A WREATH PRODUCT MULTIVARIABLE EXTENSION

Dominique Foata and Guo-Niu Han

This Note is a complement to paper [6], especially Theorem 1.3, when the total ordering imposed on the wreath product $C_l \wr \mathfrak{S}_n$ is defined as follows: the set $\{0,1,\ldots,l-1\}$ is split into two subsets I, J such that $0\in I$ and the order has the properties:

(a) $(j,i)<(j',i')$ when $i>i'$ for all j,j';

(b) $(1,i)<(2,i)<\cdots<(n,i)$ when $i\in I$;

(c) $(n,i)<\cdots<(2,i)<(1,i)$ when $i\in J$.

When $J=\emptyset$, we recover the total order used in [6]. For $l=2$, $I=\{0\}$, $J=\{1\}$, we get the natural order defined on the hyperoctahedral group B_n, namely, $-n<\cdots<-1<1<\cdots<n$ with the convention: $-j\equiv(j,1)$, $j\equiv(j,0)$ for all $j=1,2,\ldots,n$.

We keep the notations used in [6]. When using the above new order, determined by the pair (I,J), the polynomial defined in [6], (1.12) will be redefined as

$$W_n(s,t,q,(Y_i),(Z_i),I,J) := \sum_{(w,\epsilon)\in C_l\wr\mathfrak{S}_n} s^{\mathrm{fexc}(w,\epsilon)} t^{\mathrm{fdes}(w,\epsilon)} q^{\mathrm{fmaj}(w,\epsilon)} \prod_{0\le i\le l-1} Z_i^{|\epsilon|_i} Y_i^{\mathrm{fix}_i(w,\epsilon)},$$

to emphasize the presence of the partition (I,J). The computation made in [6], Section 3 can be reproduced verbatim. Instead of (3.3) and (3.2) we get

$$\sum_{n\ge0}(1+t+\cdots+t^{l-1})W_n(s,t,q,(Y_i),(Z_i),I,J)\frac{u^n}{(t^l;q^l)_{n+1}} = \sum_{r\ge0} t^r G_r(u;s,q,(Y_i),(Z_i),I,J),$$

where

$$G_r(u;s,q,(Y_i),(Z_i),I,J) := \sum_{(c,w,\epsilon)\in\mathrm{WP}(r,I,J)} u^{\lambda w} s^{\mathrm{fexc}(w,\epsilon)} q^{\mathrm{tot}\,c} \prod_{0\le i\le l-1} Y_i^{\mathrm{fix}_i(w,\epsilon)} Z_i^{|\epsilon|_i}.$$

The only difference is the following. The last sum is over the set $\text{WP}(r,I,J)$ of all wreathed permutations (c,w,ϵ) that can be expressed as juxtaposition products

$$\begin{pmatrix} c\\ w\\ \epsilon\end{pmatrix}=\begin{pmatrix} c^{(1)}\\ w^{(1)}\\ \epsilon^{(1)}\end{pmatrix}\begin{pmatrix} c^{(2)}\\ w^{(2)}\\ \epsilon^{(2)}\end{pmatrix}\cdots\begin{pmatrix} c^{(k)}\\ w^{(k)}\\ \epsilon^{(k)}\end{pmatrix},$$

where each $c^{(j)}$ is of the form $a_j^{m_j}$ $(1\le j\le k)$, $m_1+m_2+\cdots+m_k=n$ and $a_1>a_2>\cdots>a_k$. Also, $\epsilon^{(j)}=\overline{a}_j^{m_j}$, where $\overline{a}_j$ is the residue of a_j mod l. Then, each word $w^{(j)}$ is *increasing* (resp. *decreasing*) with respect of natural order of the integers, if and only if $j\in I$ (resp. $j\in J$).

