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Doubloons and q-secant numbers

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ABSTRACT. Based on the evaluation at $t = -1$ of the generating polynomial for the hyperoctahedral group by the number of descents, an observation recently made by Hirzebruch, a new q-secant number is derived by working with the Chow-Gessel q-polynomial involving the flag major index. Using the doubloon combinatorial model we show that this new q-secant number is a polynomial with positive integral coefficients, a property apparently hard to prove by analytical methods.

1. Introduction

This paper, in harmony with our previous two papers on doubloons [FH09a, FH09b], is motivated by our intention of finding a combinatorial connection between the Eulerian polynomials, on the one hand, and the trigonometric functions, tangent and secant, on the other hand, when the connection is further carried over to a q-analog environment.

Let $(t;q)_n := (1-t)(1-tq)\cdots(1-tq^{n-1})$ if $n \geq 1$ and $(t;q)_0 := 1$ be the traditional q-ascending factorial and $[\,j\,]_q := 1 + q + \cdots + q^{j-1}$ be the q-analog of the positive integer j. The *q-analogs* $A_n(t,q)$, introduced by Carlitz ([Ca54], [Ca75]), of the *Eulerian polynomials*, may be defined by the identity

$$\frac{A_n(t,q)}{(t;q)_{n+1}} = \sum_{j\geq 0} t^j([j+1]_q)^n \quad (n \geq 0). \tag{1.1}$$

For each $n \geq 0$ the q-analog $A_n(t,q)$ is a polynomial with positive integral coefficients [in short, a PIC *polynomial*], such that $A_n(t,1)$ is equal to the traditional *Eulerian polynomial* $A_n(t)$ introduced by Euler himself [Eul1755], who also derived the exponential generating function:

$$\sum_{n\geq 0} \frac{u^n}{n!} A_n(t) = \frac{1-t}{-t + \exp(u(t-1))}. \tag{1.2}$$

As $A_n(1,1) = A_n(1) = n!$, each PIC polynomial $A_n(t)$ (resp. $A_n(t,q)$) has been regarded as a generating function for the symmetric group $\mathfrak{S}_n$ by several integral-valued statistics (resp. pairs of such statistics) ([Ri58], [FZ70], [Ca75]). Note that (1.2) is easily derived from (1.1).

In the same manner, the next two identities

$$\frac{B_n(t,q)}{(t;q^2)_{n+1}} = \sum_{j\ge 0} t^j([2j+1]_q)^n \quad (n\ge 0); \tag{1.3}$$

$$\sum_{n\ge 0} \frac{u^n}{n!} B_n(t) = \frac{(1-t)\exp(u(t-1))}{-t+\exp(2u(t-1))}; \tag{1.4}$$

may serve to define two families of polynomials $(B_n(t))$, $(B_n(t,q))$ $(n\ge 0)$.

Again, both $B_n(t)$ and $B_n(t,q)$ are PIC polynomials and $B_n(t) = B_n(t,1)$. Moreover, (1.4) is easily derived from (1.3). The interpretation of $B_n(t)$ as a generating polynomial for the hyperoctahedral group B_n, together with the derivations of (1.3) for $q=1$ and (1.4), was first obtained by Reiner [Re93], also by Cohen [Co08] in the general context of the Coxeter groups of spherical type. Formula (1.3) was derived and fully interpreted by Chow and Gessel [CG07].

While studying the signatures of the toric varieties, Hirzebruch [Hi09] is led to calculate the values of both polynomials $A_n(t)$ and $B_n(t)$ at $t=-1$. He first quotes Euler's identities [Eul1755]

$$A_{2n}(-1) = 0 \ (n\ge 1); \quad (-1)^n A_{2n+1}(-1) = T_{2n+1} \ (n\ge 0), \tag{1.5}$$

where the coefficients T_{2n+1} $(n\ge 0)$ are the *tangent numbers* occurring in the Taylor expansion of $\tan u$:

$$\begin{aligned} \tan u &= \sum_{n\ge 0} \frac{u^{2n+1}}{(2n+1)!} T_{2n+1} \\ &= \frac{u}{1!}1 + \frac{u^3}{3!}2 + \frac{u^5}{5!}16 + \frac{u^7}{7!}272 + \frac{u^9}{9!}7936 + \frac{u^{11}}{11!}353792 + \cdots \end{aligned} \tag{1.6}$$

Then, he notes that

$$B_{2n+1}(-1) = 0 \ (n\ge 0); \quad (-1)^n B_{2n}(-1) = 2^{2n} E_{2n} \ (n\ge 0), \tag{1.7}$$

where the coefficients E_{2n} $(n\ge 0)$ are the *secant numbers* occurring in the Taylor expansion of $\sec u$

$$\begin{aligned} \sec u = \frac{1}{\cos u} &= \sum_{n\ge 0} \frac{u^{2n}}{(2n)!} E_{2n} \\ &= 1 + \frac{u^2}{2!}1 + \frac{u^4}{4!}5 + \frac{u^6}{6!}61 + \frac{u^8}{8!}1385 + \frac{u^{10}}{10!}50521 + \cdots \end{aligned} \tag{1.8}$$

since, by (1.4),

$$\sum_{n\ge 0} \frac{(iu)^n}{2^n n!} B_n(-1) = \frac{2}{e^{iu}+e^{-iu}} = \sec u = \sum_{n\ge 0} \frac{u^{2n}}{(2n)!} E_{2n}.$$

It so happens that (1.7) is just the relation needed to construct a new q-analog of the secant number, in parallel with what has been done already for the tangent number.

Theorem 1.1. *Let $(B_n(t,q))$ $(n \ge 0)$ be the sequence of polynomials defined by* (1.3) *and let*

$$E_{2n}(q) := (-1)^n\, q^{n^2}\, B_{2n}(-q^{-2n}, q) \quad (n \ge 1). \tag{1.9}$$

Then,

(*a*) *each $E_{2n}(q)$ is a* PIC *polynomial;*

(*b*) *it admits the factorization*

$$E_{2n}(q) = (1+q^2)(1+q^4)\cdots(1+q^{2n})F_{2n}(q), \tag{1.10}$$

where $F_{2n}(q)$ is a PIC *polynomial;*

(*c*) $E_{2n}(1) = 2^n F_{2n}(1) = 2^{2n} E_{2n}$ (E_{2n} *the secant number*);

(*d*) $B_{2n+1}(-q^{-(2n+1)}, q) = 0$ $(n \ge 0)$.

Property (*c*) follows from (1.7) and (1.9). Property (*d*) is proved in Section 5. As is often the case, it is much harder to derive the factorization shown in (*b*) and prove that the coefficients of $E_{2n}(q)$ are *positive*. It requires a long *combinatorial* development, given in the next three Sections. We reproduce the first values of the polynomials $B_n(t,q)$ and $E_{2n}(q)$ in Tables 1.1 and 1.2.

$$\begin{aligned}
&B_1(t,q) = 1 + qt; \qquad B_2(t,q) = 1 + (2q + 2q^2 + 2q^3)t + q^4t^2;\\
&B_3(t,q) = 1 + (3q + 5q^2 + 7q^3 + 5q^4 + 3q^5)t\\
&\qquad\qquad + (3q^4 + 5q^5 + 7q^6 + 5q^7 + 3q^8)t^2 + q^9t^3;\\
&B_4(t,q) = 1 + (4q + 9q^2 + 16q^3 + 18q^4 + 16q^5 + 9q^6 + 4q^7)t\\
&\quad + (6q^4 + 16q^5 + 30q^6 + 40q^7 + 46q^8 + 40q^9 + 30q^{10} + 16q^{11} + 6q^{12})t^2\\
&\quad + (4q^9 + 9q^{10} + 16q^{11} + 18q^{12} + 16q^{13} + 9q^{14} + 4q^{15})t^3 + q^{16}t^4.
\end{aligned}$$

Table 1.1. The polynomials $B_n(t,q)$.

$$\begin{aligned}
&E_2(q) = (1+q^2)2; \qquad E_4(q) = (1+q^2)(1+q^4)(6+8q+6q^2);\\
&E_6(q) = (1+q^2)(1+q^4)(1+q^6)(20 + 60q + 104q^2 + 120q^3 + 104q^4\\
&\qquad\qquad + 60q^5 + 20q^6);\\
&E_8(q) = (1+q^2)(1+q^4)(1+q^6)(1+q^8)(70 + 336q + 910q^2 + 1760q^3\\
&\qquad\qquad + 2702q^4 + 3440q^5 + 3724q^6 + 3440q^7 + 2702q^8 + 1760q^9 + 910q^{10}\\
&\qquad\qquad + 336q^{11} + 70q^{12}).
\end{aligned}$$

Table 1.2. The polynomials $E_{2n}(q)$.

