THE DECREASE VALUE THEOREM WITH AN APPLICATION TO PERMUTATION STATISTICS

Dominique Foata and Guo-Niu Han

Dedicated to Dennis Stanton,
on the occasion of his sixtieth birthday.

ABSTRACT. The decrease value theorem is restated and given a specialization more adapted to Permutation Statistic Calculus. As an application, the computation of a factorial multivariable generating function for the wreath product of the cyclic group of finite order by the symmetric group is given in full detail.

1. Introduction

In one of our recent papers [FH07] we have derived the *decrease value theorem*, that makes the calculation of a fundamental *mutivariable statistical* distribution on words possible. The multivariable statistic in question involves the basic notions of *decrease* and *increase*, whose definitions are now recalled, together with the classical *descent* and *rise*.

Let $[0,r]^*$ be the set of all words, whose letters belong to the finite alphabet $[0,r]=\{0,1,\ldots,r\}$ and let $v=y_1y_2\cdots y_n$ be such a word. An integer $i\in[1,n]$ is said to be a *descent* (or *descent place*) of v if $y_i>y_{i+1}$; it is a *decrease* of v if $y_i=y_{i+1}=\cdots=y_j>y_{j+1}$ for some j such that $i\le j\le n-1$. The letter y_i is said to be a *descent value* and a *decrease value* of v, respectively. The set of all decreases (resp. descents) is denoted by $\mathrm{DEC}(v)$ (resp. $\mathrm{DES}(v)$). Each descent is a decrease, so that $\mathrm{DES}(v)\subset\mathrm{DEC}(v)$.

In parallel with the notion of decrease, an integer $i\in[1,n]$ is said to be an *increase* (resp. a *rise*) of v if $y_i=y_{i+1}=\cdots=y_j<y_{j+1}$ for some j such that $i\le j\le n$ (resp. if $y_i<y_{i+1}$). By convention, $y_{n+1}=+\infty$. The letter y_i is said to be *an increase value* (resp. *a rise value*) of v. Thus, the rightmost letter y_n is always a rise and also an increase value. The set of all increases (resp. rises) of v is denoted by $\mathrm{INC}(v)$ (resp. $\mathrm{RISE}(v)$). Each rise is an increase, so that $\mathrm{RISE}(v)\subset\mathrm{INC}(v)$.

2000 *Mathematics Subject Classification.* Primary 05A15, 05A30, 05E15, 33D15.
Key words and phrases. Decrease value theorem, wreath product, flag-major index, flag-excedance number, flag-descent number, signed permutations, Lyndon factorization, decreases.

Furthermore, a position i $(1 \le i \le n)$ is said to be a *record* if $y_j \le y_i$ for all j such that $1 \le j \le i-1$. The letter y_i is said to be a *record value.* The set of all records of v is denoted by $\mathrm{REC}(v)$.

The multivariable statistic is now defined by means of six sequences of commuting variables (X_i), (Y_i), (Z_i), (T_i), (Y'_i), (T'_i) $(i = 0, 1, 2, \ldots)$: for each word $v = y_1 y_2 \ldots y_n$ from $[0, r]^*$ the *weight* $\psi(v)$ is defined to be

$$\begin{aligned}(1.1)\quad \psi(v) := {}& \prod_{i \in \mathrm{DES}} X_{y_i} \prod_{i \in \mathrm{RISE} \setminus \mathrm{REC}} Y_{y_i} \prod_{i \in \mathrm{DEC} \setminus \mathrm{DES}} Z_{y_i} \\ & \times \prod_{i \in (\mathrm{INC} \setminus \mathrm{RISE}) \setminus \mathrm{REC}} T_{y_i} \prod_{i \in \mathrm{RISE} \cap \mathrm{REC}} Y'_{y_i} \prod_{i \in (\mathrm{INC} \setminus \mathrm{RISE}) \cap \mathrm{REC}} T'_{y_i},\end{aligned}$$

where the argument "(v)" has not been written for typographic reasons. For example, $i \in \mathrm{RISE} \setminus \mathrm{REC}$ stands for $i \in \mathrm{RISE}(v) \setminus \mathrm{REC}(v)$.

It is important to note that for each word $v = y_1 y_2 \cdots y_n$ *every* integer $i \in [1, n]$ belongs to *one and only one* of the sets $\mathrm{DES}(v)$, $(\mathrm{RISE} \setminus \mathrm{REC})(v)$, $(\mathrm{DEC} \setminus \mathrm{DES})(v)$, $((\mathrm{INC} \setminus \mathrm{RISE}) \setminus \mathrm{REC})(v)$, $(\mathrm{RISE} \cap \mathrm{REC})(v)$, $((\mathrm{INC} \setminus \mathrm{RISE}) \cap \mathrm{REC})(v)$.

Example. For the word $v = 3\,2\,4\,4\,5\,5\,5\,3\,1\,1\,1\,4\,1\,3\,5$ the sets DES, DEC, INC, RISE, REC of v are indicated by bullets.

w =	3	2	4	4	5	5	5	3	1	1	1	4	1	3	5
DES =	•						•	•				•			
DEC =	•				•	•	•	•				•			
RISE =		•		•							•		•	•	•
INC =		•	•	•					•	•	•		•	•	•
REC =	•		•	•	•	•	•								•

We have $\psi(v) = X_3 Y_2 T'_4 Y'_4 Z_5 Z_5 X_5 X_3 T_1 T_1 Y_1 X_4 Y_1 Y_3 Y'_5$.

Now, let C be the $(r+1) \times (r+1)$ matrix

$$(1.2)\qquad C = \begin{pmatrix} 0 & \dfrac{X_1}{1-Z_1} & \dfrac{X_2}{1-Z_2} & \cdots & \dfrac{X_{r-1}}{1-Z_{r-1}} & \dfrac{X_r}{1-Z_r} \\ \dfrac{Y_0}{1-T_0} & 0 & \dfrac{X_2}{1-Z_2} & \cdots & \dfrac{X_{r-1}}{1-Z_{r-1}} & \dfrac{X_r}{1-Z_r} \\ \dfrac{Y_0}{1-T_0} & \dfrac{Y_1}{1-T_1} & 0 & \cdots & \dfrac{X_{r-1}}{1-Z_{r-1}} & \dfrac{X_r}{1-Z_r} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \dfrac{Y_0}{1-T_0} & \dfrac{Y_1}{1-T_1} & \dfrac{Y_2}{1-T_2} & \cdots & 0 & \dfrac{X_r}{1-Z_r} \\ \dfrac{Y_0}{1-T_0} & \dfrac{Y_1}{1-T_1} & \dfrac{Y_2}{1-T_2} & \cdots & \dfrac{Y_{r-1}}{1-T_{r-1}} & 0 \end{pmatrix}.$$

Theorem 1.1 (Decrease Value Theorem). *The generating function for the set $[0,r]^*$ by the weight ψ is given by*

$$\sum_{w\in[0,r]^*}\psi(w)=\frac{\displaystyle\prod_{0\le j\le r}\Bigl(1+\frac{Y'_j}{1-T'_j}\Bigr)}{\det(I-C)},\tag{1.3}$$

where I is the identity matrix of order $(r+1)$.

The proof of the Decrease Value Theorem is fully given in our previous paper [FH07], together with its two equivalent formulations:

$$\sum_{w\in[0,r]^*}\psi(w)=\frac{\dfrac{\displaystyle\prod_{0\le j\le r}\frac{1-Z_j}{1-Z_j+X_j}}{\displaystyle\prod_{0\le j\le r}\frac{1-T'_j}{1-T'_j+Y'_j}}}{1-\displaystyle\sum_{0\le k\le r}\frac{\displaystyle\prod_{0\le j\le k-1}\frac{1-Z_j}{1-Z_j+X_j}}{\displaystyle\prod_{0\le j\le k-1}\frac{1-T_j}{1-T_j+Y_j}}\,\frac{X_k}{1-Z_k+X_k}},\tag{1.4}$$

$$\sum_{w\in[0,r]^*}\psi(w)=\frac{\dfrac{\displaystyle\prod_{1\le j\le r}\frac{1-Z_j}{1-Z_j+X_j}}{\displaystyle\prod_{0\le j\le r}\frac{1-T'_j}{1-T'_j+Y'_j}}}{1-\displaystyle\sum_{1\le k\le r}\frac{\displaystyle\prod_{1\le j\le k-1}\frac{1-Z_j}{1-Z_j+X_j}}{\displaystyle\prod_{0\le j\le k-1}\frac{1-T_j}{1-T_j+Y_j}}\,\frac{X_k}{1-Z_k+X_k}}.\tag{1.5}$$