Example. Let $l=3$, $I=\{0,2\}$, $J=\{1\}$. Then

$$\begin{pmatrix}\text{Id}\\ c\\ w\\ \epsilon\end{pmatrix}=\left(\begin{array}{cc|c|ccc|cccc|cc|c} 1&2&3&4&5&6&7&\mathbf{8}&9&10&\mathbf{11}&12&13\\ 12&12&11&9&9&9&7&7&7&7&4&4&2\\ 1&2&13&5&7&12&10&\mathbf{8}&4&3&\mathbf{11}&9&6\\ 0&0&2&0&0&0&1&1&1&1&1&1&2\end{array}\right),$$

is such a wreathed permutation. The factors of w above the factors $1,1,1,1$ and $1,1$ of ϵ are *decreasing.*

The important property on such wreathed permutations (which appears to be instrumental in the construction of the next bijection) is the fact each *decreasing* factor $w^{(j)}$ such that $\epsilon^{(j)}=\overline{a}_j^{m_j}$ and $\overline{a}_j\in J$ has *at most one* fixed point of (w,ϵ). In the previous example there is one fixed point in each of the two factors $w^{(j)}$ such that $\overline{a}_j=1$, namely **8** and **11**.

Let $\text{WP}'_n(r,I,J)$ denote the subset of $\text{WP}_n(r,I,J)$ of all wreathed permutations having *no i-fixed points* when $i\in J$ and let $\text{WP}'(r,I,J)$ be the union of all $\text{WP}'_n(r,I,J)$. Also, let $\text{DW}_n^{(i)}(r)$ be the subset of $\text{NIW}_n(r)$ of all *strictly decreasing* words, whose letters are congruent to i mod l and $\text{DW}^{(i)}(r)$ be the union of such $\text{DW}_n^{(i)}(r)$.

Start with a wreathed permutation (c,w,ϵ) from $\text{WP}(r,I,J)$ and for each $i\in J$ let $j_1<j_2<\cdots<j_{h(i)}$ be the increasing sequence of all integers j such that j is an i-fixed point: $x_j=j$ and $\epsilon_j=i$; define $v^{(i)}:=c_{j_1}c_{j_2}\cdots c_{j_{h(i)}}$. As the word c is monotonic non-increasing and since there is at most one i-fixed point in each factor $w^{(j)}$ of w when $i\in J$, the word $v^{(i)}$ is strictly decreasing. More essentially,

$$v^{(i)}\in\text{DW}^{(i)}(r)\quad\text{if}\quad(c,w,\epsilon)\in\text{WP}(r,I,J).$$

For all $i\in J$ remove all the columns $\left(\begin{smallmatrix} c_j\\ x_j\\ \epsilon_j\end{smallmatrix}\right)$ such that j is an i-fixed point and form the family $(v^{(i)})_{i\in J}$. After removal the remaining columns form a wreathed permutation on a subset K of $[1,n]$. Using the unique increasing bijection ϕ of K onto the interval $[1,|K|]$, next obtain a wreathed permutation of order $|K|$. Each factor $w^{(j)}$ of the original wreathed permutation (c,w,ϵ) is then transformed into a word $\phi(w^{(j)})$. The final step consists of replacing each $\phi(w^{(j)})$ into its mirror-image, whenever $i \in J$. Let (c',w',ϵ') be the wreathed permutation ultimately obtained. Then

$$(c',w',\epsilon') \in \mathrm{WP}'(r,I,J).$$

In the previous example the removals of columns 8 and 11 yield

$$\begin{pmatrix} \begin{array}{cc|c|ccc|ccc|c|c} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 9 & 10 & 12 & 13 \\ 12 & 12 & 11 & 9 & 9 & 9 & 7 & 7 & 7 & 4 & 2 \\ 1 & 2 & 13 & 5 & 7 & 12 & 10 & 4 & 3 & 9 & 6 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 \end{array} \end{pmatrix}, \quad \text{and} \quad v^{(1)} := c_8 c_{11} = 7,4.$$

After reduction we obtain

$$\begin{pmatrix} \begin{array}{cc|c|ccc|ccc|c|c} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 12 & 12 & 11 & 9 & 9 & 9 & 7 & 7 & 7 & 4 & 2 \\ 1 & 2 & 11 & 5 & 7 & 10 & 9 & 4 & 3 & 8 & 6 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 \end{array} \end{pmatrix}.$$

and, finally,

$$\begin{pmatrix} \mathrm{Id} \\ c' \\ w' \\ \epsilon' \end{pmatrix} = \begin{pmatrix} \begin{array}{cc|c|ccc|ccc|c|c} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 12 & 12 & 11 & 9 & 9 & 9 & 7 & 7 & 7 & 4 & 2 \\ 1 & 2 & 11 & 5 & 7 & 10 & 3 & 4 & 9 & 8 & 6 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 \end{array} \end{pmatrix}.$$

