n$ transposes the two rows of each doubloon $\delta\in\mathcal{D}_{2n+1}$.

Lemma 4.5. *For each normalized doubloon δ*

$$
\mathrm{cmaj}\,\delta = \mathrm{cmaj}\,\phi_0\Phi_1(\delta). \tag{4.5}
$$

Proof. First, $\operatorname{cmaj}\delta = \operatorname{cmaj}\phi_0(\delta) - n$ by the previous lemma. Furthermore, as $\phi_0\Phi_1(\delta)$ remains normalized at 1, relation (4.2) implies that $\operatorname{cmaj}\Phi_1(\phi_0\Phi_1(\delta)) - \operatorname{cmaj}\phi_0\Phi_1(\delta) = n-1+1$. Thus,

$$\begin{aligned}\operatorname{cmaj}\phi_0\Phi_1(\delta) &= \operatorname{cmaj}\phi_0(\delta) - n\\ &= \operatorname{cmaj}\delta + n - n = \operatorname{cmaj}\delta. \quad \square\end{aligned}$$

We next study the joint action of the operators T_{-j} (introduced in Lemma. 3.2) ϕ_0 and Φ_1.

Proposition 4.6. *The transformation $T_{-j}\phi_0\Phi_1 : \delta \mapsto \delta'$ is a bijection of $\mathcal{N}^0_{2n+1,j}$ onto $\mathcal{N}^0_{2n+1,2n+2-j}$ having the property that*

$$\operatorname{cmaj}\delta - \operatorname{cmaj}\delta' = n+1-j. \tag{4.6}$$

Proof. Let $\delta \in \mathcal{N}^0_{2n+1,j}$. Then, $\phi_0\Phi_1(\delta)$ is normalized at each $i = 1, 2, \ldots, n$ and also $T_{-j}\phi_0\Phi_1(\delta)$ by the remark made before Lemma 3.2. Thus, $\delta' \in \mathcal{N}^0_{2n+1,2n+2-j}$ and the map $\delta \mapsto \delta'$ is bijective. Furthermore, $\operatorname{cmaj}\delta = \operatorname{cmaj}\phi_0\Phi_1(\delta)$ by Lemma 4.5 and $\operatorname{cmaj}\phi_0\Phi_1(\delta) = \operatorname{cmaj}\delta' + n + 1 - j$ by Lemma 3.2. $\square$

Example. Again consider the four normalized doubloons of order 5. We get the display of Fig. 4.2. with the value of "cmaj" next to each doubloon.

$$\begin{array}{cccc}
\binom{0\,4\,3}{2\,1\,5}, 1 & \binom{0\,5\,4}{3\,2\,1}, 0 & \binom{0\,4\,2}{3\,1\,5}, 1 & \binom{0\,5\,3}{4\,2\,1}, 0\\
\big\downarrow{\scriptstyle \phi_0\Phi_1} & \big\downarrow{\scriptstyle \phi_0\Phi_1} & \big\downarrow{\scriptstyle \phi_0\Phi_1} & \big\downarrow{\scriptstyle \phi_0\Phi_1}\\
\binom{2\,1\,5}{0\,4\,3}, 1 & \binom{3\,2\,1}{0\,5\,4}, 0 & \binom{3\,1\,5}{0\,4\,2}, 1 & \binom{4\,2\,1}{0\,5\,3}, 0\\
\big\downarrow{\scriptstyle T_{-2}} & \big\downarrow{\scriptstyle T_{-3}} & \big\downarrow{\scriptstyle T_{-3}} & \big\downarrow{\scriptstyle T_{-4}}\\
\binom{0\,5\,3}{4\,2\,1}, 0 & \binom{0\,5\,4}{3\,2\,1}, 0 & \binom{0\,4\,2}{3\,1\,5}, 1 & \binom{0\,4\,3}{2\,1\,5}, 1
\end{array}$$

Fig. 4.2. The transformation $T_{-j}\phi_0\Phi_1$

Corollary 4.7. *For $2 \le j \le 2n$ we have:*

$$d_{n,j}(q) = q^{n+1-j}\, d_{n,2n+2-j}(q). \tag{4.7}$$

The corollary immediately follows from Proposition 4.6. In its turn the next corollary is a consequence of both Corollaries 4.3 and 4.7.