There is a specialization of (1.5) that deserves a special attention, which is the following. For convenience, introduce three sequences of commuting variables (ξ_i), (η_i), (ζ_i) $(i=0,1,2,\ldots)$ and make the following substitutions:

$$X_i\leftarrow\xi_i,\ Z_i\leftarrow\xi_i,\ Y_i\leftarrow\eta_i,\ T_i\leftarrow\eta_i,\ Y'_i\leftarrow\zeta_i,\ T'_i\leftarrow\zeta_i,\quad (i=0,1,2,\ldots).$$

The new weight $\psi'(v)$ attached to each word $v=y_1y_2\cdots y_n$ is then

$$\psi'(v)=\prod_{i\in\mathrm{DEC}(v)}\xi_i\prod_{i\in(\mathrm{INC}\setminus\mathrm{REC})(v)}\eta_i\prod_{i\in(\mathrm{INC}\cap\mathrm{REC})(v)}\zeta_i,\tag{1.6}$$

and identity (1.5) becomes:

$$\sum_{v\in[0,r]^*}\psi'(v)=\frac{\dfrac{\prod_{1\le j\le r}(1-\xi_j)}{\prod_{0\le j\le r}(1-\zeta_j)}}{1-\sum_{1\le k\le r}\dfrac{\prod_{1\le j\le k-1}(1-\xi_j)}{\prod_{0\le j\le k-1}(1-\eta_j)}\xi_k}. \tag{1.7}$$

The further specializations of (1.6) and (1.7) require the following notations. For each word $v=y_1y_2\cdots y_n$ let λv designate its *length* ($\lambda v=n$) and $\operatorname{tot} v=y_1+y_2+\cdots+y_n$ the *sum* of its letters. Next, given a positive integer l, let $\operatorname{dec}_l v$ be the number of letters of v, which are *decrease values and multiple* of l; finally, for $i=0,1,\ldots,l-1$ let $|v|_{i \bmod l}$ denote the number of letters of v *congruent to* i mod l and $\operatorname{inrec}_i v$ the *number* of letters of v, *congruent to* i mod l, which are also increase *and* record values.

Now, let u, s, Y_i, Z_i ($i=0,1,2,\ldots$) be a new set of variables and let γ denote the homomorphism defined by the following substitutions of variables:

$$\begin{aligned}
&\xi_j\leftarrow\begin{cases} uq^js^lZ_0, & \text{if } j\equiv 0 \bmod l;\\ uq^js^iZ_i, & \text{if } j\equiv i \bmod l \text{ and } 1\le i\le l-1;\end{cases}\\
&\eta_j\leftarrow uq^js^iZ_i, \qquad \text{if } j\equiv i \bmod l \text{ and } 0\le i\le l-1;\\
&\zeta_j\leftarrow uq^js^iY_iZ_i, \quad \text{if } j\equiv i \bmod l \text{ and } 0\le i\le l-1.
\end{aligned} \tag{1.8}$$

It follows from (1.6) and (1.8) that

$$\gamma\,\psi'(v)=u^{\lambda v}q^{\operatorname{tot} v}s^{l\operatorname{dec}_l v}\prod_{0\le i\le l-1}(s^iZ_i)^{|v|_{i \bmod l}}Y_i^{\operatorname{inrec}_i v}. \tag{1.9}$$

For each $r\ge 0$ consider the following multivariable generating function for the set $[0,r]^*$:

$$F_r(u;q,s,(Y_i),(Z_i)) :=\sum_{v\in[0,r]^*}u^{\lambda v}q^{\operatorname{tot} v}s^{l\operatorname{dec}_l v}\prod_{0\le i\le l-1}(s^iZ_i)^{|v|_{i \bmod l}}Y_i^{\operatorname{inrec}_i v}. \tag{1.10}$$

Using the traditional notations for the *q-ascending factorials* $(x;q)_n=1$ if $n=0$ and $(x;q)_n=(1-x)(1-qx)\cdots(1-q^{n-1}x)$ if $n\ge 1$, the first goal of the paper is to show that $F_r(u;q,s,(Y_i),(Z_i))$ can be expressed as an explicit hypergeometric series in the variable u, as stated in the next theorem, where it is assumed that $Z_l\equiv Z_0$.

Theorem 1.2. *For each $r \geq 0$ the following evaluation holds:*

$$F_r(u;q,s,(Y_i),(Z_i)) = \frac{\prod\limits_{1\leq i\leq l} (uq^is^iZ_i;q^l)_{\lfloor (r-i)/l\rfloor+1}}{\prod\limits_{0\leq i\leq l-1} (uq^is^iY_iZ_i;q^l)_{\lfloor (r-i)/l\rfloor+1}} Z_0(1-q^ls^l) \tag{1.11}$$

$$\times \Bigl(Z_0 - Z_0q^ls^l + \sum_{i=1}^{l} q^is^iZ_i - \sum_{i=1}^{l} q^is^iZ_i \frac{(uq^ls^lZ_0;q^l)_{\lfloor (r-i)/l\rfloor+1}}{(uZ_0;q^l)_{\lfloor (r-i)/l\rfloor+1}}\Bigr)^{-1}.$$

The proof of Theorem 1.2 is given in Section 2. It is based on the decrease value theorem and makes use of the traditional techniques of q-telescoping. The above identity on *word* generating series is next used to show that the infinite series $\sum_{r\geq 0} t^r F_r(u;q,s,(Y_i),(Z_i))$ can be expressed as a factorial series $\sum_{n\geq 0} A_n(s,t,q,(Y_i),(Z_i))\, u^n/(t^l;q^l)_{n+1}$, where each coefficient $A_n(s,t,q,(Y_i),(Z_i))$ is a generating *polynomial* for an algebraic structure by a well-defined multivariable statistic.

This algebraic structure is the following. Let l be a positive integer and consider the *wreath product* $C_l \wr \mathfrak{S}_n$ of the cyclic group C_l of order l by the symmetric group $\mathfrak{S}_n$ of order n (see, e.g., [RR06] for a complete description). The elements of $C_l \wr \mathfrak{S}_n$ may be viewed as ordered pairs (w,ϵ), where $w = x_1x_2\cdots x_n$ is a permutation of $12\cdots n$ and $\epsilon = \epsilon_1\epsilon_2\cdots\epsilon_n$ a word of length n, whose letters belong to $\{0,1,\ldots,l-1\}$. Finally, $C_l \wr \mathfrak{S}_n$ is equipped with the *total* order "$<$" defined by:

$$(j,i) < (j',i') \text{ if and only if either } i > i,' \text{ or } i = i' \text{ and } j < j'.$$

For each argument A let $\chi(A) = 1$ or 0, depending on whether A is true or false and let $|\epsilon|_i$ denote the number of letters equal to i in ϵ, so that $1\cdot|\epsilon|_1 + 2\cdot|\epsilon|_2 + \cdots + (l-1)\cdot|\epsilon|_{l-1} = \operatorname{tot}\epsilon$. The statistics associated with (w,ϵ) are the following:

$$\begin{aligned}
\operatorname{exc}(w,\epsilon) &:= \#\{j : 1\leq j\leq n,\, x_j > j,\, \epsilon_j = 0\};\\
\operatorname{fexc}(w,\epsilon) &:= l\cdot \operatorname{exc}(w,\epsilon) + \operatorname{tot}\epsilon;\\
\operatorname{des}(w,\epsilon) &:= \#\{j : 1\leq j\leq n-1,\, (x_j,\epsilon_j) > (x_{j+1},\epsilon_{j+1})\};\\
\operatorname{fdes}(w,\epsilon) &:= l\cdot \operatorname{des}(w,\epsilon) + \epsilon_1;\\
\operatorname{maj}(w,\epsilon) &:= \sum_{1\leq j\leq n-1} j\,\chi\bigl((x_j,\epsilon_j) > (x_{j+1},\epsilon_{j+1})\bigr);\\
\operatorname{fmaj}(w,\epsilon) &:= l\cdot \operatorname{maj}(w,\epsilon) + \operatorname{tot}\epsilon;\\
\operatorname{fix}_i(w,\epsilon) &:= \#\{j : 1\leq j\leq n,\, (x_j,\epsilon_j) = (j,i)\} \quad (0\leq i\leq l-1).
\end{aligned}$$

For each $n\geq 0$ consider the generating polynomial for $C_l \wr \mathfrak{S}_n$:

$$\begin{aligned}
&W_n(s,t,q,(Y_i),(Z_i)) \\
&\quad := \sum_{(w,\epsilon)\in C_l\wr\mathfrak{S}_n} s^{\operatorname{fexc}(w,\epsilon)} t^{\operatorname{fdes}(w,\epsilon)} q^{\operatorname{fmaj}(w,\epsilon)} \prod_{0\leq i\leq l-1} Z_i^{|\epsilon|_i} Y_i^{\operatorname{fix}_i(w,\epsilon)}.
\end{aligned} \tag{1.12}$$

Theorem 1.3. *Let $F_r(u;q,s,(Y_i),(Z_i))$ be given by (1.11). Then, the following identity holds*

$$\sum_{n\ge 0}(1+t+\cdots+t^{l-1})W_n(s,t,q,(Y_i),(Z_i))\frac{u^n}{(t^l;q^l)_{n+1}} = \sum_{r\ge 0} t^r\, F_r(u;q,s,(Y_i),(Z_i)). \tag{1.13}$$

The proof of Theorem 1.3 is made in two steps. First, the left-hand side of (1.13) is shown to be equal to the generating function for the so-called *wreathed permutations*, expressed as a series $\sum t^r\, G_r(u;q,s,(Y_i),(Z_i))$. This is the content of Section 3. Second, each coefficient of t^r in the above series is shown to be equal to the generating series $F_r(u;q,s,(Y_i),(Z_i))$, given in (1.10). This is accomplished, in Section 4, by means of a bijection, whose construction is directly inspired by the classical *standardization* procedure. Finally, we derive several specializations of Theorem 1.3, in particular, the following theorem, which extends MacMahon's classical result of the equidistribution of exceedances and descents on permutations (see [Lo83, Chap. 10]).