Proposition. *The mapping* $(c,w,\epsilon) \mapsto ((c',w',\epsilon'),(v^{(i)})_{i\in J})$ *is a bijection of* $\mathrm{WP}(r,I,J)$ *onto* $\mathrm{WP}'(r,I,J) \times \prod_{i\in J} \mathrm{DW}^{(i)}(r)$ *having the properties:*

(1) $\lambda w = \lambda w' + \sum_{i\in J} \lambda v^{(i)}$;

(2) $\operatorname{tot} c = \operatorname{tot} c' + \sum_{i\in J} \operatorname{tot} v^{(i)}$;

(3) $|\epsilon|_i = |\epsilon'|_i$ *for* $i \in I$; $\quad |\epsilon|_i = |\epsilon'|_i + \lambda v^{(i)}$ *for* $i \in J$;

(4) $\mathrm{fix}_i(w,\epsilon) = \mathrm{fix}_i(w',\epsilon')$ *for* $i \in I$;

(5) $\mathrm{fix}_i(w,\epsilon) = \lambda v^{(i)}$ *for* $i \in J$;

(6) $\mathrm{exc}(w,\epsilon) = \mathrm{exc}(w',\epsilon')$;

(7) $\mathrm{fexc}(w,\epsilon) = \mathrm{fexc}(w',\epsilon') + \sum_{i\in J} i \cdot \lambda v^{(i)}$.

The proof of the Proposition requires no further techniques. ∎

Now, by using the identity

$$(-u;q)_{N+1}=\sum_{n\ge 0}\sum_{v\in \mathrm{DW}_n(r)} q^{\mathrm{tot}\,v},$$

we have:

$$\begin{aligned}\sum_{n\ge 0}u^n\sum_{v\in \mathrm{DW}_n^{(i)}(r)} q^{\mathrm{tot}\,v}&=\sum_{n\ge 0}(uq^i)^n\sum_{v'\in \mathrm{DW}_n(\lfloor (r-i)/l\rfloor)} q^{l\,\mathrm{tot}\,v'}\\&=(-uq^i;q^l)_{\lfloor (r-i)/l\rfloor+1}.\end{aligned}$$

Let $A(r,J)$ be the set of all sets $(w^{(i)})_{i\in J}$, where $w^{(i)}\in \mathrm{DW}^{(i)}(r)$ $(i\in J)$. Summing over all pairs $((w^{(i)})_{i\in J},(c,w,\epsilon))$ from $A(r,J)\times \mathrm{WP}(r,I,J)$ we get:

$$\begin{aligned}&\sum_{(w^{(i)})_{i\in J},(c,w,\epsilon)}\prod_{i\in J}(us^iZ_i)^{\lambda w^{(i)}}q^{\mathrm{tot}\,w^{(i)}}\\&\qquad\times u^{\lambda(c)}q^{\mathrm{tot}\,c}s^{\mathrm{fexc}(w,\epsilon)}\prod_{i\in I+J}Z_i^{|\epsilon|_i}\prod_{i\in I}Y_i^{\mathrm{fix}_i(w,\epsilon)}\prod_{i\in J}Y_i^{\mathrm{fix}_i(w,\epsilon)}\\&=\prod_{i\in J}(-us^iq^iZ_i;q^l)_{\lfloor (r-i)/l\rfloor+1}\\&\qquad\times\sum_{(c,w,\epsilon)\in \mathrm{WP}(r,I,J)}u^{\lambda(c)}q^{\mathrm{tot}\,c}s^{\mathrm{fexc}(w,\epsilon)}\prod_{i\in I+J}Z_i^{|\epsilon|_i}\prod_{i\in I}Y_i^{\mathrm{fix}_i(w,\epsilon)}\prod_{i\in J}Y_i^{\mathrm{fix}_i(w,\epsilon)}\\&=\prod_{i\in J}(-us^iq^iZ_i;q^l)_{\lfloor (r-i)/l\rfloor+1}\times G_r(u;s,q,(Y_i),(Z_i),I,J).\end{aligned}$$