Following the method developed in [FH09a] and [FH09b], the proof of Theorem 1.1 (*a*) and (*b*) will consist of making the polynomial $E_{2n+2}(q)$, defined in (1.9), appear as a generating function by an appropriate statistic "smaj," combined with a sign "sgn"

$$E_{2n+2}(q) = \sum_{w \in B_{2n+2}} \operatorname{sgn} w\, q^{\operatorname{smaj} w} \qquad (n \ge 1)$$

and constructing a sign-reversing involution on B_{2n+2}, in such a way that after its application the remaining terms in the sum have positive signs. We leave out the banal case: $E_2(q) = 2(1+q^2)$.

The final step is then to prove the identity

$$E_{2n+2}(q) = (1+q^2)(1+q^4)\cdots(1+q^{2n+2}) \sum_{w\in \mathcal{SN}_{2n+2}} q^{\operatorname{smaj} w}, \tag{1.11}$$

where the sum is over a specific class $\mathcal{SN}_{2n+2}$ of signed permutations, called *normalized signed doubloons* (see Section 4).

More importantly, the generating polynomial for $\mathcal{SN}_{2n+2}$ occurring in (1.11) will be explicitly calculated by means of the *doubloon polynomials* $(d_{n,j}(q))$ $(n\ge 1,\ 2\le j\le 2n)$, which are defined by the recurrence

$(D1)$ $d_{0,j}(q) = \delta_{1,j}$ (Kronecker symbol);
$(D2)$ $d_{n,j}(q) = 0$ for $n\ge 1$ and $j\le 1$ or $j\ge 2n+1$;
$(D3)$ $d_{n,2}(q) = \sum\limits_j q^{j-1}\, d_{n-1,j}(q)$ for $n\ge 1$;
$(D4)$ $d_{n,j}(q) - 2\,d_{n,j-1}(q) + d_{n,j-2}(q)$

$$= -(1-q)\sum_{i=1}^{j-3} q^{n+i+1-j}\, d_{n-1,i}(q)$$
$$-(1+q^{n-1})\, d_{n-1,j-2}(q) + (1-q)\sum_{i=j-1}^{2n-1} q^{i-j+1}\, d_{n-1,i}(q)$$

for $n\ge 2$ and $3\le j\le 2n$;

the first values being:

$d_{1,2}(q)=1$; $\quad d_{2,2}(q)=q$; $\quad d_{2,3}(q)=q+1$; $\quad d_{2,4}(q)=1$;
$d_{3,2}(q)=2q^3+2q^2$; $\quad d_{3,3}(q)=2q^3+4q^2+2q$; $\quad d_{3,4}(q)=q^3+4q^2+4q+1$;
$d_{3,5}(q)=2q^2+4q+2$; $\quad d_{3,6}(q)=2q+2$;
$d_{4,2}(q)=5q^6+12q^5+12q^4+5q^3$; $\quad d_{4,3}(q)=5q^6+17q^5+24q^4+17q^3+5q^2$;
$d_{4,4}(q)=3q^6+15q^5+29q^4+29q^3+15q^2+3q$;
$d_{4,5}(q)=q^6+9q^5+25q^4+34q^3+25q^2+9q+1$;
$d_{4,6}(q)=3q^5+15q^4+29q^3+29q^2+15q+3$;
$d_{4,7}(q)=5q^4+17q^3+24q^2+17q+5$; $\quad d_{4,8}(q)=5q^3+12q^2+12q+5$.

Those polynomials were introduced and used in [FH09b] to evaluate a new q-analog

$$T_{2n+1}(q) := (-1)^n q^{\binom{n}{2}} A_{2n+1}(-q^{-n},q) \tag{1.12}$$

of the *tangent number* based on the Carlitz q-Eulerian polynomial $A_n(t,q)$ defined in (1.1). It was shown that $T_{2n+1}(q)$ was a PIC polynomial equal to

$$T_{2n+1}(q) = (1+q)(1+q^2)\cdots(1+q^n)\sum_{k=2}^{2n+2} d_{n,k}(q). \tag{1.13}$$

The parallel expression for the PIC polynomials $E_{2n+2}(q)$ is next stated.

Theorem 1.2. *For each $n \geq 1$ the polynomial $E_{2n+2}(q)$ has the following expression:*

$$E_{2n+2}(q) = (1+q^2)(1+q^4)\cdots(1+q^{2n+2})\sum_{k=2}^{2n} d_{n,k}(q^2)P_{n,k}(q), \tag{1.14}$$

where the coefficients $P_{n,k}(q)$ ($n \geq 1$, $2 \leq j \leq 2n$) are defined by

$$P_{n,k}(q) := \sum_{i=0}^{2n+1-k} q^{n-1-2i} \sum_{l=i+1}^{i+k} \binom{2n+2}{l} q^l. \tag{1.15}$$

The quantities $Q_{n,k}(q) := q^{n+1-k}P_{n,k}(q)$ are PIC polynomials. Their first values are listed in Table 1.3.

$Q_{1,2}(q) = 6+8q+6q^2$;
$Q_{2,2}(q) = 15+26q+30q^2+26q^3+15q^4$;
$Q_{2,3}(q) = 20+30q+32q^2+30q^3+20q^4$; $\quad Q_{2,4}(q) = Q_{2,2}(q)$
$Q_{3,2}(q) = 28+64q+98q^2+112q^3+98q^4+64q^5+28q^6$;
$Q_{3,3}(q) = 56+98q+120q^2+126q^3+120q^4+98q^5+56q^6$;
$Q_{3,4}(q) = 70+112q+126q^2+128q^3+126q^4+112q^5+70q^6$;
$Q_{3,5}(q) = Q_{3,3}(q)$; $\quad Q_{3,6}(q) = Q_{3,2}(q)$;
$Q_{4,2}(q) = 45+130q+255q^2+372q^3+420q^4+372q^5+255q^6+130q^7+45q^8$;
$Q_{4,3}(q) = 120+255q+382q^2+465q^3+492q^4+465q^5+382q^6+255q^7+120q^8$;
$Q_{4,4}(q) = 210+372q+465q^2+502q^3+510q^4+502q^5+465q^6+372q^7+210q^8$;
$Q_{4,5}(q) = 252+420q+492q^2+510q^3+512q^4+510q^5+492q^6+420q^7+252q^8$;
$Q_{4,6}(q) = Q_{4,4}(q)$; $\quad Q_{4,7}(q) = Q_{4,3}(q)$; $\quad Q_{4,8}(q) = Q_{4,2}(q)$.

Table 1.3. The polynomials $Q_{n,k}(q)$.

The proofs of Theorems 1.1 and 1.2 are given in Sections 3 and 4. In the last Section we obtain a global expression for the generating polynomial for the group B_n by a five-variable statistic, which takes the two classical *descent* definitions into account.

To end this introduction we point out that the identity

$$T_{2n+1} = 2^n \sum_{k=2}^{2n} d_{n,k}, \tag{1.16}$$

which is the $q=1$ version of (1.13), is originally due to Christiane Poupard [Po89], who worked out the recurrence for the now called *Poupard triangle* $d_{n,k} := d_{n,k}(1)$ ($n \geq 1$, $2 \leq k \leq 2n$), obtainable from $(D1)$–$(D4)$ for $q=1$.

We reproduce the first values of the Poupard triangle $(d_{n,k})$, together with the first values of

$$Q_{n,k} := Q_{n,k}(1) = P_{n,k}(1) = \sum_{i=0}^{2n+1-k} \sum_{l=i+1}^{i+k} \binom{2n+2}{l}. \tag{1.17}$$

Both $d_{n,k}$ and $Q_{n,k}$ are displayed in triangles ($2 \le k \le 2n$, $1 \le n \le 4$), as shown in Fig. 1.4.