Theorem 1.4. *The two statistics* "fdes" *and* "fexc" *are equidistributed over $C_l \wr \mathfrak{S}_n$. Let $W^{(l)}_{n,k}$ denote the number of elements (w,ϵ) from $C_l\wr\mathfrak{S}_n$ such that* $\mathrm{fdes}(w,\epsilon)=k$. *Then, the following recurrence formula holds*

$$W^{(l)}_{n,k} = (k+1)W^{(l)}_{n-1,k} + W^{(l)}_{n-1,k-1} + \cdots + W^{(l)}_{n-1,k-(l-1)} + (ln-k)W^{(l)}_{n-1,k-l} \tag{1.14}$$

for $n\ge 2$ and $0\le k\le nl-1$ and the initial conditions: $W^{(l)}_{0,0}=1$, $W^{(l)}_{0,k}=0$ for $k\ne 0$; $W^{(l)}_{1,k}=1$ for $k=0,1,\ldots,l-1$ and 0 for any other value of k.

2. Proof of Theorem 1.2

To obtain the right-hand side of (1.11) it suffices to calculate the image under γ of the right-hand side of (1.7). First, remembering that $Z_l\equiv Z_0$,

$$\begin{aligned}\gamma\prod_{1\le j\le r}(1-\xi_j) &= \gamma\prod_{i=1}^{l}\prod_{j=0}^{\lfloor (r-i)/l\rfloor}(1-\xi_{jl+i})\\ &= \gamma\prod_{i=1}^{l-1}\prod_{j=0}^{\lfloor (r-i)/l\rfloor}(1-\xi_{jl+i})\times\prod_{j=0}^{\lfloor (r-l)/l\rfloor}(1-\xi_{jl+l})\\ &= \prod_{i=1}^{l-1}\prod_{j=0}^{\lfloor (r-i)/l\rfloor}(1-uq^{jl+i}s^iZ_i)\times\prod_{j=0}^{\lfloor (r-l)/l\rfloor}(1-uq^{jl+l}s^lZ_l)\end{aligned}$$

$$= \prod_{i=1}^{l} \prod_{j=0}^{\lfloor (r-i)/l \rfloor} (1 - uq^i s^i Z_i (q^l)^j)$$

$$= \prod_{i=1}^{l} (uq^i s^i Z_i; q^l)_{\lfloor (r-i)/l \rfloor + 1}.$$

In the same manner,

$$\gamma \prod_{0 \le j \le r} (1 - \zeta_j) = \prod_{i=0}^{l-1} (uq^i s^i Y_i Z_i; q^l)_{\lfloor (r-i)/l \rfloor + 1};$$

$$\gamma \prod_{1 \le j \le k-1} (1 - \xi_j) = \prod_{i=1}^{l} (uq^i s^i Z_i; q^l)_{\lfloor (k-1-i)/l \rfloor + 1};$$

$$\gamma \prod_{0 \le j \le k-1} (1 - \eta_j) = \prod_{i=0}^{l-1} (uq^i s^i Z_i; q^l)_{\lfloor (k-1-i)/l \rfloor + 1};$$

so that the image of (1.7) under γ becomes

$$\sum_{v \in [0,r]^*} \gamma \psi'(v) = \frac{\prod_{i=1}^{l} (uq^i s^i Z_i; q^l)_{\lfloor (r-i)/l \rfloor + 1}}{\prod_{i=0}^{l-1} (uq^i s^i Y_i Z_i; q^l)_{\lfloor (r-i)/l \rfloor + 1}} \times (1 - S)^{-1}, \tag{2.1}$$

where

$$S = \sum_{1 \le k \le r} \frac{(uq^l s^l Z_0; q^l)_{\lfloor (k-1)/l \rfloor}}{(uZ_0; q^l)_{\lfloor (k-1)/l \rfloor + 1}} \gamma(\xi_k),$$

which can be rewritten:

$$S = \sum_{i=1}^{l} \sum_{j=0}^{\lfloor (r-i)/l \rfloor} \frac{(uq^l s^l Z_0; q^l)_j}{(uZ_0; q^l)_{j+1}} uq^{lj} q^i s^i Z_i.$$

Introduce

$$G(m) = \sum_{0 \le j \le m} \frac{(uq^l s^l Z_0; q^l)_j}{(uZ_0; q^l)_{j+1}} uq^{lj},$$

so that

$$S = \sum_{i=1}^{l} q^i s^i Z_i \, G(\lfloor (r-i)/l \rfloor).$$

As $\dfrac{(uq^l s^l Z_0; q^l)_{j+1}}{(uZ_0; q^l)_{j+1}} - \dfrac{(uq^l s^l Z_0; q^l)_j}{(uZ_0; q^l)_j} = \dfrac{(uq^l s^l Z_0; q^l)_j}{(uZ_0; q^l)_{j+1}} uq^{lj} \, Z_0 (1 - q^l s^l)$, we have:

$$G(m) = \frac{1}{Z_0 (1 - q^l s^l)} \Big(\frac{(uq^l s^l Z_0; q^l)_{m+1}}{(uZ_0; q^l)_{m+1}} - 1 \Big),$$

and then

$$S = \sum_{i=1}^{l} \frac{q^i s^i Z_i}{Z_0 (1 - q^l s^l)} \Big(\frac{(uq^l s^l Z_0; q^l)_{\lfloor (r-i)/l \rfloor + 1}}{(uZ_0; q^l)_{\lfloor (r-i)/l \rfloor + 1}} - 1 \Big).$$

We obtain the righthand side of (1.11) by substituting the above S into (2.1). $\square$

3. Wreathed permutations

The generating polynomial $W_n(s,t,q,(Y_i),(Z_i))$ for the group $C_l\wr\mathfrak{S}_n$ has been defined in (1.12). Recall the notation for the q-binomial coefficient ${n\brack k}_q := (q;q)_n/((q;q)_k\,(q;q)_{n-k})$ for $0\le k\le n$ and the classical identities

$$\frac{1}{(t;q)_{n+1}}=\sum_{r\ge 0}{n+r\brack r}_q t^r;\qquad {n+r\brack r}_q=\sum_{b\in \mathrm{NIW}_n(r)} q^{\operatorname{tot} b};$$

where $\mathrm{NIW}_n(r)$ (resp. NIW_n) denotes the set of all words $b=b_1b_2\cdots b_n$ of length n, whose letters are nonnegative integers satisfying $r\ge b_1\ge b_2\ge\cdots b_n\ge 0$ (resp. $b_1\ge b_2\ge\cdots b_n\ge 0$) (see [An76, Chap. 3]). Using those two identities we have:

$$\begin{aligned}
\frac{1+t+\cdots+t^{l-1}}{(t^l;q^l)_{n+1}}&=\sum_{r'\ge 0}(t^{lr'}+t^{lr'+1}+\cdots+t^{lr'+l-1}){n+r'\brack r'}_{q^l}\\
&=\sum_{r\ge 0}t^r{n+\lfloor r/l\rfloor\brack \lfloor r/l\rfloor}_{q^l}=\sum_{r\ge 0}t^r\sum_{b\in\mathrm{NIW}_n(\lfloor r/l\rfloor)}q^{l\operatorname{tot} b}\\
&=\sum_{r\ge 0}t^r\sum_{\substack{b\in\mathrm{NIW}_n,\\ lb_1\le r}}q^{l\operatorname{tot} b}.
\end{aligned}$$

