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THE FLAG-DESCENT AND -EXCEDANCE NUMBERS

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We deal with the wreath product $C_l \wr \mathfrak{S}_n$, where C_l is the cyclic group of order l and $\mathfrak{S}_n$ the symmetric group of order n. The elements of $C_l \wr \mathfrak{S}_n$ are viewed as ordered pairs (w,ϵ), where $w = x_1x_2\cdots x_n$ is a permutation of $12\ldots n$ and $\epsilon = \epsilon_1\epsilon_2\cdots\epsilon_n$ is a word of length n, all letters of which belong to $\{0,1,\ldots,l-1\}$. As a total order on $C_l \wr \mathfrak{S}_n$ we take:

$$(1,l-1)<(2,l-1)<\cdots<(n,l-1)<(1,l-2)<\cdots<(n,l-2)$$
$$<\cdots<(1,0)<(2,0)<\cdots<(n,0).$$

For each argument A let $\chi(A)=1$ or 0, depending on whether A is true or false. Also, for each word $\epsilon=\epsilon_1\epsilon_2\cdots\epsilon_n$, whose letters ϵ_i are nonnegative letters, let $\operatorname{tot}\epsilon := \epsilon_1+\epsilon_2+\cdots+\epsilon_n$. Now, for each element $(w,\epsilon)=(x_1,\epsilon_1)(x_2,\epsilon_2)\cdots(x_n,\epsilon_n)$ from $C_l \wr \mathfrak{S}_n$ define

$$\begin{aligned}
\operatorname{tot}\epsilon &:= \epsilon_1+\epsilon_2+\cdots+\epsilon_n;\\
\operatorname{exc} w &:= \#\{j : 1\le j\le n,\, x_j>j, \epsilon_j=0\};\\
\operatorname{des}(w,\epsilon) &:= \#\{j : 1\le j\le n-1, (x_j,\epsilon_j)>(x_{j+1},\epsilon_{j+1})\};\\
\operatorname{maj}(w,\epsilon) &:= \sum_{1\le j\le n-1} j\,\chi((x_j,\epsilon_j)>(x_{j+1},\epsilon_{j+1}));
\end{aligned}$$

as well as the flag-statistics:

$$\begin{aligned}
\operatorname{fexc}(w,\epsilon) &:= l.\operatorname{exc} w+\operatorname{tot}\epsilon;\\
\operatorname{fdes}(w,\epsilon) &:= l.\operatorname{des}(w,\epsilon)+\epsilon_1;\\
\operatorname{fmaj}(w,\epsilon) &:= l.\operatorname{maj}(w,\epsilon)+\operatorname{tot}\epsilon.
\end{aligned}$$

Let $W_n^l(t,q)=\sum_{(w,\epsilon)\in C_l\wr\mathfrak{S}_n} t^{\operatorname{fdes}(w,\epsilon)}q^{\operatorname{fmaj}(w,\epsilon)}$ be the generating polynomial for $C_l \wr \mathfrak{S}_n$ by the pair (fdes, fmaj). In [6] (Theorem 5.3) it is shown that the generating function for the polynomials $W_n^l(t,q)$ can be expressed as:

$$(1)\quad \sum_{n\ge0}(1+t+\cdots+t^{l-1})W_n^l(t,q)\frac{u^n}{(t^l;q^l)_{n+1}} = \sum_{r\ge0} t^r\frac{1}{1-u[r+1]}$$
$$= \sum_{n\ge0}u^n\sum_{r\ge0}t^r[r+1]^n.$$

In the present Note we first obtain four other *equivalent* definitions for the polynomials $W_n^l(t,q)$ (Section 1). Then, let $W_n^l(t,1):=\sum_k W_{n,k}^l t^k$. In the next Sections the two identities

$$W_{n,k}^l = \#\{(w,\epsilon)\in C_l\wr\mathfrak{S}_n : \mathrm{fdes}(w,\epsilon)=k\};\tag{2}$$

$$W_{n,k}^l = \#\{(w,\epsilon)\in C_l\wr\mathfrak{S}_n : \mathrm{fexc}(w,\epsilon)=k\};\tag{3}$$

are given *combinatorial proofs.*

1. Four equivalent definitions

Identity (1) is also equivalent to:

$$\frac{W_n^l(t,q)}{(1-t)(t^lq^l;q^l)_n}=\sum_{r\ge 0}t^r[r+1]^n\quad (l\ge 1,\, n\ge 0).\tag{4}$$