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & 2 & 1 & & \\ & 4 & 8 & 10 & 8 & 4 & \\ 34 & 68 & 94 & 104 & 94 & 68 & 34 \end{array} \qquad \begin{array}{ccccccc} & & & 20 & & & \\ & & 112 & 132 & 112 & & \\ & 492 & 674 & 744 & 674 & 492 & \\ 2024 & 2936 & 3608 & 3860 & 3608 & 2936 & 2024 \end{array}$$

The Poupard triangle $(d_{n,k})$ — The coefficients $(Q_{n,k})$

Fig. 1.4.

The $q = 1$ version of identity (1.14) reads:

$$2^{n+1}\, E_{2n+2} = \sum_{k=2}^{2n} d_{n,k} Q_{n,k}. \tag{1.18}$$

For instance, (1.18) for $n = 2$ yields: $2^3 E_6 = 8 \times 61 = 488 = 1 \times 112 + 2 \times 132 + 1 \times 112$. There exists a rich formulary of relations for tangent and secant numbers (see, e.g., the old monograph by Nielsen [Ni23]). Identities (1.16) and (1.18) provide a new parametrization of those coefficients by means of the Poupard triangle $(d_{n,k})$.

2. Statistics on the hyperoctahedral group

The elements of the hyperoctahedral group B_n, usually called *signed permutations*, may be viewed as *words* $w = x_1 x_2 \cdots x_n$, where each x_i belongs to the set $\{-n, \ldots, -1, 1, \ldots, n\}$ and $|x_1||x_2| \cdots |x_n|$ is a permutation of $12 \ldots n$. The *set* (resp. the *number*) of *negative* letters among the x_i's is denoted by $\operatorname{Neg} w$ (resp. $\operatorname{neg} w$). In the same manner, let $\operatorname{Pos} w$ (resp. $\operatorname{pos} w$) be the set (resp. the number) of all positive letters in w. It is convenient to write $\bar{i} := -i$ for each integer i. There are $2^n n!$ signed permutations of order n. The symmetric group $\mathfrak{S}_n$ may be considered as the subset of all w from B_n such that $\operatorname{Neg} w = \emptyset$.

For each statement A let $\chi(A) = 1$ or 0 depending on whether A is true or not. The usual *number of descents* and *major index* of each word $w = x_1 x_2 \cdots x_n$ are defined by

$$\operatorname{des} w := \sum_{i=1}^{n-1} \chi(x_i > x_{i+1}); \tag{2.1}$$

$$\operatorname{maj} w := \sum_{i=1}^{n-1} i\, \chi(x_i > x_{i+1}). \tag{2.2}$$

When B_n is regarded as a Coxeter group, an extra descent is counted, when the first letter x_1 of the signed permutation $w = x_1 x_2 \cdots x_n$ is *negative.*

In the literature two definitions are then used:

$$\mathrm{des}_B\, w := \chi(x_1 < 0) + \mathrm{des}\, w; \tag{2.3}$$

$$\mathrm{fdes}\, w := \chi(x_1 < 0) + 2\,\mathrm{des}\, w. \tag{2.4}$$

Furthermore, a *flag major index* "fmaj" defined by

$$\mathrm{fmaj}\, w := 2\,\mathrm{maj}\, w + \mathrm{neg}\, w, \tag{2.5}$$

has been adopted for B_n, because it is equidistributed with the *Coxeter length* "ℓ" for B_n (see, e.g., [ABR01], [FH07]), a property that extends the corresponding property for the symmetric group $\mathfrak{S}_n$, which says that the major index "maj" and the number of inversions "inv" (the Coxeter length for $\mathfrak{S}_n$) are equidistributed.

Proposition 2.1. *The polynomial* $B_n(t, q)$ *defined by* (1.3) *has the following combinatorial interpretation:*

$$B_n(t, q) = \sum_{w \in B_n} t^{\mathrm{des}_B\, w} q^{\mathrm{fmaj}\, w}. \tag{2.6}$$

In other words, $B_n(t, q)$ *is the generating polynomial for the hyperoctahedral group* B_n *by the pair* $(\mathrm{des}_B, \mathrm{fmaj})$.

The proof of the proposition can be found in [CG07]. This is also a consequence of Theorem 6.2, that takes both "des_B" and "fdes" into account (see (6.15) and (6.16)).

From the definition of the polynomials $E_{2n+2}(q)$ given in (1.9) and (2.6) it follows that

$$E_{2n+2}(q) = (-1)^{n+1} q^{(n+1)^2} B_{2n+2}(-q^{-2n+2}, q)$$

may be expressed as

$$E_{2n+2}(q) = (-1)^{n+1} \sum_{w = x_1 \cdots x_{2n+2} \in B_{2n+2}} (-1)^{\chi(x_1<0) + \mathrm{des}\, w} q^{\mathrm{smaj}\, w}, \tag{2.7}$$

where "smaj" is a new statistic — call it *signed major index* — defined for each signed permutation $w = x_1 x_2 \cdots x_{2n+2} \in B_{2n+2}$ by

$$\mathrm{smaj}\, w := (n+1)^2 - 2(n+1)\big(\chi(x_1 < 0) + \mathrm{des}\, w\big) + 2\,\mathrm{maj}\, w + \mathrm{neg}\, w. \tag{2.8}$$

A *compressed major index* "cmaj" was defined in [FH09a], [FH09b] on the symmetric group $\mathfrak{S}_n$. Extend its definition to each $w \in B_{2n+2}$, as follows

$$\mathrm{cmaj}\, w := \mathrm{maj}\, w - (n+1)\,\mathrm{des}\, w + (n-1)n/2. \tag{2.9}$$

The next lemma only needs a straightforward calculation.

Lemma 2.2. *For each $w = x_1x_2\cdots x_{2n+2}\in B_{2n+2}$ we have:*

$$(2.10)\quad \operatorname{smaj} w - 2\operatorname{cmaj} w = 3n+1+\operatorname{neg} w - 2(n+1)\chi(x_1<0);$$

so that

$$(2.11)\quad \operatorname{smaj} w - 2\operatorname{cmaj} w = n+\operatorname{neg} w - 1,\qquad \textit{if } x_1<0.$$

The *mirror image* of a signed permutation $w = x_1x_2\cdots x_{2n+2}$ is defined by $\mathbf{r}\,w := x_{2n+2}\cdots x_2x_1$. It is easily verified that

$$(2.12)\qquad \operatorname{des}\mathbf{r}\,w = (2n+1) - \operatorname{des} w;$$

$$(2.13)\qquad \operatorname{maj}\mathbf{r}\,w = (2n+2)(2n+1)/2 - (2n+2)\operatorname{des} w + \operatorname{maj} w.$$

Those two relations suffice to prove the next lemma.

Lemma 2.3. *For each $w = x_1x_2\cdots x_{2n+2}\in B_{2n+2}$ we have:*

$$(2.14)\quad \operatorname{smaj}\mathbf{r}\,w - \operatorname{smaj} w = 2(n+1)\bigl(\chi(x_1<0)-\chi(x_{2n+2}<0)\bigr);$$

$$(2.15)\quad (-1)^{\operatorname{des}\mathbf{r}\,w+\chi(x_{2n+2}<0)}\times(-1)^{\operatorname{des} w+\chi(x_1<0)} = -(-1)^{\chi(x_1<0)+\chi(x_{2n+2}<0)}.$$

The sum displayed in (2.7) may be decomposed into four subsums:

$$\begin{aligned}\sum_{w=x_1\cdots x_{2n+2}\in B_{2n+2}} &= \sum_{\substack{x_1x_{2n+2}>0,\\ x_1<x_{2n+2}}} + \sum_{\substack{x_1x_{2n+2}>0,\\ x_1>x_{2n+2}}} + \sum_{\substack{x_1<0,\\ x_{2n+2}>0}} + \sum_{\substack{x_1>0,\\ x_{2n+2}<0}}\\ &= \sum_{\substack{x_1x_{2n+2}>0,\\ x_1<x_{2n+2}}} + \mathbf{r}\sum_{\substack{x_1x_{2n+2}>0,\\ x_1<x_{2n+2}}} + \sum_{\substack{x_1<0,\\ x_{2n+2}>0}} + \mathbf{r}\sum_{\substack{x_1<0,\\ x_{2n+2}>0}}.\end{aligned}$$

It follows from Lemma 2.3 that the sum of the first two subsums vanishes, and the fourth one is equal to the product of the third one by q^{2n+2}. Thus,

$$(2.16)\qquad E_{2n+2}(q) = (-1)^{n+1}(1+q^{2n+2})\sum_{\substack{w=x_1\cdots x_{2n+2}\in B_{2n+2},\\ x_1<0<x_{2n+2}}}(-1)^{\operatorname{des} w+1}q^{\operatorname{smaj} w}$$

since $\chi(x_1<0)=1$ for every w occurring in the sum. To pursue the calculation of $E_{2n+2}(q)$ we use the doubloon calculus, as developed in our previous two papers.