Hence,

$$\begin{aligned}
(3.1)\quad &\frac{1+t+\cdots+t^{l-1}}{(t^l;q^l)_{n+1}}W_n(s,t,q,(Y_i),(Z_i))\\
&=\sum_{r\ge 0}t^r\sum_{\substack{b\in\mathrm{NIW}_n,\\ lb_1\le r}}q^{l\operatorname{tot} b}\sum_{(w,\epsilon)\in C_l\wr\mathfrak{S}_n}s^{\operatorname{fexc}(w,\epsilon)}t^{\operatorname{fdes}(w,\epsilon)}q^{\operatorname{fmaj}(w,\epsilon)}\prod_{0\le i\le l-1}Y_i^{\operatorname{fix}_i(w,\epsilon)}Z_i^{|\epsilon|_i}\\
&=\sum_{r\ge 0}t^r\sum_{\substack{b\in\mathrm{NIW}_n,\,(w,\epsilon)\in C_l\wr\mathfrak{S}_n\\ lb_1+\operatorname{fdes}(w,\epsilon)\le r}}s^{\operatorname{fexc}(w,\epsilon)}q^{l\operatorname{tot} b+\operatorname{fmaj}(w,\epsilon)}\prod_{0\le i\le l-1}Y_i^{\operatorname{fix}_i(w,\epsilon)}Z_i^{|\epsilon|_i}.
\end{aligned}$$

For each element $(w,\epsilon)\in C_l\wr\mathfrak{S}_n$ form the word $z=z_1z_2\cdots z_n$, where z_j is defined to be the number of k such that $j\le k\le n-1$ and $(x_k,\epsilon_k)>(x_{k+1},\epsilon_{k+1})$ with respect to the order imposed on $C_l\wr\mathfrak{S}_n$. In other words, z_j is the number of *descents* in the right factor $(x_j,\epsilon_j)(x_{j+1},\epsilon_{j+1})\cdots(x_n,\epsilon_n)$. The next proposition is easy to verify.

Proposition 3.1. *We have:* $\operatorname{des}(w,\epsilon)=z_1$, $\operatorname{maj}(w,\epsilon)=\operatorname{tot} z$.

Now, let $b=b_1b_2\cdots b_n\in\mathrm{NIW}_n$ and $(w,\epsilon)=(x_1,\epsilon_1)(x_2,\epsilon_2)\cdots(x_n,\epsilon_n)\in C_l\wr\mathfrak{S}_n$ be given. We define the word $c=c_1c_2\cdots c_n$ by

$$(3.2)\qquad c_j:=l(b_j+z_j)+\epsilon_j\quad(1\le j\le n).$$

As both words b and z are monotonic nonincreasing, we have $(b_j+z_j)\ge (b_{j+1}+z_{j+1})$ $(1\le j\le n-1)$. If the inequality is strict, then $c_j>c_{j+1}$ since $0\le \epsilon_{j+1}\le l-1$. If $b_j+z_j=b_{j+1}+z_{j+1}$, then $z_j=z_{j+1}$ and, consequently, $(x_j,\epsilon_j)<(x_{j+1},\epsilon_{j+1})$, so that $\epsilon_j\ge\epsilon_{j+1}$. Therefore, $c_j\ge c_{j+1}$. The word c is then *monotonic nonincreasing.* We further have the following properties:

(i) $c_j=c_{j+1}\Rightarrow (x_j,\epsilon_j)<(x_{j+1},\epsilon_{j+1})$;
(ii) $c_j\equiv \epsilon_j \pmod l$;
(iii) $c_1=l\,b_1+\mathrm{fdes}(w,\epsilon)$;
(iv) $\mathrm{tot}\,c=l\,\mathrm{tot}\,b+\mathrm{fmaj}(w,\epsilon)$.

Note that (iii) and (iv) are immediate consequences of Proposition 3.1 and the definition of c.

A triple (c,w,ϵ) such that $(w,\epsilon)=(x_1,\epsilon_1)(x_2,\epsilon_2)\cdots(x_n,\epsilon_n)\in C_l\wr\mathfrak{S}_n$ and $c=c_1c_2\cdots c_n\in \mathrm{NIW}_n$ and such that properties (i) and (ii) hold is called a *wreathed permutation* of order n. The set of all wreathed permutations (c,w,ϵ) of order n is denoted by WP_n and the subset of WP_n of all (c,w,ϵ) such that $c_1\le r$ by $\mathrm{WP}_n(r)$.

It follows from (3.2) that, for each $r\ge 0$ the mapping

$$(b,w,\epsilon)\mapsto(c,w,\epsilon)$$

provides a bijection of the set of all triples (b,w,ϵ) such that $b\in\mathrm{NIW}_n$, $(w,\epsilon)\in C_l\wr\mathfrak{S}_n$ and $l\,b_1+\mathrm{fdes}(w,\epsilon)\le r$ onto $\mathrm{WP}_n(r)$, having properties (iii) and (iv).

By (3.1)

$$\frac{1+t+\cdots+t^{l-1}}{(t^l;q^l)_{n+1}}\,W_n(s,t,q,(Y_i),(Z_i)) = \sum_{r\ge 0} t^r \sum_{(c,w,\epsilon)\in\mathrm{WP}_n(r)} s^{\mathrm{fexc}(w,\epsilon)}q^{\mathrm{tot}\,c}\prod_{0\le i\le l-1} Y_i^{\mathrm{fix}_i(w,\epsilon)}Z_i^{|\epsilon|_i},$$

so that, if we let

$$G_r(u;s,q,(Y_i),(Z_i)) := \sum_{(c,w,\epsilon)\in\mathrm{WP}(r)} u^{\lambda w}s^{\mathrm{fexc}(w,\epsilon)}q^{\mathrm{tot}\,c}\prod_{0\le i\le l-1} Y_i^{\mathrm{fix}_i(w,\epsilon)}Z_i^{|\epsilon|_i}, \tag{3.3}$$

we have the identity:

$$\sum_{n\ge 0}(1+t+\cdots+t^{l-1})W_n(s,t,q,(Y_i),(Z_i))\frac{u^n}{(t^l;q^l)_{n+1}} = \sum_{r\ge 0} t^r G_r(u;s,q,(Y_i),(Z_i)). \tag{3.4}$$

Example. With $l=3$ the monotonic nonincreasing word c is calculated from the triple (b,w,ϵ):

$$\begin{array}{rcccccccccc}
\mathrm{Id} = & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10\\
b = & 5 & 5 & 4 & 2 & 2 & 2 & 0 & 0 & 0 & 0\\
w = & 1 & 8 & 7 & 4 & 10 & 2 & 6 & 9 & 5 & 3\\
\epsilon = & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 2 & 1 & 0\\
z = & 3 & 3 & 2 & 1 & 1 & 1 & 0 & 0 & 0 & 0\\
c = & 24 & 24 & 19 & 10 & 10 & 9 & 2 & 2 & 1 & 0
\end{array}$$

Note that

$$\begin{gathered}
\mathrm{des}(w,\epsilon)=3=z_1;\quad \mathrm{maj}(w,\epsilon)=2+3+6=11;\\
\mathrm{tot}\,b=20;\quad \mathrm{tot}\,\epsilon=8;\\
\mathrm{fdes}(w,\epsilon)=3\cdot\mathrm{des}(w,\epsilon)+\epsilon_1=9;\\
24=c_1=3\cdot b_1+\mathrm{fdes}(w,\epsilon)=3\cdot 5+9;\\
\mathrm{fmaj}(w,\epsilon)=3\cdot\mathrm{maj}(w,\epsilon)+\mathrm{tot}\,\epsilon=3\cdot 11+8=41;\\
101=\mathrm{tot}\,c=3\cdot\mathrm{tot}\,b+\mathrm{fmaj}(w,\epsilon)=3\cdot 20+41.
\end{gathered}$$

4. Standardization

By comparison with (1.13) we see that Theorem 1.3 is proved if we show that $F_r(u;s,q,(Y_i),(Z_i))$, given by (1.10), is equal to $G_r(u;s,q,(Y_i),(Z_i))$, given by (3.3), for all $r\ge 0$, that is, if we prove

$$\begin{aligned}
\sum_{v\in[0,r]^*} u^{\lambda v} q^{\mathrm{tot}\,v} s^{l\,\mathrm{dec}_l\,v} &\prod_{0\le i\le l-1} (s^i Z_i)^{|v|_{i\bmod l}} Y_i^{\mathrm{inrec}_i\,v}\\
&= \sum_{(c,w,\epsilon)\in\mathrm{WP}(r)} u^{\lambda w} s^{\mathrm{fexc}(w,\epsilon)} q^{\mathrm{tot}\,c} \prod_{0\le i\le l-1} Y_i^{\mathrm{fix}_i(w,\epsilon)} Z_i^{|\epsilon|_i}.
\end{aligned}$$

To do this, it suffices to construct a bijection

$$v\mapsto (c,w,\epsilon)$$

of $[0,r]^n$ (the set of all words of length n with letters from the alphabet $[0,r]$) onto $\mathrm{WP}_n(r)$ having the properties:
(i) $\mathrm{tot}\,v=\mathrm{tot}\,c$;
(ii) $\mathrm{dec}_l\,v=\mathrm{exc}(w,\epsilon)$;
(iii) $|v|_{i\bmod l}=|\epsilon|_i$ for $i=0,1,\ldots,l-1$;
(iv) $\mathrm{inrec}_i\,v=\mathrm{fix}_i(w,\epsilon)$ for $i=0,1,\ldots,l-1$.