Other definitions can be worked out in the following way. Starting with (4) we have:

$$\begin{aligned}\frac{W_n^l(t,q)}{(1-t)(t^lq^l;q^l)_n}&=\sum_{r\ge 0}t^r\frac{1-q^{r+1}}{1-q}[r+1]^{n-1}\\
&=\frac{1}{1-q}\sum_{r\ge 0}t^r[r+1]^{n-1}-\frac{q}{1-q}\sum_{r\ge 0}(tq)^r[r+1]^{n-1}\\
&=\frac{1}{1-q}\frac{W_{n-1}^l(t,q)}{(1-t)(t^lq^l;q^l)_{n-1}}-\frac{q}{1-q}\frac{W_{n-1}^l(tq,q)}{(1-tq)(t^lq^{2l};q^l)_{n-1}}.\end{aligned}$$

This can be rewritten

$$\begin{aligned}&(1-q)(1-tq)W_n^l(t,q)\\
&\qquad=(1-tq)(1-t^lq^l)W_{n-1}^l(t,q)-q(1-t)(1-t^lq^l)W_{n-1}^l(tq,q),\end{aligned}$$

further simplified into

$$\begin{aligned}(1-q)W_n^l(t,q-&=(1-t^lq^l)W_{n-1}^l(t,q)\\
&\qquad -q(1-t)(1+tq+\cdots+t^{l-1}q^{l-1})W_{n-1}^l(tq,q)\end{aligned}$$

or still

$$\begin{aligned}(1-q)W_n^l(t,q)&=(1-t^lq^l)W_{n-1}^l(t,q)\\
&\quad-(-q+tq(1-q)+\cdots+t^{l-1}q^{l-1}(1-q)+t^lq^l)W_{n-1}^l(tq,q).\end{aligned}\tag{5}$$

Now, look for the coefficients of t^k on both sides. With $W_n^l(t,q):=\sum_k W_{n,k}^l(q)t^k$ we get:

$$\begin{aligned}(1-q)W_{n,k}^l(q)&=W_{n-1,k}^l(q)-q^{nl}W_{n-1,k-l}^l(q)\\
&\quad-q^{k+1}W_{n-1,k}^l(q)+q^{k+1}(1-q)W_{n-1,k-1}^l(q)\\
&\quad+\cdots+q^{k+l-1}(1-q)W_{n-1,k-(l-1)}^l(q)+q^kW_{n-1,k-l}^l(q),\end{aligned}$$

so that, by dividing both sides by $(1-q)$,

$$\begin{aligned}(6)\quad W^l_{n,k}(q) &= [k+1]_q W^l_{n-1,k}(q) + q^{k+1} W^l_{n-1,k-1}(q)\\ &+ q^{k+2} W^l_{n-1,k-2}(q) + \cdots + q^{k+l-1} W^l_{n-1,k-(l-1)}(q) + q^k [nl-k]_q W^l_{n-1,k-l}(q).\end{aligned}$$

Thus, formulas (1), (4), (5), (6) are four equivalent definitions for the statistical distribution of the pair (fdes, fmaj) over $C_l \wr \mathfrak{S}_n$. There is a fifth one, that may be regarded as the *finite analog* of (4), which reads:

$$(7)\quad \frac{1}{1+t+\cdots+t^{l-1}} \sum_{j=1}^{m} t^j ([j]_q)^n$$
$$= t \frac{W^l_n(t,q)}{(t^l;q^l)_{n+1}} - t^{m+1} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} [m]_q^{n-k} \frac{W^l_k(tq^{n-k},q)}{(t^l q^{(n-k)l};q^l)_{k+1}}.$$

We do not reproduce the proof of the equivalence, as it is quite similar to the proof of (5.2) in [1]. Remember that $W^l_n(t,q) = A_n(t,q)$, the Carlitz q-Eulerian polynomial, for $l=1$.