Corollary 4.8. *For $2 \le j \le 2n$ we have:*

$$d_{n,j}(q) = q^{\binom{n+1}{2}+1-j}\, d_{n,j}(q^{-1}). \tag{4.8}$$

As could be seen in Fig. 1.2 for the first values, the polynomial $d_{n,j}(q)$ is a multiple of $d_{n,2n+2-j}(q)$ (Corollary 4.7) and Corollary 4.8 indicates a symmetry between its coefficients.

5. The recurrence itself

In this section we prove Theorem 1.2, that is, we show that relations $(D1)-(D4)$ hold for $d_{n,j}(q)=\sum_{\delta\in\mathcal{N}^0_{2n+1,j}} q^{\operatorname{cmaj}\delta}$. The first two relations $(D1)-(D2)$ are evidently true. For proving that relations $(D3)-(D4)$ hold for such a $d_{n,j}(q)$ we start with a doubloon $\delta\in\mathcal{N}^0_{2n+1}$, drop its leftmost column and compare the compressed major indices of δ and of the doubloon obtained after deletion. In so doing we get the following result.

Lemma 5.1. *To each doubloon $\delta\in\mathcal{N}^0_{2n+1,j}$ there corresponds a unique triplet (k,l,δ') such that $k\ge j-1$, $l\le j-2$ and $\delta'\in\mathcal{N}^k_{2n-1,l}$ having the property that*

$$\operatorname{cmaj}\delta=\operatorname{cmaj}\delta'+(n-1). \tag{5.1}$$

Proof. Let $\delta=\left(\begin{smallmatrix}0 & a_1 & \cdots & a_n\\ j & b_1 & \cdots & b_n\end{smallmatrix}\right)\in\mathcal{N}^0_{2n+1,j}$ and define

$$a'_i=\begin{cases}a_i-1, & \text{if } a_i<j;\\ a_i-2, & \text{if } j<a_i\le 2n+1;\end{cases}$$

with an analogous definition for the b'_i, the b's replacing the a's. Then, $\delta'=\left(\begin{smallmatrix}a'_1 & \cdots & a'_n\\ b'_1 & \cdots & b'_n\end{smallmatrix}\right)$ is a normalized doubloon of order $(2n+1)$, but its left-top corner a'_1 is not necessarily 0. Call it the *reduction* of δ. As $a_1>j>b_1$, we see that $b'_1=b_1-1<j-1$ and $a'_1=a_1-2>j-2$, that is, $b'_1\le j-2<j-1\le a'_1$. Thus $\delta'\in\mathcal{N}^k_{2n-1,l}$ with $k\ge j-1$ and $l\le j-2$. Conversely, given the triplet (δ',k,l) with the above properties, we can uniquely reconstruct the normalized doubloon δ.

As δ is normalized at 1, the inequalities $0<b_1<j<a_1$ hold, so that $\rho(\delta)$ starts and ends with a rise: $0<a_1$, $b_1<j$. In particular, $\operatorname{des}\rho(\delta)=\operatorname{des}\rho(\delta')$. However, $\operatorname{maj}\rho(\delta)=\operatorname{maj}\rho(\delta')+\operatorname{des}\delta$, as the first letter 0 is dropped when going from $\rho(\delta)$ to $\rho(\delta')$. Hence, $\operatorname{cmaj}\delta-\operatorname{cmaj}\delta'=\operatorname{des}\delta-((n+1)-n)\operatorname{des}\delta+\binom{n}{2}-\binom{n-1}{2}=n-1$. ∎

The previous Lemma yields the first passage from the generating polynomial $d_{n,j}(q)$ for the normalized doubloons $\delta\in\mathcal{N}^0_{2n+1,j}$ by "cmaj" to the generating polynomial for doubloons of order $(2n-1)$, as expressed in the next Corollary.

Corollary 5.2. *For $2\le j\le 2n$ we have:*

$$d_{n,j}(q)=q^{n-1}\sum_{k\ge j-1,\, l\le j-2} q^{n-k}\, d_{n-1,2n+l-k}(q). \tag{5.2}$$

Proof. By the previous Lemma we can write:

$$d_{n,j}(q) = \sum_{\delta \in \mathcal{N}^0_{2n+1,j}} q^{\operatorname{cmaj}\delta} = q^{n-1} \sum_{k \geq j-1,\, l \leq j-2} \; \sum_{\delta' \in \mathcal{N}^k_{2n-1,l}} q^{\operatorname{cmaj}\delta'}.$$