For such a bijection the word c is to be the monotonic nonincreasing rearrangement of v, the permutation w an adequate *labelling* from 1 to n of the n letters of v, and ϵ a word whose letters are the residues of the letters of v mod l. Such a bijection appears to be a *standardization* of each word by a certain element from $C_l \wr \mathfrak{S}_n$. Earlier standardizations by the symmetric group $\mathfrak{S}_n$ (resp. the hyperoctahedral group B_n) have been constructed by Gessel and Reutenauer [GR93] (resp. in [FH09]). The procedure we now develop proceeds from the same principle.

Recall that a nonempty word $v = y_1 y_2 \cdots y_n$ is a *Lyndon word*, if either $n = 1$, or $n \geq 2$ and, with respect to the lexicographic order, the inequality $y_1 y_2 \cdots y_n > y_i y_{i+1} \cdots y_n y_1 \cdots y_{i-1}$ holds for every i such that $2 \leq i \leq n$. Let v, v' be two nonempty primitive words (none of them can be written as v_0^a for $a \geq 2$ and some word v_0). We write $v \preceq v'$ if and only if $v^a \leq v'^a$ with respect to the lexicographic order for an integer a large enough. As shown for instance in [Lo83, Theorem 5.1.5] (also see [Ch58], [Sch65]) each nonempty word v can be written uniquely as a product $l_1 l_2 \cdots l_k$, called its *Lyndon factorization*, where each l_i is a Lyndon word and $l_1 \preceq l_2 \preceq \cdots \preceq l_k$. In the example below the Lyndon factorization of v has been materialized by vertical bars.

Now, start with the Lyndon factorization $l_1 l_2 \cdots l_k$ of a word v from $[0, r]^n$. With such a v associate a permutation σ from $\mathfrak{S}_n$ in the following manner: each letter y_i of v belongs to a Lyndon word factor l_h, so that $l_h = v' y_i v''$. Then, form the infinite word $A(y_i) := y_i v'' v' y_i v'' v' \cdots$ If y_i and $y_{i'}$ are two letters of v, say that y_i *precedes* $y_{i'}$ if $A(y_i) > A(y_{i'})$ for the lexicographic order, or if $A(y_i) = A(y_{i'})$ and y_i is to the right of $y_{i'}$ in the word v. This precedence determines a total order on the n letters of v. The letter that precedes all the other ones is given label 1, the next one label 2, and so on. When each letter y_i of v is replaced by its label, say, $\mathrm{lab}(y_i)$, each Lyndon word factor l_j becomes a new word τ_j. The essential property is that each τ_j starts with its *minimum* element and those minimum elements read from left to right are in *decreasing order*. We can then interpret each τ_j as the *cycle* of a permutation and the (juxtaposition) product $\tau_1 \tau_2 \cdots \tau_k$ as the (functional) product of *disjoint* cycles. This product, said to be written *in canonical form*, defines a unique permutation σ from $\mathfrak{S}_n$ ([Lo83], § 10.2).

For example,

$$\begin{array}{l}
v = 2 \mid 3\ \ 2\ \ 1\ \ 1 \mid 3 \mid 5 \mid 6\ 4\ 2\ \ 1\ \ 3\ \ 2\ 3 \mid 6\ 6\ 3\ \ 1\ 6\ 6\ 2 \mid 6 \\
\sigma = 16 \mid 12\ 18\ 22\ 21 \mid 10 \mid 7 \mid 4\ 8\ 17\ 20\ 11\ 15\ 9 \mid 2\ 5\ 13\ 19\ 3\ 6\ 14 \mid 1
\end{array}$$

The labels on the second row are obtained as follows: read the letters equal to 6 (the maximal letter) from left to right and form their associated infinite words: $64213236421\cdots$, $66316626631\cdots$, $6316621131\cdots$, $662663166\cdots$, $62663166\cdots$, $66666\cdots$ Those letters 6 read from left to right will be given the labels 4, 2, 5, 3, 6, 1. We continue the labellings by reading the letters equal to 5, then 4, ... in the above word v.

No decrease y_i in v can be the rightmost letter of a Lyndon word factor l_h. We have then $l_h = \cdots y_i y_{i+1} \cdots y_j y_{j+1} \cdots$ with $y_i \ge y_{i+1} \ge \cdots \ge y_j > y_{j+1}$. Consequently, $A(y_i) > A(y_{i+1})$ and $\mathrm{lab}(y_i) < \mathrm{lab}(y_{i+1})$. Conversely, if $\mathrm{lab}(y_i) < \mathrm{lab}(y_{i+1})$ and y_i, y_{i+1} belong to the same Lyndon factor, then y_i is a decrease in v. To each decrease y_i in v there corresponds a unique cycle τ_h of σ and a pair $\mathrm{lab}(y_i)\,\mathrm{lab}(y_{i+1})$ of *successive* letters of τ_h such that $\mathrm{lab}(y_i) < \mathrm{lab}(y_{i+1})$ and $\mathrm{lab}(y_{i+1}) = \sigma(\mathrm{lab}(y_i))$.

Consider the monotonic nonincreasing rearrangement $c = c_1 c_2 \cdots c_n$ of v and form the three-row matrix

$$\begin{array}{cccc} 1 & 2 & \cdots & n \\ c_1 & c_2 & \cdots & c_n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{array}$$

Then, if y_i is a decrease in v, the $\mathrm{lab}(y_i)$-th column of the previous matrix is of the form

$$\begin{array}{c} \mathrm{lab}(y_i) \\ y_i \\ \mathrm{lab}(y_{i+1}) \end{array} \qquad \text{with} \quad \mathrm{lab}(y_i) < \mathrm{lab}(y_{i+1}).$$

Consequently, y_i is a decrease in v if and only if $\sigma(\mathrm{lab}(y_i)) > \mathrm{lab}(y_i)$. As $\sigma(i) > \sigma(i+1) \Rightarrow c_i > c_{i+1}$, the above three-row matrix can be expressed as

$$\begin{array}{rccc|ccc|c|ccc} \mathrm{Id} = & 1 & \cdots & m_1 & m_1{+}1 & \cdots & m_1{+}m_2 & \cdots & m_1{+}\cdots{+}m_{k-1}{+}1 & \cdots & n \\ c = & a_1 & \cdots & a_1 & a_2 & \cdots & a_2 & \cdots & a_k & \cdots & a_k \\ \sigma = & \sigma(1) & \cdots & \sigma(m_1) & \sigma(m_1{+}1) & \cdots & \sigma(m_1{+}m_2) & \cdots & \sigma(m_1{+}\cdots{+}m_{k-1}\ {+}1) & \cdots & \sigma(n) \end{array}$$

where $a_1 > a_2 > \cdots > a_k \ge 0$ and $m_1 \ge 1$, $m_2 \ge 1$, $\ldots$, $m_k \ge 1$ and $\sigma(1) < \cdots < \sigma(m_1)$, $\sigma(m_1+1) < \cdots < \sigma(m_1+m_2)$, $\ldots$, $\sigma(m_1+\cdots+m_{k-1}+1) < \cdots < \sigma(n)$. For each $i = 1, \ldots, k$ let $\overline{a}_i$ be the residue of a_i mod l and let

$$\binom{w}{\epsilon} := \begin{pmatrix} \sigma(1)\cdots\sigma(m_1) & \sigma(m_1{+}1)\cdots\sigma(m_1{+}m_2) & \cdots & \sigma(m_1{+}\cdots{+}m_{k-1}+1)\cdots\sigma(n) \\ \overline{a}_1\ \cdots\ \overline{a}_1 & \overline{a}_2\ \cdots\ \overline{a}_2 & \cdots & \overline{a}_k\ \cdots\ \overline{a}_k \end{pmatrix}$$

It then follows that (c, w, ϵ) is a wreathed permutation and properties (i), (ii) and (iii) hold. For the proof of (iv) we note that a letter y_i of v is an increase and record value if and only if y_i is a one-letter factor in the Lyndon factorization of v, that is, if and only if $\mathrm{lab}\, y_i$ is a cycle of length 1 of w, or equivalently, a fixed point of w. All the steps previously described are perfectly reversible. This achieves the proof of Theorem 1.3