2. The combinatorial proof for the flag-descent number

Let $q=1$ in identity (4). Remembering that $W^l_n(t,1) := \sum_k W^l_{n,k} t^k$ we obtain:

$$(8)\quad W^l_{n,k} = (k+1) W^l_{n-1,k} + W^l_{n-1,k-1} + \cdots + W^l_{n-1,k-(l-1)} + (ln-k) W^l_{n-1,k-l},$$

for $n \geq 2$ and $0 \leq k \leq nl-1$ and the initial conditions: $W^l_{0,0} = 1$, $W^l_{0,k} = 0$ for $k \neq 0$; $W^l_{1,k} = 1$ for $k = 0,1,\ldots,l-1$ and 0 for any other value of k. In this Section let $W^l_{n,k}$ denote the number of elements (w,ϵ) from $C_l \wr \mathfrak{S}_n$ such that $\mathrm{fdes}(w,\epsilon) = k$. Our intention is to prove that $W^l_{n,k}$ satisfies recurrence (8).

Note that $C_l \wr \mathfrak{S}_n$ is generated from $C_l \wr \mathfrak{S}_{n-1}$ (from $l^{n-1}(n-1)!$ elements to $l^{n-1}(n-1)! \times l\,n = l^n \cdot n!$ elements) by inserting the biletter $\binom{n}{i}$ $(0 \leq i \leq l-1)$ to the left of each element $\binom{w}{\epsilon} = \binom{x_1\,x_2\,\cdots\,x_{n-1}}{\epsilon_1\,\epsilon_2\,\cdots\,\epsilon_{n-1}}$ from $C_l \wr \mathfrak{S}_{n-1}$, or between two consecutive letters $\binom{x_j}{\epsilon_j}$, $\binom{x_{j+1}}{\epsilon_{j+1}}$, or still to the right of $\binom{w}{\epsilon}$. Let us examine those three possibilities.

1. *Insertion to the left of* $\binom{w}{\epsilon}$. We have:

$$\mathrm{fdes}\binom{n\,w}{i\,\epsilon} = \begin{cases} \mathrm{fdes}\binom{w}{\epsilon} + l + i - \epsilon_1, & \text{if } 0 \leq i \leq \epsilon_1;\\ \mathrm{fdes}\binom{w}{\epsilon} + i - \epsilon_1, & \text{if } 0 \leq \epsilon_1 < i.\end{cases}$$

Next, let $1 \leq j \leq n-1$ and

$$\binom{w'}{\epsilon'} := \binom{x_1 \cdots x_j\; n\; x_{j+1} \cdots x_{n-1}}{\epsilon_1 \cdots \epsilon_j\; i\; \epsilon_{j+1} \cdots \epsilon_{n-1}}.$$

2. *Insertion into a descent of* $\binom{w}{\epsilon}$. By assumption, $\binom{x_j}{\epsilon_j} > \binom{x_{j+1}}{\epsilon_{j+1}}$, so that $\epsilon_j < \epsilon_{j+1}$, or $\epsilon_j = \epsilon_{j+1}$ and $x_j > x_{j+1}$. A simple verification leads to:

$$\operatorname{fdes}\binom{w'}{\epsilon'} = \begin{cases} \operatorname{fdes}\binom{w}{\epsilon}, & \text{if } i \le \epsilon_j < \epsilon_{j+1}, \text{ or } i \le \epsilon_j = \epsilon_{j+1} \text{ and } x_j > x_{j+1}, \\ & \text{or if } \epsilon_j < \epsilon_{j+1} < i, \\ & \qquad \text{or } \epsilon_j = \epsilon_{j+1} < i \text{ and } x_j > x_{j+1}; \\ \operatorname{fdes}\binom{w}{\epsilon} + l, & \text{if } \epsilon_j < i \le \epsilon_{j+1}. \end{cases}$$

3. *Insertion into a rise of* $\binom{w}{\epsilon}$. By assumption, $\binom{x_j}{\epsilon_j} < \binom{x_{j+1}}{\epsilon_{j+1}}$, so that $\epsilon_j > \epsilon_{j+1}$, or $\epsilon_j = \epsilon_{j+1}$ and $x_j < x_{j+1}$. Again, we get:

$$\operatorname{fdes}\binom{w'}{\epsilon'} = \begin{cases} \operatorname{fdes}\binom{w}{\epsilon} + l, & \text{if } i \le \epsilon_{j+1} < \epsilon_j, \text{ or } i \le \epsilon_j = \epsilon_{j+1} \text{ and } x_j < x_{j+1}, \\ & \text{or if } \epsilon_{j+1} < \epsilon_j < i, \\ & \qquad \text{or } \epsilon_{j+1} = \epsilon_j < i \text{ and } x_j < x_{j+1}; \\ \operatorname{fdes}\binom{w}{\epsilon}, & \text{if } \epsilon_j \ge i > \epsilon_{j+1}. \end{cases}$$