3. Doubloons

A *doubloon* of order $(2n+1)$ is defined to be a permutation of the word $012\cdots(2n+1)$, represented as a $2\times(n+1)$-matrix $\delta = \binom{a_0\,\cdots\,a_n}{b_0\,\cdots\,b_n}$. The word

$a_0 \cdots a_n b_n \cdots b_0$ is called the *reading* $\rho(\delta)$ of δ. Define $\operatorname{stat}\delta := \operatorname{stat}\rho(\delta)$, whenever "stat" is equal to "des," "maj," "fmaj," "cmaj," or "smaj." Let $F\,\delta := a_0$, $L\,\delta := b_0$. The set of all doubloons of order $(2n+1)$ is denoted by $\mathcal{D}_{2n+1}$. The subset of all doubloons δ such that $L\,\delta = j$ (resp. $F\,\delta = i$ and $L\,\delta = j$) is denoted by $\mathcal{D}_{2n+1,j}$ (resp. $\mathcal{D}^i_{2n+1,j}$).

Each doubloon $\delta = \binom{a_0 \cdots a_n}{b_0 \cdots b_n}$ from $\mathcal{D}_{2n+1}$ is said to be *interlaced* (resp. *normalized*), if for every $k = 1, 2, \ldots, n$ the sequence $(a_{k-1}, a_k, b_{k-1}, b_k)$ or one of its three *cyclic rearrangements* is monotonic increasing or decreasing (resp. decreasing). Let $\mathcal{I}^i_{2n+1}$ (resp. $\mathcal{I}^i_{2n+1,j}$, resp. $\mathcal{N}^i_{2n+1}$, resp. $\mathcal{N}^i_{2n+1,j}$) denote the set of all doubloons δ from $\mathcal{D}^i_{2n+1}$, which are interlaced (resp. interlaced with $L\,\delta = j$, resp. normalized, resp. normalized with $L\,\delta = j$).

For instance, the doubloon $\delta = \binom{0\,4\,3}{2\,1\,5}$ is normalized, since both sequences $(4, 2, 1, 0)$ and $(5, 4, 3, 1)$, which are cyclic rearrangements of $(0, 4, 2, 1)$ and $(4, 3, 1, 5)$, respectively, are decreasing.

The geometry of interlaced and normalized doubloons has been studied in [FH09a]. The connection between interlaced doubloons and *split-pair arrangements*, introduced by Graham and Zang [GZ08], is explicitly made in [FH09b].

We now recall several properties on doubloons already proved in [FH09a], [FH09b]. For each doubloon $\delta = \binom{a_0\, a_1 \cdots a_n}{b_0\, b_1 \cdots b_n}$ from $\mathcal{D}_{2n+1}$ and each integer h let $\delta + h$ be the doubloon

$$\delta + h := \begin{pmatrix} a_0 + h & a_1 + h & \cdots & a_n + h \\ b_0 + h & b_1 + h & \cdots & b_n + h \end{pmatrix}, \tag{3.1}$$

where each entry is expressed as a residue $\operatorname{mod}(2n+2)$.

Property 3.1. *The mapping $\delta \mapsto \delta + h$ is a bijection of $\mathcal{I}^i_{2n+1,j}$ (resp. $\mathcal{N}^i_{2n+1,j}$) onto $\mathcal{I}^{i+h}_{2n+1,j+h}$ (resp. $\mathcal{N}^{i+h}_{2n+1,j+h}$) (superscript and subscript being taken* $\operatorname{mod}(2n+2)$*).*

See [FH09b], Proposition 2.1.

Property 3.2. *Let $0 \le i < j$ and $\delta = \binom{a_0\, a_1 \cdots a_n}{b_0\, b_1 \cdots b_n}$ be a doubloon from $\mathcal{D}^i_{2n+1,j}$, so that $\delta - i = \binom{0 \;\; a_1 - i \cdots a_n - i}{j-i\; b_1 - i \cdots b_n - i}$ belongs to $\mathcal{D}^0_{2n+1,j-i}$. Then,*

$$\operatorname{des}(\delta - i) = \operatorname{des}\delta, \qquad \operatorname{cmaj}(\delta - i) = \operatorname{cmaj}\delta + i. \tag{3.2}$$

See [FH09b], Lemma 3.2.

Property 3.3. *For each integer k there is a sign-reversing involution on $\mathcal{D}^0_{2n+1,k} \setminus \mathcal{I}^0_{2n+1,k}$ having the property that*

$$\sum_{\delta \in \mathcal{D}^0_{2n+1,k}} (-1)^{n + \operatorname{des}\delta} q^{\operatorname{cmaj}\delta} = \sum_{\delta \in \mathcal{I}^0_{2n+1,k}} q^{\operatorname{cmaj}\delta}. \tag{3.3}$$

Moreover,

$$\sum_{\delta\in\mathcal{I}^0_{2n+1,k}} q^{\operatorname{cmaj}\delta} = (1+q)(1+q^2)\cdots(1+q^n)\sum_{\delta'\in\mathcal{N}^0_{2n+1,k}} q^{\operatorname{cmaj}\delta}; \tag{3.4}$$

Proof. Refer to the proofs of Theorems 4.2 and 1.6 in [FH09a], and observe that the first column $\binom{0}{k}$ is left invariant under each macro flip. ∎

4. Signed doubloons

Now, we extend the notion of doubloon to the group of signed permutations and speak of *signed doubloons*, but only for those signed permutations $w = x_1x_2\cdots x_{2n+2}\in B_{2n+2}$ occurring in the summation displayed in (2.16). They have the property that $F\,w := x_1 < 0 < x_{2n+2} =: L\,w$. We represent them as $2\times(n+1)$-matrices $w = \begin{pmatrix} x_1 & x_2 & \cdots x_{n+1}\\ x_{2n+2} & x_{2n+1} & \cdots x_{n+2}\end{pmatrix}$. The set of all those signed doubloons will be denoted by $\mathcal{SD}_{2n+2}$.

For each $w = \begin{pmatrix} x_1 & x_2 & \cdots x_{n+1}\\ x_{2n+2} & x_{2n+1} & \cdots x_{n+2}\end{pmatrix}$ from $\mathcal{SD}_{2n+2}$ let ϕ_w be the increasing bijection of $\{x_1, x_2, \ldots, x_{2n+2}\}$ onto $\{0, 1, 2, \ldots, 2n+1\}$ and form the (unsigned) doubloon $\delta_w := \begin{pmatrix} \phi_w(x_1) & \phi_w(x_2) & \cdots \phi_w(x_{n+1})\\ \phi_w(x_{2n+2}) & \phi_w(x_{2n+1}) & \cdots \phi_w(x_{n+2})\end{pmatrix}$. The signed doubloon w is characterized by the pair $(\delta_w, -\operatorname{Neg} w)$. Moreover, $\operatorname{stat} w = \operatorname{stat}\delta_w$ whenever "stat" is equal to "des," "maj," "fmaj," "cmaj," or "smaj." The signed doubloon w is said to be *interlaced* (resp. *normalized*), if δ_w is interlaced (resp. normalized).

As $F\,w < 0 < L\,w$ when w belongs to $\mathcal{SD}_{2n+2}$, the mapping

$$w \mapsto (\delta_w, -\operatorname{Neg} w) \tag{4.1}$$

is a bijection of the set $\mathcal{SD}_{2n+2}$ onto the set of pairs (δ, J) such that $\delta\in\mathcal{D}_{2n+1}$, and J a subset of $\{1, 2, \ldots, 2n+2\}$ such that $F\,\delta + 1 \le \#J \le L\,\delta$.