As $\mathrm{fexc}(w,\epsilon)=\mathrm{fexc}(w',\epsilon')+\sum_{i\in J}i\,\lambda v^{(i)}$, the initial expression can also be summed over triples $((w^{(i)})_{i\in J},(c',w',\epsilon'),(v^{(i)})_{i\in J})$ from $A(r,J)\times \mathrm{WP}'(r,I,J)\times A(r,J)$ in the form

$$\begin{aligned}&\sum_{(w^{(i)})_{i\in J},(c',w',\epsilon'),(v^{(i)})_{i\in J}}u^{\lambda\,c'}q^{\mathrm{tot}\,c'}s^{\mathrm{fexc}(w',\epsilon')}\prod_{i\in I+J}Z_i^{|\epsilon'|_i}\prod_{i\in I}Y_i^{\mathrm{fix}_i(w',\epsilon')}\\&\qquad\times\prod_{i\in J}(us^iY_iZ_i)^{\lambda\,v^{(i)}}q^{\mathrm{tot}\,v^{(i)}}\times\prod_{i\in J}(us^iZ_i)^{\lambda w^{(i)}}q^{\mathrm{tot}\,w^{(i)}}\\&=\sum_{(v^{(i)})_{i\in J}}\prod_{i\in J}(us^iY_iZ_i)^{\lambda\,v^{(i)}}q^{\mathrm{tot}\,v^{(i)}}\\&\qquad\times\sum_{(c',w',\epsilon'),(w^{(i)})_{i\in J}}u^{\lambda\,c'}q^{\mathrm{tot}\,c'}s^{\mathrm{fexc}(w',\epsilon')}\prod_{i\in I+J}Z_i^{|\epsilon'|_i}\prod_{i\in I}Y_i^{\mathrm{fix}_i(w',\epsilon')}\\&\qquad\times\prod_{i\in J}(us^iZ_i)^{\lambda\,w^{(i)}}q^{\mathrm{tot}\,w^{(i)}}\end{aligned}$$

$$= \prod_{i\in J}(-us^iq^iY_iZ_i;q^l)_{\lfloor (r-i)/l\rfloor+1}$$
$$\times \sum_{(c,w,\epsilon)\in \mathrm{WP}(r,I,J)} u^{\lambda\, c} q^{\mathrm{tot}\, c} s^{\mathrm{fexc}(w,\epsilon)} \prod_{i\in I+J} Z_i^{|\epsilon|_i} \prod_{i\in I} Y_i^{\mathrm{fix}_i(w,\epsilon)}.$$

But the last sum is also equal to

$$\sum_{(c,w,\epsilon)\in \mathrm{WP}(r)} u^{\lambda\, c} q^{\mathrm{tot}\, c} s^{\mathrm{fexc}(w,\epsilon)} \prod_{i\in I+J} Z_i^{|\epsilon|_i} \prod_{i\in I} Y_i^{\mathrm{fix}_i(w,\epsilon)}$$
$$= G_r(u;s,q,(Y_i),(Z_i)) \,|_{Y_i=1 \text{ for all } i\in J},$$

where $G_r(u;s,q,(Y_i),(Z_i))$, itself expressed in [6] (3.2), has been proved to be equal to

$$\frac{\prod\limits_{1\le i\le l} (uq^is^iZ_i;q^l)_{\lfloor (r-i)/l\rfloor+1}}{\prod\limits_{0\le i\le l-1} (uq^is^iY_iZ_i;q^l)_{\lfloor (r-i)/l\rfloor+1}}\, H_r(u;q,s,(Z_i)),$$

where

$$H_r(u;q,s,(Z_i)) = Z_0(1-q^ls^l)$$
$$\times \Big(Z_0 - Z_0q^ls^l + \sum_{i=1}^{l} q^is^iZ_i - \sum_{i=1}^{l} q^is^iZ_i \frac{(uq^ls^lZ_0;q^l)_{\lfloor (r-i)/l\rfloor+1}}{(uZ_0;q^l)_{\lfloor (r-i)/l\rfloor+1}}\Big)^{-1}.$$