4. *Insertion to the right of* $\binom{w}{\epsilon}$. Clearly,

$$\operatorname{fdes}\binom{w\,n}{\epsilon\,i} = \begin{cases} \operatorname{fdes}\binom{w}{\epsilon} + l, & \text{if } \epsilon_{n-1} < i \le l-1; \\ \operatorname{fdes}\binom{w}{\epsilon}, & \text{if } 0 \le i \le \epsilon_{n-1}. \end{cases}$$

For each pair (k, n) let $\mathcal{W}^l_{n,k}$ denote the set of all elements from $C_l \wr \mathfrak{S}_n$, whose "fdes" are equal to k. The previous analysis shows that the insertion of $\binom{n}{i}$ increases "fdes" by $1, 2, \ldots$ or l in procedure 1, and keeps it invariant or increases it by l in procedures 2, 3 and 4. Hence, for each $h = 1, 2, \ldots, l$ each element from $\mathcal{W}_{n-1,k-h}$ gives rise to one and only one element from $\mathcal{W}_{n,k}$, if the l insertions to the left are applied (procedure 1).

Let $k = k'l + r$ $(0 \le r \le l-1)$ and let $\binom{w}{\epsilon}$ belong to $\mathcal{W}^l_{n-1,k}$, so that $\operatorname{des}\binom{w}{\epsilon} = k'$ and $\epsilon_1 = r$. The number of insertions of $\binom{n}{i}$ into $\binom{w}{\epsilon}$, using procedures 2, 3 and 4, which keep "fdes" invariant is equal to

$$\begin{aligned} &\sum_{j=1}^{n-2} \big((\epsilon_j + 1) + (l - 1 - \epsilon_{j+1})\big)\chi\big(\tbinom{x_j}{\epsilon_j} > \tbinom{x_{j+1}}{\epsilon_{j+1}}\big) \\ &\qquad\qquad + \sum_{j=1}^{n-2} \big(\epsilon_j - \epsilon_{j+1}\big)\chi\big(\tbinom{x_j}{\epsilon_j} < \tbinom{x_{j+1}}{\epsilon_{j+1}}\big) + (\epsilon_{n-1} + 1) \\ &\quad = \epsilon_1 - \epsilon_2 + \epsilon_2 - \epsilon_3 + \cdots + \epsilon_{n-2} - \epsilon_{n-1} + \epsilon_{n-1} + k' \cdot l + 1 \\ &\quad = k'l + \epsilon_1 + 1 = k + 1. \end{aligned}$$

Finally, suppose that $\operatorname{fdes}\binom{w}{\epsilon} = k - l = k'l + \epsilon_1$. The number of insertions of $\binom{n}{i}$ into $\binom{w}{\epsilon}$, using procedures 1, 2, 3 and 4, which increase "fdes" by l is equal to

$$\begin{aligned}
&1+\sum_{j=1}^{n-2}\big((\epsilon_{j+1}-\epsilon_j)\big)\chi\big(\tbinom{x_j}{\epsilon_j}>\tbinom{x_{j+1}}{\epsilon_{j+1}}\big)\\
&\qquad\qquad+\sum_{j=1}^{n-2}\big((\epsilon_{j+1}+1)+(l-1-\epsilon_j)\big)\chi\big(\tbinom{x_j}{\epsilon_j}<\tbinom{x_{j+1}}{\epsilon_{j+1}}\big)+(l-1-\epsilon_{n-1})\\
&=1+\epsilon_2-\epsilon_1+\epsilon_3-\epsilon_2+\cdots+\epsilon_{n-1}-\epsilon_{n-2}+(n-2-k')l+l-1-\epsilon_{n-1}\\
&=-\epsilon_1+nl-l-k'l\\
&=ln-(k'l+\epsilon_1+l)=ln-k.
\end{aligned}$$

This establishes the above recurrence holds when

$$W_{n,k}^l=\#\{(w,\epsilon)\in C_l\wr\mathfrak{S}_n:\operatorname{fdes}(w,\epsilon)=k\}.$$