For instance, if $\delta = \binom{042}{315}\in\mathcal{D}_5$, then $F\delta + 1 = 1 \le \#J \le 3 = L\delta$. Take $J = \{3\}, \{1, 3\}, \{2, 3, 5\}$ for example, the three signed doubloons $w\in\mathcal{SD}_6$ associated with those three subsets J are the following:

$$\begin{aligned}
\binom{\bar{3}52}{416} &\mapsto \left(\binom{042}{315}, \{3\}\right),\\
\binom{\bar{3}52}{4\bar{1}6} &\mapsto \left(\binom{042}{315}, \{1,3\}\right),\\
\binom{\bar{5}4\bar{2}}{1\bar{3}6} &\mapsto \left(\binom{042}{315}, \{2,3,5\}\right).
\end{aligned}$$

If $(\delta_w, -\operatorname{Neg} w) = (\delta, J)$, then (see (2.11))

$$\operatorname{des} w = \operatorname{des}\delta;\quad \operatorname{smaj} w = 2\operatorname{cmaj}\delta + \#J + n - 1. \tag{4.2}$$

We next make the composition product of the two mappings described in (3.1) and (4.1).

Theorem 4.1. *For each pair (i,k) of integers such that $1\le k\le 2n$ and $0\le i\le 2n+1-k$ the mapping*

$$w\mapsto(\delta_w-i,\ -\operatorname{Neg} w) \tag{4.3}$$

is a bijection of the set $\mathcal{SD}^i_{2n+2,i+k}$ of the signed doubloons w satisfying $F\,\delta_w=i$, $L\,\delta_w=i+k$ onto the set of pairs (δ,J) such that $\delta\in\mathcal{D}^0_{2n+1,k}$ and $J\subset[1,2n+2]$ with $i+1\le \#J\le i+k$. Moreover, if w is interlaced (resp. normalized), so is δ_w-i, and conversely. Finally, if $\delta=\delta_w-i$, then

$$\operatorname{des} w=\operatorname{des}\delta;\quad \operatorname{smaj} w=2\operatorname{cmaj}\delta-2i+\#J+n-1. \tag{4.4}$$

Proof. The theorem is a consequence of Properties 3.1 and 3.2 and the properties of the bijection $w\mapsto(\delta_w,-\operatorname{Neg} w)$ given in (4.2). ∎

Identity (2.16) may be rewritten as

$$\begin{aligned}E_{2n+2}(q)&=(-1)^{n+1}(1+q^{2n+2})\sum_{w\in\mathcal{SD}_{2n+2}}(-1)^{\operatorname{des} w+1}q^{\operatorname{smaj} w}\\&=(1+q^{2n+2})\sum_{k=1}^{2n}\sum_{i=0}^{2n+1-k}\sum_{w\in\mathcal{SD}^i_{2n+2,i+k}}(-1)^{n+\operatorname{des} w}q^{\operatorname{smaj} w}.\end{aligned}$$

Let

$$P_{n,i,k}(q):=q^{n-1-2i}\sum_{l=i+1}^{i+k}\binom{2n+2}{l}q^l. \tag{4.5}$$

Using the preceding theorem and Property 3.3 we evaluate the third sum as follows.

$$\begin{aligned}&\sum_{w\in\mathcal{SD}^i_{2n+2,i+k}}(-1)^{n+\operatorname{des} w}q^{\operatorname{smaj} w}\\&\qquad=\sum_{\delta\in\mathcal{D}^0_{2n+1,k}}\ \sum_{i+1\le\#J\le i+k}(-1)^{n+\operatorname{des}\delta}q^{2\operatorname{cmaj}\delta-2i+\#J+n-1}\\&\qquad=q^{n-1-2i}\sum_{\delta\in\mathcal{D}^0_{2n+1,k}}(-1)^{n+\operatorname{des}\delta}q^{2\operatorname{cmaj}\delta}\sum_{l=i+1}^{i+k}\binom{2n+2}{l}q^l\\&\qquad=P_{n,i,k}(q)\sum_{\delta\in\mathcal{D}^0_{2n+1,k}}(-1)^{n+\operatorname{des}\delta}q^{2\operatorname{cmaj}\delta}\\&\qquad=P_{n,i,k}(q)\sum_{\delta\in\mathcal{I}^0_{2n+1,k}}q^{2\operatorname{cmaj}\delta}\\&\qquad=(1+q^2)\cdots(1+q^{2n})P_{n,i,k}(q)\sum_{\delta\in\mathcal{N}^0_{2n+1,k}}q^{2\operatorname{cmaj}\delta}\\&\qquad=(1+q^2)\cdots(1+q^{2n})P_{n,i,k}(q)\,d_{n,k}(q^2),\end{aligned}\tag{4.6}$$

where the last equality follows from [FH09b], Theorem 1.2. By multiplying (4.6) by $(1+q^{2n+2})$ and summing over all pairs (k,i) such that $1\le k\le 2n$ and $0\le i\le 2n+1-k$ we derive identity (1.14), keeping in mind that $P_{n,k}(q)=\sum\limits_{0\le i\le 2n+1-k} P_{n,i,k}$.

This achieves the proofs of both Theorems 1.1 and 1.2, except part (d).

Let $\mathcal{SN}^i_{2n+2,i+k}$ be the set of the normalized signed doubloons w satisfying $F\,\delta_w=i$, $L\,\delta_w=i+k$. It also follows from Theorem 4.1 that

$$\sum_{w\in\mathcal{SN}^i_{2n+2,i+k}} q^{\operatorname{smaj} w}=P_{n,i,k}(q)\sum_{\delta\in\mathcal{N}^0_{2n+1,k}} q^{2\operatorname{cmaj}\delta}. \tag{4.7}$$

From (4.6) it follows that

$$\sum_{w\in\mathcal{SD}^i_{2n+2,i+k}}(-1)^{n+\operatorname{des} w}q^{\operatorname{smaj} w}=(1+q^2)\cdots(1+q^{2n})\sum_{w\in\mathcal{SN}^i_{2n+2,i+k}} q^{\operatorname{smaj} w}. \tag{4.8}$$

By multiplying (4.8) by $(1+q^{2n+2})$ and summing over all pairs (k,i) such that $1\le k\le 2n$ and $0\le i\le 2n+1-k$ we derive identity (1.11).

5. Proof of Theorem 1.1 (d)

Recall that for each $w=x_1x_2\cdots x_{2n+1}\in B_{2n+1}$ we have used the notations $F\,w:=x_1$ and $L\,w:=x_{2n+1}$. As $B_{2n+1}(t,q)=\sum\limits_{w\in B_{2n+1}} t^{\operatorname{des}_B w}q^{\operatorname{fmaj} w}$, we may write

$$B_{2n+1}(-q^{-(2n+1)},q)=\sum_{w\in B_{2n+1}}(-1)^{\operatorname{sgn} w}q^{\operatorname{smaj} w}, \tag{5.1}$$

where

$$\operatorname{sgn} w:=(-1)^{\operatorname{des} w+\chi(F\,w<0)}; \tag{5.2}$$

$$\operatorname{smaj} w:=2\operatorname{maj} w+\operatorname{neg} w-(2n+1)(\operatorname{des} w+\chi(F\,w<0), \tag{5.3}$$

as there is no ambiguity to adopt this definition of “smaj” for signed permutations from B_{2n+1}.

For proving the identity $A_{2n}(-q^{-n},q)=0$ in [FH09a] we had recourse to the classical properties of the dihedral group acting on $\mathfrak{S}_{2n}$. Actually, the mirror image $\mathbf{r}$ provided the sign-reversing involution that was needed. With the group B_{2n+1} the supplementary descent to be counted, when the first letter is negative, makes it necessary to include another dihedral group involution, as well as a sign change operation.

In this section the elements of B_{2n+1} will be regarded as two-row matrices $w=\binom{|w|}{\epsilon}:=\binom{|x_1|\,|x_2|\,\cdots\,|x_{2n+1}|}{\epsilon_1\ \epsilon_2\ \cdots\ \epsilon_{2n+1}}$, where $|w|:=|x_1||x_2|\cdots|x_{2n+1}|$

becomes an *ordinary* permutation and $\epsilon := \epsilon_1\epsilon_2\cdots\epsilon_{2n+1}$ is the sign word defined by $\epsilon_i := 1$ or -1, dependeing on whether x_i is positive or negative $(1 \le i \le 2n+1)$.