With the running example take $l = 3$ we can form the table:

$$\begin{array}{rc|cccc|c|c|ccccccc|ccccccc|c} v = & \mathit{2} & \mathbf{3} & 2 & 1 & 1 & \mathit{3} & \mathit{5} & \mathbf{6} & 4 & 2 & 1 & \mathbf{3} & 2 & 3 & \mathbf{6} & \mathbf{6} & \mathbf{3} & 1 & \mathbf{6} & \mathbf{6} & 2 & \mathit{6} \\ \sigma = & \mathit{16} & 12 & 18 & 22 & 21 & \mathit{10} & \mathit{7} & 4 & 8 & 17 & 20 & 11 & 15 & 9 & 2 & 5 & 13 & 19 & 3 & 6 & 14 & \mathit{1} \end{array}$$

$$\begin{array}{rcccccc|c|c|ccccc|ccccc|cccc} \mathrm{Id} = & \mathit{1} & 2 & 3 & 4 & 5 & 6 & \mathit{7} & 8 & 9 & \mathit{10} & 11 & 12 & 13 & 14 & 15 & \mathit{16} & 17 & 18 & 19 & 20 & 21 & 22 \\ c = & 6 & 6 & 6 & 6 & 6 & 6 & 5 & 4 & 3 & 3 & 3 & 3 & 3 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\ w = & \mathit{1} & \mathbf{5} & \mathbf{6} & \mathbf{8} & \mathbf{13} & \mathbf{14} & \mathit{7} & 17 & 4 & \mathit{10} & \mathbf{15} & \mathbf{18} & \mathbf{19} & 2 & 9 & \mathit{16} & 20 & 22 & 3 & 11 & 12 & 21 \\ \epsilon = & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \end{array}$$

The word v is given, with its corresponding Lyndon factorization; the word σ is obtained from v by replacing each letter y_i by its label $\mathrm{lab}(y_i)$ and to be regarded as the product of the cycles duly materialized; c is just the monotonic nonincreasing rearrangement of v; w is the sequence $\sigma(1)\sigma(2)\cdots\sigma(n)$; ϵ is derived from c by replacing each c_i by its residue mod 3.

The eight decreases of v which are multiple of 3, and the eight excedances of (w,ϵ) have been reproduced in boldface. The four increase and record values of v are reproduced in italic, together with the four one-letter factors of σ, and the four fixed points of w, that is, 16, 10, 7, 1.

5. Specializations

Let $W_n(s,t,q)$ (resp. $F_r(u;q,s)$) be the polynomial $W_n(s,t,q,(Y_i),(Z_i))$ (resp. the series $F(u;q,s,(Y_i),(Z_i))$), when $Y_i=Z_i=1$ for all i, so that

$$(5.1)\quad W_n(s,t,q)=\sum_{(w,\epsilon)\in C_l\wr\mathfrak{S}_n} s^{\mathrm{fexc}(w,\epsilon)}t^{\mathrm{fdes}(w,\epsilon)}q^{\mathrm{fmaj}(w,\epsilon)};$$

$$\begin{aligned}(5.2)\quad F_r(u;q,s)&=\frac{(uq^ls^l;q^l)_{\lfloor r/l\rfloor}}{(u;q^l)_{\lfloor r/l\rfloor+1}}\,(1-q^ls^l)\\ &\quad\times\Bigl(1-q^ls^l+\sum_{i=1}^{l}q^is^i-\sum_{i=1}^{l}q^is^i\frac{(uq^ls^l;q^l)_{\lfloor (r-i)/l\rfloor+1}}{(u;q^l)_{\lfloor (r-i)/l\rfloor+1}}\Bigr)^{-1}.\end{aligned}$$

Theorem 1.3 yields the following result.

Theorem 5.1. *The generating function for the polynomials $W_n(s,t,q)$ reads:*

$$(5.3)\qquad \sum_{n\ge0}(1+t+\cdots+t^{l-1})W_n(s,t,q)\frac{u^n}{(t^l;q^l)_{n+1}}=\sum_{r\ge0}t^r\,F_r(u;q,s).$$

When n tends to infinity, remember that $1/(u;q)_n$ tends to the q-exponential series $e_q(u)=\sum_{n\ge0}u^n/(q;q)_n$ (see, e.g., [An76], chap. 2). Accordingly, when r tends to infinity, $F_r(u;q,s)$ tends to

$$\begin{aligned}F_\infty(u;q,s)&=\frac{e_{q^l}(u)}{e_{q^l}(uq^ls^l)}\,(1-q^ls^l)\Bigl(1-q^ls^l+\sum_{i=1}^{l}q^is^i-\sum_{i=1}^{l}q^is^i\frac{e_{q^l}(u)}{e_{q^l}(uq^ls^l)}\Bigr)^{-1}\\ &=\frac{e_{q^l}(u)}{e_{q^l}(uq^ls^l)}\,(1-q^ls^l)\Bigl((1-q^ls^l)\\ &\qquad\qquad+\frac{qs(1-q^ls^l)}{1-qs}\Bigl(1-\frac{e_{q^l}(u)}{e_{q^l}(uq^ls^l)}\Bigr)\Bigr)^{-1}\end{aligned}$$

$$= \frac{e_{q^l}(u)}{e_{q^l}(uq^l s^l)}\,(1-qs)\Bigl((1-qs)+qs\Bigl(1-\frac{e_{q^l}(u)}{e_{q^l}(uq^l s^l)}\Bigr)\Bigr)^{-1}$$
$$= \frac{e_{q^l}(u)}{e_{q^l}(uq^l s^l)}\,(1-qs)\Bigl(1-qs\frac{e_{q^l}(u)}{e_{q^l}(uq^l s^l)}\Bigr)^{-1}$$
$$= \frac{(1-qs)e_{q^l}(u)}{e_{q^l}(uq^l s^l)-qse_{q^l}(u)}.$$

Theorem 5.2. *Let*

$$W_n(s,1,q)=\sum_{(w,\epsilon)\in C_l\wr\mathfrak{S}_n} s^{\mathrm{fexc}(w,\epsilon)}q^{\mathrm{fmaj}(w,\epsilon)}.$$

Then, the following identity holds:

$$\sum_{n\ge 0} W_n(s,1,q)\frac{u^n}{(q^l;q^l)_n}=\frac{(1-qs)e_{q^l}(u)}{e_{q^l}(uq^l s^l)-qse_{q^l}(u)}. \tag{5.4}$$

Proof. There suffices to multiply both sides of identity (5.3) by $(1-t)$ and let t tend to 1. The right-hand side tends to $F_\infty(u;q,s)$, which has just been calculated. ∎

Let $u=(1-q^l)u$ and then $q\to 1$ in (5.4). This leads to the identity:

$$\sum_{n\ge 0} W_n(s,1,1)\frac{u^n}{n!}=\frac{1-s}{-s+\exp(u(s^l-1))}, \tag{5.5}$$

where

$$W_n(s,1,1)=\sum_{(w,\epsilon)\in C_l\wr\mathfrak{S}_n} s^{\mathrm{fexc}(w,\epsilon)}. \tag{5.6}$$

The next step is to calculate $F_r(u;q,1)$. First, note that

$$\frac{(uq^l;q^l)_m}{(u;q^l)_{m+1}}=\frac{1}{1-u}\quad\text{and}\quad\frac{(uq^l;q^l)_m}{(u;q^l)_m}=\frac{1-uq^{ml}}{1-u}.$$

Now, let $s=1$ in (5.2). We get:

$$F_r(u;q,1)=\frac{(uq^l;q^l)_{\lfloor r/l\rfloor}}{(u;q^l)_{\lfloor r/l\rfloor+1}}\,(1-q^l)$$
$$\times\Bigl(1-q^l+\sum_{i=1}^{l}q^i-\sum_{i=1}^{l}q^i\frac{(uq^l;q^l)_{\lfloor (r-i)/l\rfloor+1}}{(u;q^l)_{\lfloor (r-i)/l\rfloor+1}}\Bigr)^{-1}$$
$$=\frac{1-q^l}{1-u}\times\Bigl(1-q^l+\sum_{i=1}^{l}q^i-\sum_{i=1}^{l}q^i\frac{1-uq^{l(\lfloor (r-i)/l\rfloor+1)}}{1-u}\Bigr)^{-1}$$

$$= \frac{1-q^l}{1-u} \times \Bigl(1-q^l-u\sum_{i=1}^{l} q^i \frac{1-q^{l(\lfloor (r-i)/l\rfloor+1)}}{1-u}\Bigr)^{-1}$$

$$= \Bigl(1-u-u\sum_{i=1}^{l} q^i \frac{1-q^{l(\lfloor (r-i)/l\rfloor+1)}}{1-q^l}\Bigr)^{-1}.$$

Let $r = kl+s$ with $0 \le s \le l-1$. Then

$$\begin{aligned} l(\lfloor (r-i)/l\rfloor+1) &= l(\lfloor (kl+s-i)/l\rfloor+1)\\ &= l(k+\lfloor (s-i)/l\rfloor+1)\\ &= \begin{cases} l(k+0+1), & \text{if } 1\le i\le s;\\ l(k-1+1), & \text{if } s+1\le i\le l.\end{cases}\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{i=1}^{l} q^i \frac{1-q^{l(\lfloor (r-i)/l\rfloor+1)}}{1-q^l} &= \sum_{i=1}^{s} q^i \frac{1-q^{l(k+1)}}{1-q^l} + \sum_{i=s+1}^{l} q^i \frac{1-q^{lk}}{1-q^l}\\ &= \frac{1-q^{l(k+1)}}{1-q^l}\frac{q-q^{s+1}}{1-q} + \frac{1-q^{lk}}{1-q^l}\frac{q^{s+1}-q^{l+1}}{1-q}\\ &= \frac{1}{1-q^l}\frac{1}{1-q}\bigl(q(1-q^l)-q^{lk+s+1}(1-q^l)\bigr)\\ &= \frac{q}{1-q}(1-q^{lk+s}) = q\frac{1-q^r}{1-q}.\end{aligned}$$

Consequently, $F_r(u;q,1) = \Bigl(1-u-uq\dfrac{1-q^r}{1-q}\Bigr)^{-1} = \Bigl(1-u\dfrac{1-q^{r+1}}{1-q}\Bigr)^{-1}$, so that, using the traditional notation for the q-analogs of integers,

$$F_r(u;q,1) = \bigl(1-u[r+1]_q\bigr)^{-1}. \tag{5.7}$$

The following theorem has then be proved.