3. The combinatorial proof for the flag-excedance number

In this Section $W_{n,k}^l$ denotes the number of elements (w,ϵ) from $C_l\wr\mathfrak{S}_n$ such that $\operatorname{fexc}(w,\epsilon)=k$ and the purpose is to prove that recurrence (8) still holds. We start with an element from $C_l\wr\mathfrak{S}_{n-1}$ written as a three-row matrix:

$$\begin{pmatrix}\mathrm{Id}\\ w\\ \epsilon\end{pmatrix}=\begin{pmatrix}1\ \cdots\ i\ \cdots n-1\\ x_1\cdots x_i\cdots x_{n-1}\\ \epsilon_1\ \cdots\ \epsilon_i\ \cdots\ \epsilon_{n-1}\end{pmatrix}.$$

1. If $1\le i\le n-1$ and $\epsilon_i=0$ form:

$$\begin{array}{ccccccc}
\mathrm{Id} & =1 & \cdots & i & \cdots & n-1 & n\\
w' & =x_1 & \cdots & n & \cdots & x_{n-1} & x_i\\
\epsilon^{(0)} & =\epsilon_1 & \cdots & 0 & \cdots & \epsilon_{n-1} & 0\\
\epsilon^{(1)} & =\epsilon_1 & \cdots & 1 & \cdots & \epsilon_{n-1} & l-1\\
\epsilon^{(2)} & =\epsilon_1 & \cdots & 2 & \cdots & \epsilon_{n-1} & l-2\\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots\\
\epsilon^{(l-1)} & =\epsilon_1 & \cdots & l-1 & \cdots & \epsilon_{n-1} & 1
\end{array}$$

Note that $\epsilon_1^{(0)}+\epsilon_n^{(0)}=0$, $\epsilon_1^{(j)}+\epsilon_n^{(j)}=l$ for $j=1,2,\ldots,l-1$. Furthermore, if $x>i$, then $\operatorname{fexc}(w',\epsilon^{(j)})=\operatorname{fexc}(w,\epsilon)$ for $j=0,1,2,\ldots,l-1$; but if $x_i\le i$, then $\operatorname{fexc}(w',\epsilon^{(j)})=\operatorname{fexc}(w,\epsilon)+l$ for $j=0,1,2,\ldots,l-1$.

2. If $1 \le i \le n-1$ and $\epsilon_i = j$ with $1 \le j \le l-1$ form:

$$\begin{array}{rlllll}
\mathrm{Id} & = 1 & \cdots & i & \cdots n-1 & n \\
w' & = x_1 & \cdots & n & \cdots x_{n-1} & x_i \\
\phi^{(1)} & = \epsilon_1 & \cdots & 1 & \cdots \epsilon_{n-1} & j-1 \\
\phi^{(2)} & = \epsilon_1 & \cdots & 2 & \cdots \epsilon_{n-1} & j-2 \\
\vdots & \ \vdots & \ddots & \vdots & \ddots \ \vdots & \vdots \\
\phi^{(j)} & = \epsilon_1 & \cdots & j & \cdots \epsilon_{n-1} & 0 \\
 & & & & & \\
\phi^{(j+1)} & = \epsilon_1 & \cdots & j+1 & \cdots \epsilon_{n-1} & l-1 \\
\vdots & \ \vdots & \ddots & \vdots & \ddots \ \vdots & \vdots \\
\phi^{(l-1)} & = \epsilon_1 & \cdots & l-1 & \cdots \epsilon_{n-1} & j+1 \\
\phi^{(l)} & = \epsilon_1 & \cdots & 0 & \cdots \epsilon_{n-1} & j
\end{array}$$

Note that $\phi_i^{(h)} + \phi_n^{(h)} = h$ for $h = 1, 2, \ldots, j$, while $\phi_i^{(h)} + \phi_n^{(h)} = j + l$ for $h = j+1, \ldots, l-1, l$. Also, $\mathrm{fexc}(w', \phi^{(h)}) = \mathrm{fexc}(w, \epsilon)$ for $h = 1, 2, \ldots, j$, while $\mathrm{fexc}(w', \phi^{(h)}) = \mathrm{fexc}(w, \epsilon) + l$ for $h = j+1, \ldots, l-1, l$.