Three operations $\mathbf{r}$, $\mathbf{c}$, $\mathbf{s}$ are now introduced and further extended to all of B_{2n+1}: first, the *mirror image*

$$\mathbf{r} : y_1y_2\cdots y_{2n+1} \mapsto y_{2n+1}\cdots y_2y_1,$$

defined for every *arbitrary word*; second, the *complement to* $(2n+2)$, defined for each *permutation* from $\mathfrak{S}_{2n+1}$, by

$$\mathbf{c} : y_1y_2\cdots y_{2n+1} \mapsto (2n+2-y_1)(2n+2-y_2)\cdots(2n+2-y_{2n+1});$$

third, the *sign change* $\mathbf{s}$, defined for each *binary word*, such as $\epsilon = \epsilon_1\epsilon_2\cdots\epsilon_{2n+1}$, whose letters are equal to $+1$ or -1, by

$$\mathbf{s} : \epsilon_1\epsilon_2\cdots\epsilon_{2n+1} \mapsto \overline{\epsilon}_1\overline{\epsilon}_2\cdots\overline{\epsilon}_{2n+1}.$$

We use the same symbols for their extensions to B_{2n+1}:

$$\mathbf{r} : \binom{|w|}{\epsilon} = \binom{|x_1|\,|x_2|\cdots|x_{2n+1}|}{\epsilon_1\ \epsilon_2\ \cdots\ \epsilon_{2n+1}} \mapsto \binom{\mathbf{r}\,|w|}{\mathbf{r}\,\epsilon} = \binom{|x_{2n+1}|\cdots|x_2||x_1|}{\epsilon_{2n+1}\ \cdots\ \epsilon_2\ \epsilon_1}; \tag{5.4}$$

$$\mathbf{c} : \binom{|w|}{\epsilon} = \binom{|x_1|\cdots|x_{2n+1}|}{\epsilon_1\ \cdots\ \epsilon_{2n+1}} \mapsto \binom{\mathbf{c}\,|w|}{\epsilon} = \binom{(2n+2-|x_1|)\cdots(2n+2-|x_{2n+1}|)}{\epsilon_1\ \cdots\ \epsilon_{2n+1}}; \tag{5.5}$$

$$\mathbf{s} : \binom{|w|}{\epsilon} = \binom{|x_1|\,|x_2|\cdots|x_{2n+1}|}{\epsilon_1\ \epsilon_2\ \cdots\ \epsilon_{2n+1}} \mapsto \binom{|w|}{\mathbf{s}\,\epsilon} = \binom{|x_1|\,|x_2|\cdots|x_{2n+1}|}{\overline{\epsilon}_1\ \overline{\epsilon}_2\ \cdots\ \overline{\epsilon}_{2n+1}}. \tag{5.6}$$

Note that the three involutions $\mathbf{r}$, $\mathbf{c}$, $\mathbf{s}$, defined on B_{2n+1} by (5.4), (5.5) and (5.6) *commute*. The composition product $\mathbf{b} := \mathbf{c}\,\mathbf{s}\,\mathbf{r}$ can also be written as

$$\mathbf{b} : \binom{|w|}{\epsilon} = \binom{|x_1|\cdots|x_{2n+1}|}{\epsilon_1\ \cdots\ \epsilon_{2n+1}} \mapsto \binom{\mathbf{c}\,\mathbf{r}\,|w|}{\mathbf{r}\,\mathbf{s}\,\epsilon} = \binom{(n+2-|x_{2n+1}|)\cdots(n+2-|x_1|)}{\overline{\epsilon}_{2n+1}\ \cdots\ \overline{\epsilon}_1}. \tag{5.7}$$

Theorem 5.1. *The composition product* $\mathbf{b}$ *defined in* (5.7) *is a sign-reversing involution of* B_{2n+1}, *i.e.,*

$$(\mathrm{sgn}, \mathrm{smaj})\binom{|w|}{\epsilon} = (-\,\mathrm{sgn}, \mathrm{smaj})\mathbf{b}\binom{|w|}{\epsilon}. \tag{5.8}$$

The proof of the theorem is based on the next three lemmas. The first two ones being easy to verify are given without proofs.

Lemma 5.2. *For each* $w = \binom{|w|}{\epsilon} \in B_{2n+1}$ *we have*

$$\mathrm{sgn}\,\mathbf{r}\,w = \mathrm{sgn}\,w \cdot (-1)^{\chi(L\,w<0)+\chi(F\,w<0)}; \tag{5.9}$$

$$\mathrm{smaj}\,\mathbf{r}\,w = \mathrm{smaj}\,w + (2n+1)\big(\chi(F\,w<0) - \chi(L\,w<0)\big). \tag{5.10}$$

Lemma 5.3. *For each* $w = \binom{|w|}{\epsilon} \in B_{2n+1}$ *we have:*

$$\mathrm{sgn}\,\mathbf{s}\,w = -\,\mathrm{sgn}\,w; \tag{5.11}$$

$$\mathrm{smaj}\,\mathbf{s}\,w = -\,\mathrm{smaj}\,w. \tag{5.12}$$

The third lemma requires a careful analysis.

Lemma 5.4. *For each* $w = \binom{|w|}{\epsilon} = \binom{|x_1|\,|x_2|\,\cdots\,|x_{2n+1}|}{\epsilon_1\ \epsilon_2\ \cdots\ \epsilon_{2n+1}} \in B_{2n+1}$ *we have:*

$$\operatorname{sgn} \mathbf{c}\, w = (-1)^{\chi(F\,w<0)-\chi(L\,w<0)} \operatorname{sgn} w; \tag{5.13}$$

$$\operatorname{smaj} \mathbf{c}\, w = -\operatorname{smaj} w - (2n+1)\big(\chi(F\,w<0) - \chi(L\,w<0)\big). \tag{5.14}$$

Proof. If $\binom{|x_i|}{\epsilon_i} > \binom{|x_{i+1}|}{\epsilon_{i+1}}$ (resp. $\binom{|x_i|}{\epsilon_i} < \binom{|x_{i+1}|}{\epsilon_{i+1}}$) say that i is an *interior* descent (resp. rise), if $|x_i| > |x_{i+1}|$ (resp. $|x_i| < |x_{i+1}|$) *and* $\epsilon_i = \epsilon_{i+1}$. Denote the set of all descents (resp. rises) of w by $\mathrm{DES}\, w$ (resp. $\mathrm{RISE}\, w$), the set of all *interior* descents (resp. rises) being designated by $\mathrm{DES}^i\, w$ (resp. $\mathrm{RISE}^i\, w$), so that $\mathrm{DES}\, w = \mathrm{DES}^i\, w + \mathrm{DES}\,\epsilon$ and $\mathrm{RISE}\, w = \mathrm{RISE}^i\, w + \mathrm{RISE}\,\epsilon$.

First, $\mathrm{DES}^i\, w = \mathrm{RISE}^i\, \mathbf{c}\, w$ and $\mathrm{RISE}^i\, w = \mathrm{DES}^i\, \mathbf{c}\, w$. Hence,

$$\begin{aligned}
\operatorname{des} w + \operatorname{des} \mathbf{c}\, w &= (\#\,\mathrm{DES}\,\epsilon + \#\,\mathrm{DES}^i\, w) + (\#\,\mathrm{DES}\,\epsilon + \#\,\mathrm{DES}^i\, \mathbf{c}\, w)\\
&= (\#\,\mathrm{DES}^i\, w + \#\,\mathrm{DES}\,\epsilon) + (\#\,\mathrm{RISE}^i\, w + \#\,\mathrm{RISE}\,\epsilon)\\
&\qquad\qquad + (\#\,\mathrm{DES}\,\epsilon - \#\,\mathrm{RISE}\,\epsilon)\\
&= 2n + (\#\,\mathrm{DES}\,\epsilon - \#\,\mathrm{RISE}\,\epsilon)\\
&:= 2n + \operatorname{drise} \epsilon.
\end{aligned}$$

In the same way, let $\mathrm{DRISE}\,\epsilon := \sum_i i\,\big(\chi(i \in \mathrm{DES}\,\epsilon) - \chi(i \in \mathrm{RISE}\,\epsilon)\big)$. Then,

$$\begin{aligned}
\operatorname{maj} w + \operatorname{maj} \mathbf{c}\, w &= \sum_i \big(i\,\chi(i \in \mathrm{DES}\,\epsilon) + i\,\chi(i \in \mathrm{DES}^i\, w)\big)\\
&\qquad + \sum_i \big(i\,\chi(i \in \mathrm{DES}\,\epsilon) + i\,\chi(i \in \mathrm{DES}^i\, \mathbf{c}\, w)\big)\\
&= \sum_i \big(i\,\chi(i \in \mathrm{DES}\, w) + i\,\chi(i \in \mathrm{RISE}\, w)\big)\\
&\qquad + \sum_i i\,\big(\chi(i \in \mathrm{DES}\,\epsilon) - \chi(i \in \mathrm{RISE}\,\epsilon)\big)\\
&= (1 + 2 + \cdots + 2n) + \mathrm{DRISE}\,\epsilon\\
&= n(2n+1) + \mathrm{DRISE}\,\epsilon.
\end{aligned}$$

Let $d_1 < d_2 < \cdots$ (resp. $r_1 < r_2 < \cdots$) denote the sequence of the descents (resp. rises) of ϵ, when reading the word ϵ from left to right. Four cases are now considered.