With each $\delta' \in \mathcal{N}^k_{2n-1,l}$ associate $\delta'' = T_{-k}\delta' = \delta' - k$. By Lemma 3.2 we get $\delta'' \in \mathcal{N}^0_{2n-1,2n+l-k}$ (note that the residue $\operatorname{mod}(2n+2)$ must be considered in the subscript for $l < k$) and $\operatorname{cmaj}\delta' - \operatorname{cmaj}\delta'' = n - k$. Hence,

$$\sum_{\delta' \in \mathcal{N}^k_{2n-1,l}} q^{\operatorname{cmaj}\delta'} = q^{n-k} \sum_{\delta'' \in \mathcal{N}^0_{2n-1,2n+l-k}} q^{\operatorname{cmaj}\delta''} = q^{n-k} d_{n-1,2n+l-k}(q). \qquad \square$$

When $j = 2$ in (5.2), we get:

$$\begin{aligned} d_{n,2}(q) &= \sum_{k \geq 2} q^{2n-k-1} d_{n-1,2n-k}(q) \\ &= \sum_{k \geq 2} q^{i-1} d_{n-1,i}(q) \ \text{[by the change of variables } i = 2n-k\text{]} \end{aligned}$$

and also

$$d_{n,2}(q) = q^{n-1} \sum_{i \geq 2} d_{n-1,i}(q) \qquad \text{[by using (4.7).]}$$

Consequently, relation $(D3)$ holds for the polynomial $d_{n,j}(q) = \sum_{\delta \in \mathcal{N}^0_{2n+1,j}} q^{\operatorname{cmaj}\delta}$.

Using (5.2) it is also easy to derive the identity:

$$q\, d_{n,3}(q) = (q+1)\, d_{n,2}(q).$$

Proposition 5.2. *For $2 \leq j \leq 2n$ we have:*

$$d_{n,j}(q) = \sum_i \frac{q^{\max(0,i+1-j)} - q^{\min(i,2n+1-j)}}{1-q} d_{n-1,i}(q). \tag{5.3}$$

Proof. Let $i = 2n + l - k$ with $0 \leq l \leq j-2$ and $j - 1 \leq k \leq 2n-1$. This implies $0 \leq i + k - 2n \leq j - 2$, or still $2n - i \leq k \leq 2n + j - i - 2$. Taking the two relations keeping k within bounds into account we get the double inequality:

$$\max(2n-i, j-1) \leq k \leq \min(2n+j-i-2, 2n-1). \tag{5.4}$$

Identity (5.2) may then be rewritten as

$$d_{n,j}(q) = q^{2n-1} \sum_i d_{2n-1,i}(q) \sum_k q^{-k}, \tag{5.5}$$

with k ranging over the interval defined in (5.4). The geometric sum over k is equal to

$$\frac{q^{-\max(2n-i,j-1)} - q^{-\min(2n+j-i-2,2n-1)-1}}{1-q^{-1}}.$$

Now, $2n - \max(2n-i, j-1) = 2n + \min(-2n+i, -j+1) = \min(i, 2n-j+1)$ and $2n-1-\min(2n+j-i-2, 2n-1) = \max(0, -j+i+1)$. Accordingly,

$$q^{2n-1} \sum_{k \text{ subject to } (5.4)} q^{-k} = \frac{q^{\max(0,i+1-j)} - q^{\min(i,2n+1-j)}}{1-q}.$$

This proves (5.3) when reporting the latter expression into (5.5). $\square$

For getting rid of "max" and "min" from identity (5.3) we decompose the sum into four subsums, assuming $j \geq 3$ (when $j = 2$ identity (5.3) gives back $(D3)$). We obtain:

$$\begin{aligned} (5.6) \quad (1-q)d_{n,j}(q) = \sum_{i=0}^{j-1} d_{n-1,i}(q) + \sum_{i=j}^{2n+1} q^{i+1-j} d_{n-1,i}(q) \\ - \sum_{i=0}^{2n-j} q^i d_{n-1,i}(q) - \sum_{i=2n-j+1}^{2n+1} q^{2n+1-j} d_{n-1,i}(q). \end{aligned}$$