Theorem 5.3. *The factorial generating function for the polynomials*

$$W_n(1,t,q) = \sum_{(w,\epsilon)\in C_l\wr\mathfrak{S}_n} t^{\mathrm{fdes}(w,\epsilon)} q^{\mathrm{fmaj}(w,\epsilon)} \quad (n\ge 0) \tag{5.8}$$

reads

$$\sum_{n\ge 0}(1+t+\cdots+t^{l-1})W_n(1,t,q)\frac{u^n}{(t^l;q^l)_{n+1}} = \sum_{r\ge 0} t^r\,\bigl(1-u[r+1]_q\bigr)^{-1}. \tag{5.9}$$

Several authors ([AR01], [ABR01], [ABR05], [ABR06], [CHGe07], [HLR05], [FH09]) have derived identity (5.9) in the particular case $l = 2$

(hyperoctahedral group). Theorem 5.3 has several consequences. Write $W^{(l)}(1,t,q) := W_n(1,t,q)$ to indicate that the polynomial also depends on l. In particular,

$$W_n^{(1)}(1,t,q) = A_n(t,q) = \sum_{\sigma\in\mathfrak{S}_n} t^{\operatorname{des}\sigma} q^{\operatorname{maj}\sigma}, \tag{5.10}$$

which is precisely the *q-Eulerian polynomial* introduced by Carlitz [Ca54], and also combinatorially interpreted by him [Ca75]. As (5.9) holds for every l, we also have

$$\sum_{n\ge 0} A_n(t,q)\frac{u^n}{(t;q)_{n+1}} = \sum_{r\ge 0} t^r\left(1-u[r+1]_q\right)^{-1}, \tag{5.11}$$

so that

$$W_n^{(l)}(1,t,q) = \frac{(t^l q^l;q^l)_n}{(t;q)_n} A_n(t,q) \quad (n\ge 0). \tag{5.12}$$

In view of (5.12) there is no use working out the other formulas for $W^{(l)}(1,t,q)$ from scratch. We just have to report to Carlitz's original paper [Ca54], dealing with the polynomials $A_n(t,q)$, and use (5.12). First,

$$\begin{aligned}(1-q)W_n^{(l)}(1,t,q) &= (1-t^lq^l)W_{n-1}^{(l)}(1,t,q)\\ &\quad -(-q+tq(1-q)+\cdots+t^{l-1}q^{l-1}(1-q)+t^lq^l)W_{n-1}^{(l)}(1,tq,q).\end{aligned} \tag{5.13}$$

Next, let $W_n^{(l)}(1,t,q) = \sum_k W_{n,k}^{(l)}(q)t^k$ and look for the coefficients of t^k on both sides. We get:

$$\begin{aligned}W_{n,k}^{(l)}(q) &= [k+1]_q W_{n-1,k}^{(l)}(q) + q^{k+1}W_{n-1,k-1}^{(l)}(q)\\ &+q^{k+2}W_{n-1,k-2}^{(l)}(q)+\cdots+q^{k+l-1}W_{n-1,k-(l-1)}^{(l)}(q)+q^k[nl-k]_qW_{n-1,k-l}^{(l)}(q).\end{aligned} \tag{5.14}$$

With $q=1$ in (5.14) we obtain the recurrence formula (1.14) of Theorem 1.4.

Note that for $l=1$ identity (1.14) becomes the classical recurrence formula for the *Eulerian numbers* $A_{n,k} := W_{n,k}^{(1)}$

$$A_{n,k} = (k+1)A_{n-1,k} + (n-k)A_{n-1,k-1}$$

(see, e.g., [FS70]), which are the coefficients of the classical *Eulerian polynomial* $A_n(t,1) = \sum_k A_{n,k}t^k$. For $q=1$ identity (5.12) becomes:

$$W_n^{(l)}(1,t,1) = \frac{(1-t^l)^n}{(1-t)^n} A_n(t,1). \tag{5.15}$$

As the exponential generating function for the Eulerian polynomials (see, e.g., [FS70]) reads

$$\sum_{n\geq 0} A_n(t,1)\frac{u^n}{n!} = \frac{1-t}{-t+\exp(u(t-1))}, \tag{5.16}$$

identity (5.16) implies that

$$\sum_{n\geq 0} W_n^{(l)}(1,t,1)\frac{u^n}{n!} = \frac{1-t}{-t+\exp(u(t^l-1))}. \tag{5.17}$$

From identities (5.5) and (5.17) we conclude that

$$W_n^{(l)}(t,1,1) = W_n^{(l)}(1,t,1) = \sum_{k=0}^{nl-1} W_{n,k}^{(l)} t^k, \tag{5.18}$$

the coefficients $W_{n,k}^{(l)}$ satisfying recurrence (5.15). This proves Theorem 1.4.

A good exercise of Combinatorics consists of proving directly that both coefficients $W_{n,k}^{(l,\mathrm{fdes})} := \#\{(w,\epsilon)\in C_l\wr\mathfrak{S}_n,\ \mathrm{fdes}(w,\epsilon)=k\}$ and $W_{n,k}^{(l,\mathrm{fexc})} := \#\{(w,\epsilon)\in C_l\wr\mathfrak{S}_n,\ \mathrm{fexc}(w,\epsilon)=k\}$ satisfy recurrence formula (1.14). For $W_{n,k}^{(l,\mathrm{fdes})}$ it suffices to analyze the impact of the *insertion* of the biletter $\binom{n}{i}$ into the n slots of an element from $C_l\wr\mathfrak{S}_{n-1}$. For $W_{n,k}^{(l,\mathrm{fexc})}$ the variation of "fexc" is to be analyzed when the following operation is performed: replace the j-th biletter $\binom{x_j}{\epsilon_j}$ of an element $\binom{w}{\epsilon} = \binom{x_1 \ldots x_j \ldots x_{n-1}}{\epsilon_1 \ldots \epsilon_j \ldots \epsilon_{n-1}}$ from $C_l\wr\mathfrak{S}_{n-1}$ by $\binom{n}{k}$, and insert the biletter $\binom{x_j}{k'}$ to the right. We do not reproduce the solution of this exercise!

6. Concluding remarks

The Decrease Value Theorem, stated and proved in [FH07], was regarded as our Ur-result in our studies on q-calculus of permutation statistics. The motivation of this paper has been to extend its application to another group structure, namely the wreath product $C_l\wr\mathfrak{S}_n$. This theorem makes it possible to have a full control of all *decreases* in each word. By means of a standardization procedure, the decreases can then be carried over to *excedances* of the underlying permutations.

As was done in our two papers [FH08] and [FH09] dealing with the symmetric, and hyperoctahedral group, respectively, we could as well have made use of the so-called *V-word decomposition theorem* for words, a theorem directly inspired from a result by Kim-Zeng [KZ01] valid for permutations, for deriving Theorem 1.2. The Decrease Value Theorem has the advantage of providing the adequate identity immediately without any further combinatorial construction.

There is also an extension of Theorem 1.3, where the set $\{0,1,\ldots,l-1\}$ is split into two subsets I, J such that $0\in I$ and the total order imposed on $C_l \wr \mathfrak{S}_n$ has the properties:

$$(6.1)\qquad \begin{array}{l} \text{(a) } (j,i)<(j',i') \text{ when } i>i' \text{ for all } j,j';\\ \text{(b) } (1,i)<(2,i)<\cdots<(n,i) \text{ when } i\in I;\\ \text{(c) } (n,i)<\cdots<(2,i)<(1,i) \text{ when } i\in J.\end{array}$$

When $J=\emptyset$, we recover the total order used in the previous sections. For $l=2$, $I=\{0\}$, $J=\{1\}$, we get the natural order defined on the hyperoctahedral group B_n, namely, $-n<\cdots<-1<1<\cdots<n$ with the convention: $-j\equiv(j,1)$, $j\equiv(j,0)$ for all $j=1,2,\ldots,n$.