(a) $\epsilon_1 = \epsilon_{2n+1} = -1$; the rises and descents alternate in such a way that $1 \le r_1 < d_1 < r_2 < d_2 < \cdots < r_k < d_k \le 2n$ and $k \ge 0$. Hence, $\operatorname{drise} \epsilon = 0$ and $\mathrm{DRISE}\,\epsilon = \sum_{i=1}^{k} (d_i - r_i) = \operatorname{pos} w$.

(b) $\epsilon_1 = +1$, $\epsilon_{2n+1} = -1$; the alternation becomes: $1 \le d_1 < r_1 < d_2 < r_2 < \cdots < d_k < r_k < d_{k+1} \le 2n$ $(k \ge 0)$. In this case, $\operatorname{drise} \epsilon = 1$ and $\mathrm{DRISE}\,\epsilon = \operatorname{pos} w$.

(c) $\epsilon_1 = \epsilon_{2n+1} = 1$; the sequence is then: $1 \le d_1 < r_1 < d_2 < r_2 < \cdots < d_k < r_k \le 2n$ $(k \ge 0)$. Hence, $\operatorname{drise}\epsilon = 0$ and $\operatorname{DRISE}\epsilon = -\operatorname{neg} w$.

(d) $\epsilon_1 = -1$, $\epsilon_{2n+1} = 1$; then $1 \le r_1 < d_2 < r_2 < \cdots < r_k < d_k < r_{r+1} \le 2n$ $(k \ge 0)$. Hence, $\operatorname{drise}\epsilon = -1$ and $\operatorname{DRISE}\epsilon = -\operatorname{neg} w$.

Thus, $\operatorname{sgn} w + \operatorname{sgn} \mathbf{c}\, w = \operatorname{des} w + \chi(\epsilon_1 < 0) + \operatorname{des} \mathbf{c}\, w + \chi(\epsilon_1 < 0) \equiv \operatorname{drise}\epsilon \pmod 2$, which is 0 when ϵ_1 and ϵ_{2n+1} are of the same sign (cases (a) and (c)), equal to 1 when $\epsilon_1 = 1$, $\epsilon_{2n+1} = -1$ (case (b)) and -1 when $\epsilon_1 = -1$ and $\epsilon_{2n+1} = +1$ (case (d)). Gathering in a common formula: $\operatorname{sgn} w + \operatorname{sgn} \mathbf{c}\, w \equiv \chi(\epsilon_1 = 1) - \chi(\epsilon_{2n+1} = 1)$. This implies (5.13).

Finally,

$$\begin{aligned}
\operatorname{smaj} \mathbf{c}\, w + \operatorname{smaj} w &= 2(\operatorname{maj} \mathbf{c}\, w + \operatorname{maj} w) + \operatorname{neg} \mathbf{c}\, w + \operatorname{neg} w\\
&\qquad - (2n+1)(\operatorname{des} \mathbf{c}\, w + \operatorname{des} w + 2\chi(\epsilon_1 < 0))\\
&= 2n(2n+1) + 2\operatorname{DRISE}\epsilon + 2\operatorname{neg} w\\
&\qquad - (2n+1)(2n + \operatorname{drise}\epsilon + 2\chi(\epsilon_1 < 0))\\
&= 2\operatorname{DRISE}\epsilon + 2\operatorname{neg} w\\
&\qquad - (2n+1)(\operatorname{drise}\epsilon + 2\chi(\epsilon_1 < 0))
\end{aligned}$$

$$= \begin{cases}
2\operatorname{pos} w + 2\operatorname{neg} w - (2n+1)2 = 0, & \text{in case (a);}\\
2\operatorname{pos} w + 2\operatorname{neg} w - (2n+1) = 2n+1, & \text{in case (b);}\\
-2\operatorname{neg} w + 2\operatorname{neg} w - (2n+1)0 = 0, & \text{in case (c);}\\
-2\operatorname{neg} w + 2\operatorname{neg} w - (2n+1) = -(2n+1), & \text{in case (d).}
\end{cases}$$

Altogether, $\operatorname{smaj} \mathbf{c}\, w = -\operatorname{smaj} w - (2n+1)\big(\chi(\epsilon_1 = -1) - \chi(\epsilon_{2n+1} = -1)\big)$. This proves (5.14) and also Lemma 5.4. ∎

Proof of Theorem 5.1. Let $w \in B_{2n+1}$. By the previous three lemmas

$$\begin{aligned}
\operatorname{sgn} \mathbf{r}\,\mathbf{c}\, w &= \operatorname{sgn} \mathbf{c}\, w \cdot (-1)^{\chi(L\ \mathbf{c}\, w<0)+\chi(F\ \mathbf{c}\ w<0)}\\
&= \operatorname{sgn} w(-1)^{\chi(F\,w<0)-\chi(L\,w<0)}(-1)^{\chi(L\,w<0)+\chi(F\,w<0)}\\
&= \operatorname{sgn} w;\\
\operatorname{smaj} \mathbf{r}\,\mathbf{c}\, w &= \operatorname{smaj} \mathbf{c}\, w + (2n+1)(\chi(F\,\mathbf{c}\, w < 0) - \chi(L\,\mathbf{c}\, w < 0))\\
&= \operatorname{smaj} \mathbf{c}\, w + (2n+1)(\chi(F\, w < 0) - \chi(L\, w < 0))\\
&= -\operatorname{smaj} w;\\
\operatorname{sgn} \mathbf{s}\,\mathbf{r}\,\mathbf{c}\, w &= -\operatorname{sgn} \mathbf{r}\,\mathbf{c}\, w = -\operatorname{sgn} w;\\
\operatorname{smaj} \mathbf{s}\,\mathbf{r}\,\mathbf{c}\, w &= -\operatorname{smaj} \mathbf{r}\,\mathbf{c}\, w = \operatorname{smaj} w. \quad ∎
\end{aligned}$$

6. Which descent for the hyperoctahedral group?

The purpose of this Section is to work out a *global* expression for the generating polynomial for B_n by the five-term statistic $(\operatorname{neg}, \operatorname{pos}, \Xi, \operatorname{des}, \operatorname{fmaj})$, where $\Xi\, w$ is equal to 1 or 0, depending on whether the first letter of w

is negative or positive, and to derive the specializations when the pair (Ξ, des) is replaced either by "des_B," or by "fdes," defined in (2.3) and (2.4). Our main result is the following.

Theorem 6.1. *Let*

$$B_n(X,Y,Z;t,q) = \sum_{w\in B_n} X^{\mathrm{neg}\, w} Y^{\mathrm{pos}\, w} Z^{\chi(x_1<0)} t^{\mathrm{des}\, w} q^{\mathrm{fmaj}\, w}. \tag{6.1}$$

Then,

$$\frac{B_n(X,Y,Z;t,q)}{(t;q^2)_{n+1}} = \frac{t-Z}{t-1}\sum_{s\ge 0} t^s\big((qX+Y)[s+1]_{q^2}\big)^n + \frac{Z-1}{t-1}\sum_{s\ge 0} t^s\big((qX+Y)[s+1]_{q^2} - Xq^{2s+1}\big)^n. \tag{6.2}$$

When $q=1$, write $B_n(X,Y,Z;t) := B_n(X,Y,Z;t,1)$. The exponential generating function for the latter polynomials can be derived in the following form.