By means of (5.6) we calculate $d_{n,j}(q) - d_{n,j-1}(q)$ and then $-d_{n,j-1}(q) + d_{n,j-2}(q)$, whose sum is the left-hand side of $(D4)$. In the computation of $(1-q)(d_{n,j}(q) - d_{n,j-1}(q))$ the contribution of the first subsum is simply $d_{n-1,j-1}(q)$. Next, $q^{2n-j+1} d_{n-1,2n-j+1}(q)$ is the contribution of the second subsum. For the third subsum we get

$$-q\, d_{n-1,j-1}(q) + \sum_{i=j}^{2n+1} q^{i-j+1}(1-q) d_{n-1,i}(q)$$

and for the fourth one

$$-q^{2n+1-j} d_{n-1,2n+1-j}(q) + \sum_{i=2n-j+2}^{2n+1} q^{2n+1-j}(q-1) d_{n-1,i}(q).$$

Altogether

$$\begin{aligned} (5.7) \quad & d_{n,j}(q) - d_{n,j-1}(q) \\ & = d_{n-1,j-1}(q) + \sum_{i=j}^{2n+1} q^{i-j+1} d_{n-1,i}(q) - \sum_{i=2n-j+2}^{2n+1} q^{2n+1-j} d_{n-1,i}(q); \end{aligned}$$

$$\begin{aligned} (5.8) \quad & - d_{n,j-1}(q) + d_{n,j-2}(q) \\ & = -d_{n-1,j-2}(q) - \sum_{i=j-1}^{2n+1} q^{i-j+2} d_{n-1,i}(q) + \sum_{i=2n-j+3}^{2n+1} q^{2n+2-j} d_{n-1,i}(q). \end{aligned}$$

Summing (5.7) and (5.8) we get:

$$\begin{aligned}
&d_{n,j}(q) - 2d_{n,j-1}(q) + d_{n-1,j-2}(q)\\
&\quad = d_{n-1,j-1}(q) - d_{n-1,j-2}(q)\\
&\qquad - q\, d_{n-1,j-1}(q) + (1-q)\sum_{i=j}^{2n+1} q^{i-j+1} d_{n-1,i}(q)\\
&\qquad - q^{2n+1-j}\, d_{n-1,2n-j+2}(q) - (1-q)\sum_{i=2n-j+3}^{2n+1} q^{2n+1-j} d_{n-1,i}(q).
\end{aligned}$$

Now, $-q^{2n+1-j}\, d_{n-1,2n-j+2}(q) = -q^{n-1} d_{n-1,j-2}(q)$ by Corollary 4.7. Also

$$\begin{aligned}
\sum_{i=2n-j+3}^{2n+1} q^{2n+1-j} d_{n-1,i}(q) &= \sum_{i=2n-j+3}^{2n-1} q^{2n+1-j} d_{n-1,i}(q)\\
&= \sum_{k=1}^{j-3} q^{2n+1-j} d_{n-1,2n-k}(q) \qquad [k = 2n-i]\\
&= \sum_{i=1}^{j-3} q^{n+i+1-j} d_{n-1,i}(q). \qquad \text{[by Corollary 4.7]}
\end{aligned}$$

Hence, $d_{n,j}(q) - 2\, d_{n,j-1}(q) + d_{n,j-2}(q)$

$$\begin{aligned}
&= -(1-q)\sum_{i=1}^{j-3} q^{n+i+1-j}\, d_{n-1,i}(q)\\
&\quad - (1+q^{n-1})\, d_{n-1,j-2}(q) + (1-q)\sum_{i=j-1}^{2n-1} q^{i-j+1}\, d_{n-1,i}(q)
\end{aligned}$$

for $n \geq 2$ and $3 \leq j \leq 2n$, which is precisely relation $(D4)$.

6. Concluding remarks

As mentioned in the Introduction, there has been a great number of papers dealing with the Catalan Triangle (the numbers $d_{n,j}(0)$). The recurrence for the numbers $d_{n,j}(1)$, namely the set of conditions $(P1)$–$(P4)$, is definitely due to Christiane Poupard [Po89] and was recently rediscovered by Graham and Zang [GZ08]. More exactly, the latter authors introduced the notion of *split-pair arrangement*. To show that the number of such arrangements of order n was equal to the reduced tangent number t_n they set up an algebra for the coefficients $d_{n,j}(1)$ and again produced the recurrence $(P1)$–$(P4)$, but their proof of the identity $\sum_j d_{n,j}(1) = t_n$ was more elaborate than the original one made by Christiane Poupard [Po89], shortly sketched in the Introduction.