Referring to formula (1.11) let

$$\begin{aligned} H_r(u;q,s,(Z_i)) &:= Z_0(1-q^ls^l)\\ &\quad\times\Bigl(Z_0-Z_0q^ls^l+\sum_{i=1}^{l}q^is^iZ_i-\sum_{i=1}^{l}q^is^iZ_i\frac{(uq^ls^lZ_0;q^l)_{\lfloor (r-i)/l\rfloor+1}}{(uZ_0;q^l)_{\lfloor (r-i)/l\rfloor+1}}\Bigr)^{-1},\end{aligned}$$

a series that does *not* depend on (Y_i). The extension of (1.13) for the total order defined in (6.1) reads:

$$\begin{aligned}(6.2)\quad &\sum_{n\ge 0}(1+t+\cdots+t^{l-1})W_n(s,t,q,(Y_i),(Z_i))\frac{u^n}{(t^l;q^l)_{n+1}}\\ &=\sum_{r\ge 0}t^r\,\frac{\prod_{i\in J}(-us^iq^iY_iZ_i;q^l)_{\lfloor (r-i)/l\rfloor+1}}{\prod_{i\in J}(-us^iq^iZ_i;q^l)_{\lfloor (r-i)/l\rfloor+1}}\,\frac{\prod_{i\in(I\setminus\{0\})\cup\{l\}}(uq^is^iZ_i;q^l)_{\lfloor (r-i)/l\rfloor+1}}{\prod_{i\in I}(uq^is^iY_iZ_i;q^l)_{\lfloor (r-i)/l\rfloor+1}}\\ &\qquad\times H_r(u;q,s,(Z_i)).\end{aligned}$$

We do not reproduce the proof of this extension. Note that it fully implies the result we had derived in [FH09] for the hyperoctahedral group. Also note that the first statistical study of the latter group has been made by Reiner [Re93a], [Re93b], [Re93c], Re95a].

Several papers have recently been published dealing with statistics on wreath products. The first analysis made by Bagno [B04] was followed by Haglund et al. [HLR05], who studied another aspect of permutation statistics on wreath products in connection with the theory of perfect matchings and rook placement q-counting. Bernstein [DB06] has worked out a theory for the so-called $C_a\wr S_n$ q-maj Euler-Mahonian *bivariable* polynomials and obtained a solid formulary. The definitions he took for his generalized descent and major index do not coincide with the ones presented here. Bagno and Garber [BG06] generalize results by

Ksavrelof and Zeng [KZ03] on the multidistribution of the excedance number associated with the numbers of fixed points and cycles. Some more or less explicit three-variable statistical distributions are derived. Regev and Roichman [RR06] have worked out recurrence formulas of binomial-Stirling type for wreath product statistics related to left minima. Finally, Mendes and Remmel [MR07] have developed a brick-tabloid symmetric function approach for calculating generating functions for statistics for tuples of permutations.

References

[AR01] Ron M. Adin; Yuval Roichman. The flag major index and group actions on polynomial rings, *Europ. J. Combin.*, vol. **22**, 2001, p. 431–446.

[ABR01] Ron M. Adin; Francesco Brenti; Yuval Roichman. Descent Numbers and Major Indices for the Hyperoctahedral Group, *Adv. in Appl. Math.*, vol. **27**, 2001, p. 210–224.

[ABR05] Ron M. Adin; Francesco Brenti; Yuval Roichman. Descent representations and multivariate statistics, *Trans. Amer. Math. Soc.*, vol. **357**, 2005, p. 3051–3082.

[ABR06] Ron M. Adin; Francesco Brenti; Yuval Roichman. Equi-distribution over Descent Classes of the Hyperoctahedral Group, *J. Comb. Theory, Ser. A.*, vol. **113**, 2006, p. 917–933.

[An76] George E. Andrews. *The theory of partitions*. Addison-Wesley, Reading MA, 1976 (*Encyclopedia of Math. and its Appl.*, **2**).

[DB06] Dan Bernstein. Euler-Mahonian polynomials for $C_a \wr S_n$, *Advances in Appl. Math.*, vol. **37**, 2006, p. 198–208.

[B04] Eli Bagno. Euler-Mahonian parameters on colored permutation groups, *Sém. Lothar. Combin.* **51**, 2004, Art. B51f, 16 pp.

[BG06] Eli Bagno; David Garber. On the excedance number of colored permutation groups, *Sém. Lothar. Combin.* **53**, 2006, Art. B53f.

[Ca54] Leonard Carlitz. q-Bernoulli and Eulerian numbers, *Trans. Amer. Math. Soc.*, vol. **76**, 1954, p. 332–350.

[Ca75] Leonard Carlitz. A combinatorial property of q-Eulerian numbers, *Amer. Math. Monthly*, vol. **82**, 1975, p. 51–54.

[CH58] K. T. Chen; R. H. Fox; R. C. Lyndon. Free differential calculus, IV. The quotient group of the lower central series, *Ann. of Math.*, vol. **68**, 1958, p. 81–95.

[ChGe07] Chak-On-Chow; Ira M. Gessel. On the descent numbers and major indices for the hyperoctahedral group, *Adv. in Appl. Math.*, vol. **38**, 2007, p. 275–301.

[FH07] Dominique Foata; Guo-Niu Han. Decreases and Descents on Words, *Sémin. Lothar. Combin.*, **58**, 2007, Art. B58a, 17 pages.

[FH09] Dominique Foata; Guo-Niu Han. Signed words and permutations, V; A sextuple distributio, *Ramanujan J.*, vol. **19**, 2009, p. 29–52.

[FH08] Dominique Foata; Guo-Niu Han. Fix-Mahonian Calculus, III: A quadruple distribution, *Monatshefte für Math.*, vol. **154**, 2008, p. 177-197.

[FS70] Dominique Foata; Marcel-Paul Schützenberger. *Théorie géométrique des polynômes eulériens*. Springer-Verlag, Berlin, 1970 (*Lecture Notes in Math.*, **138**).

[GR93] Ira Gessel; Christophe Reutenauer. Counting permutations with given cycle structure and descent set, *J. Combin. Theory Ser. A*, vol. **64**, 1993, p. 189–215.

[HLR05] J. Haglund; N. Loehr; J. B. Remmel. Statistics on wreath products, perfect matchings, and signed words, *Europ. J. Combin.*, vol. **26**, 2005, p. 835–868.

[KZ01] Dongsu Kim; Jiang Zeng. A new decomposition of derangements, *J. Combin. Theory Ser. A*, vol. **96**, 2001, p. 192–198.

[KZ03] G. Ksavrelof; J. Zeng. Two involutions for signed excedance numbers, *Sém. Lothar. Combin.*, **49**, 2003, Art. B49e, 8 pp.

[Lo83] M. Lothaire. *Combinatorics on Words*. Addison-Wesley, London 1983 (Encyclopedia of Math. and its Appl., **17**).

[MR07] Anthony Mendes; Jeffrey Remmel. Generating functions for statistics on $C_k \wr \mathfrak{S}_n$, *Sém. Lothar. Combin.*, **54A**, 2007, Article B54At.

[Re93a] V. Reiner. Signed permutation statistics, *Europ. J. Combinatorics*, vol. **14**, 1993, p. 553–567.

[Re93b] V. Reiner. Signed permutation statistics and cycle type, *Europ. J. Combinatorics*, vol. **14**, 1993, p. 569–579.

[Re93c] V. Reiner. Upper binomial posets and signed permutation statistics, *Europ. J. Combinatorics*, vol. **14**, 1993, p. 581–588.

[Re95a] V. Reiner. Descents and one-dimensional characters for classical Weyl groups, *Discrete Math.*, vol. **140**, 1995, p. 129–140.

[Re95b] V. Reiner. The distribution of descents and length in a Coxeter group, *Electronic J. Combinatorics*, vol. **2**, 1995, # R25.

[RR06] Amitai Regev; Yuval Roichman. Statistics on wreath products and generalized binomial-Stirling numbers, *Israel J. Math.*, vol. **151**, 2006, p. 189–221.

[Sch65] Marcel-Paul Schützenberger. On a factorization of free monoids, *Proc. Amer. Math. Soc.*, vol. **16**, 1965, p. 21–24.

Dominique Foata
Institut Lothaire
1, rue Murner
F-67000 Strasbourg, France
foata@unistra.fr

Guo-Niu Han
I.R.M.A. UMR 7501
Université de Strasbourg et CNRS
7, rue René-Descartes
F-67084 Strasbourg, France
guoniu.han@unistra.fr