Hence

$$G_r(u;s,q,(Y_i),(Z_i)) \,|_{Y_i=1 \text{ for all } i\in J}$$
$$= \frac{\prod\limits_{i\in I} (uq^is^iZ_i;q^l)_{\lfloor (r-i)/l\rfloor+1}}{\prod\limits_{i\in I} (uq^is^iY_iZ_i;q^l)_{\lfloor (r-i)/l\rfloor+1}}\, H_r(u;q,s,(Z_i)).$$

From the identity

$$\prod_{i\in J}(-us^iq^iZ_i;q^l)_{\lfloor (r-i)/l\rfloor+1} \times G_r(u;s,q,(Y_i),(Z_i),I,J)$$
$$= \prod_{i\in J}(-us^iq^iY_iZ_i;q^l)_{\lfloor (r-i)/l\rfloor+1} \times G_r((u;s,q,(Y_i),(Z_i)) \,|_{Y_i=1 \text{ for all } i\in J}$$

we deduce:

$$G_r(u;s,q,(Y_i),(Z_i),I,J)$$
$$= \frac{\prod\limits_{i\in J} (-us^iq^iY_iZ_i;q^l)_{\lfloor (r-i)/l\rfloor+1}}{\prod\limits_{i\in J} (-us^iq^iZ_i;q^l)_{\lfloor (r-i)/l\rfloor+1}} \frac{\prod\limits_{i\in I} (uq^is^iZ_i;q^l)_{\lfloor (r-i)/l\rfloor+1}}{\prod\limits_{i\in I} (uq^is^iY_iZ_i;q^l)_{\lfloor (r-i)/l\rfloor+1}}$$
$$\times H_r(u;q,s,(Z_i)).$$

Consequently, the factorial generating function for the polynomials $W_n(s,t,q,(Y_i),(Z_i),I,J)$ can be expressed in the form:

$$\sum_{n\ge 0}(1+t+\cdots+t^{l-1})W_n(s,t,q,(Y_i),(Z_i),I,J)\frac{u^n}{(t^l;q^l)_{n+1}} = \sum_{r\ge 0} t^r G_r(u;s,q,(Y_i),(Z_i),I,J).$$

When $l=2$, $I=\{0\}$, $J=\{1\}$, $Z_0=1$, $Z_1=Z$, with the convention $-j\equiv(j,1)$, $j\equiv(j,0)$ for all $j=1,2,\ldots$, we recover the following identity derived in [1] for the hyperoctahedral group:

$$\begin{aligned}\sum_{n\ge 0}(1+t)W_n(s,t,q,Y_0,Y_1,Z)\frac{u^n}{(t^2;q^2)_{n+1}} &= \sum_{r\ge 0} t^r \frac{(u;q^2)_{\lfloor r/2\rfloor+1}}{(uY_0;q^2)_{\lfloor r/2\rfloor+1}}\frac{(-usqY_1Z;q^2)_{\lfloor (r+1)/2\rfloor}}{(-usqZ;q^2)_{\lfloor (r+1)/2\rfloor}}\\ &\times \frac{(us^2q^2;q^2)_{\lfloor r/2\rfloor}\,(1-s^2q^2)\,(u;q^2)_{\lfloor (r+1)/2\rfloor}\,(u;q^2)_{\lfloor r/2\rfloor}}{(u;q^2)_{\lfloor r/2\rfloor+1}\Big((u;q^2)_{\lfloor (r+1)/2\rfloor}\big((u;q^2)_{\lfloor r/2\rfloor}-s^2q^2\,(us^2q^2;q^2)_{\lfloor r/2\rfloor}\big) + sqZ\,(u;q^2)_{\lfloor r/2\rfloor}\big((u;q^2)_{\lfloor (r+1)/2\rfloor}-(us^2q^2;q^2)_{\lfloor (r+1)/2\rfloor}\big)\Big)}.\end{aligned}$$

References

[1] Dominique Foata, Guo-Niu Han. Signed words and permutations, V; a sextuple distribution, *Ramanujan J.*, vol. **19**, 2009, p. 29–52.

[6] Dominique Foata; Guo-Niu Han. The decrease value theorem with an application to permutation statistics, 19 p.