Referring to Section 4 we say that two doubloons δ, $\delta' \in \mathcal{D}_{2n+1}$ are *equivalent* if there is a sequence $\phi_{i_1}, \phi_{i_2}, \ldots, \phi_{i_k}$ of micro flips such that

$\delta' = \phi_{i_1}\phi_{i_2}\cdots\phi_{i_k}(\delta)$. A doubloon $\delta = \binom{a_0\,a_1\,\cdots\,a_n}{b_0\,b_1\,\cdots\,b_n}$ is said to be *minimal*, if $a_k < b_k$ holds for every k. Clearly, every equivalence class of doubloons contains one and only one minimal doubloon. Also, as proved in [FH08], each doubloon $\delta \in \mathcal{D}^0_{2n+1}$ is equivalent to one and only one normalized doubloon.

The link between *split-pair arrangements* and *equivalence classes* of *interlaced* doubloons is the following: start with a *interlaced* and *minimal* doubloon $\delta = \binom{a_0\,a_1\,\cdots\,a_n}{b_0\,b_1\,\cdots\,b_n}$ and define the word $w = x_1x_2\cdots x_{2n}$ by $x_{l+1} = l$ if and only if a_k or b_k is equal to l. For instance,

$$\delta = \begin{pmatrix} 0\,6\,3\,1\,9 \\ 4\,2\,8\,7\,5 \end{pmatrix} \mapsto w = 1\,4\,2\,3\,1\,5\,2\,4\,3\,5.$$

As δ is interlaced (condition $(N1)$), exactly one of the integers a_k, b_k lies between a_{k-1} and b_{k-1}. If $a_k = l < l' = b_k$, then $x_{l+1} = x_{l'+1} = k$. If $a_{k-1} = m < a_k = l < b_{k-1} = m' < b_k = l'$, then $x_{m+1} = x_{m'+1} = k-1$ and there is exactly one letter equal to k, namely x_{l+1} between the two occurrences of $(k-1)$, namely x_{m+1} and $x_{m'+1}$. Same conclusion if $a_{k-1} < a_k < b_{k-1} < b_k$. Those words w of length $2n$ having the property that exactly one letter equal to k lies between two occurrences of $(k-1)$ for each $k = 2, 3, \ldots, n$ was called *split-pair arrangements* by Graham and Zang [GZ08]. The mapping $\delta \mapsto w$ provides a bijection between equivalence classes of interlaced doubloons and those arrangements.

A permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(2n+1)$ of $12\cdots(2n+1)$ is said to be *alternating* if $\sigma(2i) < \sigma(2i-1), \sigma(2i+1)$ for each $i = 1, 2, \ldots, n$. For each even integer $2i$ $(1 \le i \le n)$ let w'_i (resp. w''_i) be the longest right factor of $\sigma(1)\cdots\sigma(2i-1)$ (resp. longest left factor of $\sigma(2i+1)\cdots\sigma(2n+1)$), all letters of which are greater than $\sigma(2i)$. Let $\min w'_i$ (resp. $\min w''_i$) denote the minimum letter of w'_i (resp. of w''_i). Then σ is called an *alternating André* permutation if it is alternating and satifies $\min w'_i > \min w''_i$ for every $i = 1, 2, \ldots, n$. The number of alternating André permutations of order $(2n+1)$ is equal to t_n (see, e.g., [FS71], Property 2.6 and (5.4), or [Po89]). The alternating André permutations of order 5 are the following: 53412, 51423, 41523, 31524. Let $A_{2n+1,j}$ denote the set of alternating André permutations of order $(2n+1)$ *ending with* j.

Thanks to the identity

$$d_{n,j}(1) = \sum_{i\ge 0} \binom{2n+1-j}{2i+1} t_i \sum_{k=0}^{j-1} d_{n-i-1,k}(1), \tag{6.1}$$

valid for $n \ge 1$, $2 \le j \le 2n-1$ ($d_{0,j}(1) = \delta_{0,j}$), derived from the bivariable generating function (1.8), Christiane Poupard proved that

$$\#\, A_{2n+1,j} = d_{n,j}(1). \tag{6.2}$$

Three questions arise:

(1) Construct a natural (?) bijection of $\mathcal{N}^0_{2n+1,j}$ onto $A_{2n+1,j}$.

(2) Find an adequate statistic "stat" on the alternating André permutation such that holds the identity

$$d_{n,j}(q) = \sum_{\sigma \in A_{2n+1,j}} q^{\operatorname{stat} \sigma}.$$

(3) As mentioned in the Introduction, we have an expression for the generating function for $d_{n,j}(1)$ (formula (1.8)) and for $d_{n,j}(0)$ (formula (1.14)). Find the exponential (or factorial) generating function for $d_{n,j}(q)$.

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