3. Finally, form

$$\begin{array}{rllll}
\mathrm{Id} & = 1 & \cdots & n-1 & n \\
wn & = x_1 & \cdots & x_{n-1} & n \\
\psi^{(0)} & = \epsilon_1 & \cdots & \epsilon_{n-1} & 0 \\
\psi^{(1)} & = \epsilon_1 & \cdots & \epsilon_{n-1} & 1 \\
\vdots & \ \vdots & \ddots & \vdots & \vdots \\
\psi^{(l-1)} & = \epsilon_1 & \cdots & \epsilon_{n-1} & l-1
\end{array}$$

Then, $\mathrm{fexc}(wn, \psi^{(h)}) = \mathrm{fexc}(w, \epsilon) + h$ for $h = 0, 1, 2, \ldots, l-1$.

Let $(w, \epsilon) \in C_l \wr \mathfrak{S}_{n-1}$ be such that $\mathrm{exc}(w, \epsilon) = a$ and ϵ is a rearrangement of $0^{b_0} 1^{b_1} \cdots (l-1)^{b_{l-1}}$, so that $\mathrm{fexc}(w, \epsilon) = l \cdot a + \sum_{j=1}^{l-1} j \cdot b_j$. When procedure 1 is applied to (w, ϵ) at each of the a excedances, it gives rise to $a \cdot l$ elements from $C_l \wr \mathfrak{S}_n$ having the same "fexc" as (w, ϵ).

When procedure 2 is applied to each of the b_j biletters (x_i, ϵ_i) such that $\epsilon_i = j$, it gives rise to j elements from $C_l \wr \mathfrak{S}_n$ having the same "fexc" as (w, ϵ). Let $\mathrm{fexc}(w, \epsilon) = k$. Altogether, procedures 1 and 2 yield

$$l \times a + \sum_{j=1}^{l-1} j \cdot b_j = \mathrm{fexc}(w, \epsilon) = k$$

elements (w', ϵ') from $C_l \wr \mathfrak{S}_n$ such that $(w', \epsilon') = k$.

Finally, procedure 3 yields $(wn, \psi^{(0)})$, which is another element such that $\mathrm{fexc}(wn, \psi^{(0)}) = k$. Altogether, each element from $C_l \wr \mathfrak{S}_{n-1}$, whose "fexc" is equal to k, exactly produces $(k+1)$ elements from $C_l \wr \mathfrak{S}_n$ having an "fexc" equal to k.

Now, suppose that $\mathrm{fexc}(w, \epsilon) = k - l = l \cdot a + \sum_{j=1}^{l-1} j \cdot b_j$. Then, (w, ϵ) exactly has $n-1-a-b_1-b_2-\cdots-b_{l-1}$ biletters (x_i, ϵ_i) such that $x_i \le i$ and $\epsilon_i = 0$. (Remember that a counts the excedances of (w, ϵ).) When procedure 1 is applied to each of those biletters, it gives rise to l elements (w', ϵ') from $C_l \wr \mathfrak{S}_n$ such that $\mathrm{fexc}(w', \epsilon') = \mathrm{fexc}(w, \epsilon) + l = (k-l)+l = k$.

When procedure 2 is applied to a biletter (x_i, ϵ_i) such that $\epsilon_i = j \ge 1$, the $(l-j)$ elements $(w', \phi^{(j+1)})$, … , $(w', \phi^{(l-1)})$, $(w', \phi^{(l)})$ have their "fexc"'s equal to $\mathrm{fexc}(w, \epsilon) + l = (k-l)+l = k$.

Altogether, procedures 1 and 2 yield

$$\begin{aligned} l(n-1-a-b_1-b_2-\cdots-b_{l-1}) &+ \sum_{j=1}^{l-1}(l-j)b_j \\ &= ln - l - al - \sum_{j=1}^{l-1}(l-j)b_j = ln - l - (k-l) = ln - k \end{aligned}$$

elements (w', ϵ') from $C_l \wr \mathfrak{S}_n$ such that $\mathrm{fexc}(w', \epsilon') = k$.

Finally, let $1 \le h \le l-1$ and suppose $\mathrm{fexc}(w, \epsilon) = k-h$. By procedure 3 we have: $\mathrm{fexc}(wn, \psi^{(h)}) = \mathrm{fexc}(w, \epsilon) + h = k - h + h = k$, so that each such an element (w, ϵ) exactly produces one element, whose "fexc" is equal to k □

References

[6] Dominique Foata; Guo-Niu Han, The decrease value theorem with an application to permutation statistics, 19 p.