Theorem 6.2. *The following identity holds:*

$$\sum_{n\ge 0} \frac{u^n}{n!} B_n(X,Y,Z;t) = \frac{Z-t+(1-Z)\exp(uX(t-1))}{-t+\exp(u(X+Y)(t-1))}. \tag{6.3}$$

Proof of Theorem 6.1. Let $w = x_1x_2\cdots x_n$ be a signed permutation from B_n and ϕ be the unique increasing bijection of the set $\{x_1,x_2,\ldots,x_n\}$ onto the interval $[n] := \{1,2,\ldots,n\}$. The word $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n) := \phi(x_1)\phi(x_2)\cdots\phi(x_n)$ is then an (ordinary) permutation from $\mathfrak{S}_n$ and the map $w\mapsto(\mathrm{Neg}\, w, \sigma)$ a bijection of B_n onto the Cartesian product $2^{[n]}\times\mathfrak{S}_n$ having the following properties:

$$\chi(x_1<0) = \chi(\sigma(1)\le \mathrm{neg}\, w);\quad \mathrm{des}\, w = \mathrm{des}\,\sigma;\quad \mathrm{fmaj}\, w = \mathrm{fmaj}\,\sigma.$$

For convenience, introduce the polynomial

$$A_n^k(Z;t,q) := \sum_{\sigma} Z^{\chi(\sigma(1)\le k)} t^{\mathrm{des}\,\sigma} q^{\mathrm{maj}\,\sigma}\quad (\sigma = \sigma(1)\cdots\sigma(n)\in\mathfrak{S}_n)$$

and express $B_n(X,Y,Z;t,q)$ in terms of the latter polynomials, to get:

$$\begin{aligned}
B_n(X,Y,Z;t,q) &= \sum_{k=0}^{n}\sum_{|E|=k}\sum_{\mathrm{Neg}\, w=E} (qX)^{\mathrm{neg}\, w} Y^{\mathrm{pos}\, w} Z^{\chi(x_1<0)} t^{\mathrm{des}\, w} q^{2\,\mathrm{maj}\, w}\\
&= \sum_{k=0}^{n} (qX)^k Y^{n-k} \sum_{|E|=k}\sum_{(E,\sigma)} Z^{\chi(\sigma(1)\le k)} t^{\mathrm{des}\,\sigma} q^{2\,\mathrm{maj}\,\sigma}\\
&= \sum_{k=0}^{n}\binom{n}{k}(qX)^k Y^{n-k} A_n^k(Z;t,q^2).
\end{aligned}$$

Next, with each permutation $\sigma = k\,\sigma(2)\cdots\sigma(n)$ starting with k associate the permutation $\sigma' = \sigma'(1)\cdots\sigma'(n-1) := \psi(\sigma(2))\cdots\psi(\sigma(n))$, where ψ is the unique increasing bijection of $[n]\setminus\{k\}$ onto $[n-1]$. If $\sigma(2)\le k-1$, then $\operatorname{des}\sigma = \operatorname{des}\sigma'+1$, while $\operatorname{maj}\sigma = \operatorname{maj}\sigma' + \operatorname{des}\sigma'+1$ and $\sigma'(1)\le k-1$. If $\sigma(2)\ge k+1$, then $\operatorname{des}\sigma = \operatorname{des}\sigma'$, while $\operatorname{maj}\sigma = \operatorname{maj}\sigma' + \operatorname{des}\sigma'$ and $\sigma'(1)\ge k$. Hence,

$$\begin{aligned}\sum_{\substack{\sigma(1)=k,\\ \sigma(2)\le k-1}} Z^{\chi(\sigma(1)\le k)} t^{\operatorname{des}\sigma} q^{\operatorname{maj}\sigma} &= Z \sum_{\sigma'(1)\le k-1} t^{\operatorname{des}\sigma'+1} q^{\operatorname{maj}\sigma'+\operatorname{des}\sigma'+1}\\ &= Z \sum_{\sigma'(1)\le k-1} (tq)^{\chi(\sigma(1)\le k-1)} (tq)^{\operatorname{des}\sigma'} q^{\operatorname{maj}\sigma'};\end{aligned}$$

while

$$\begin{aligned}\sum_{\substack{\sigma(1)=k,\\ \sigma(2)\ge k+1}} Z^{\chi(\sigma(1)\le k)} t^{\operatorname{des}\sigma} q^{\operatorname{maj}\sigma} &= Z \sum_{\sigma'(1)\ge k} t^{\operatorname{des}\sigma'} q^{\operatorname{maj}\sigma'+\operatorname{des}\sigma'}\\ &= Z \sum_{\sigma'(1)\ge k} (tq)^{\chi(\sigma(1)\le k-1)} (tq)^{\operatorname{des}\sigma'} q^{\operatorname{maj}\sigma'}.\end{aligned}$$

Altogether

$$\sum_{\sigma(1)=k} Z^{\chi(\sigma(1)\le k)} t^{\operatorname{des}\sigma} q^{\operatorname{maj}\sigma} = Z\, A_{n-1}^{k-1}(tq;tq,q).$$

In the same manner,

$$\sum_{\sigma(1)=k} Z^{\chi(\sigma(1)\le k-1)} t^{\operatorname{des}\sigma} q^{\operatorname{maj}\sigma} = A_{n-1}^{k-1}(tq;tq,q).$$

Consequently, we have the relation:

$$A_n^k(Z;t,q) = A_n^{k-1}(Z;t,q) + (Z-1)A_{n-1}^{k-1}(tq;tq,q). \tag{6.4}$$

By iteration we are led to:

$$\begin{aligned}A_n^k(Z;t,q) &= A_n^0(Z;t,q)\\ &\quad + \frac{Z-1}{t-1}\sum_{j=1}^{k}\binom{k}{j}(t-1)(tq-1)\cdots(tq^{j-1})A_{n-j}^0(tq^j,tq^j,q).\end{aligned} \tag{6.5}$$

But, the variable Z vanishes from $A_n^k(Z;t,q)$ when $k=0$ and then $A_n^0(Z;t,q) = A_n(t,q)$, which is the Carlitz q-analog of the Eulerian polynomial ([Ca54], [Ca75]) appearing in (1.1). Hence,

$$\begin{aligned}A_n^k(Z;t,q) &= A_n(t,q) + \frac{Z-1}{t-1}\sum_{j=1}^{k}\binom{k}{j}(-1)^j\,(t;q)_j\,A_{n-j}(tq^j,q)\\ &= \frac{t-Z}{t-1}A_n(t,q) + \frac{Z-1}{t-1}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\,(t;q)_j\,A_{n-j}(tq^j,q).\end{aligned}$$

The next step is to report this new expression of $A_n^k(Z;t,q)$ into the polynomial $B_n(X,Y,Z;t,q)$. We get:

$$\begin{aligned}
B_n(X,Y,Z;t,q) &= \sum_{k=0}^{n}\binom{n}{k}(qX)^kY^{n-k}A_n^k(Z;t,q^2)\\
&= \sum_{k=0}^{n}\binom{n}{k}(qX)^kY^{n-k}\Bigl(\frac{t-Z}{t-1}A_n(t,q^2)\\
&\quad + \frac{Z-1}{t-1}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\,(t;q^2)_j\,A_{n-j}(tq^{2j},q^2)\Bigr)\\
&= \frac{t-Z}{t-1}(qX+Y)^nA_n(t,q^2)\\
&\quad + \frac{Z-1}{t-1}\sum_{\substack{j,l,m\ge 0\\ j+l+m=n}}\frac{n!}{j!\,l!\,m!}(qX)^{j+l}Y^m(-1)^j\,(t;q^2)_j\,A_{l+m}(tq^{2j},q^2),
\end{aligned}$$

where $k=j+l$.

Next, with $r=l+m$ we get

$$\begin{aligned}
\frac{B_n(X,Y,Z;t,q)}{(t;q^2)_{n+1}} &= \frac{t-Z}{t-1}(qX+Y)^n\frac{A_n(t,q^2)}{(t;q^2)_{n+1}}\\
&\quad + \frac{Z-1}{t-1}\sum_{j+r=n}\frac{n!}{r!\,j!}(-qX)^j\frac{A_r(tq^{2j},q^2)}{(tq^{2j};q^2)_{r+1}}\sum_{l+m=r}\frac{r!}{l!\,m!}(qX)^lY^m\\
&= \frac{t-Z}{t-1}(qX+Y)^n\frac{A_n(t,q^2)}{(t;q^2)_{n+1}}\\
&\qquad + \frac{Z-1}{t-1}\sum_{j+r=n}\frac{n!}{r!\,j!}(-qX)^j\frac{A_r(tq^{2j},q^2)}{(tq^{2j};q^2)_{r+1}}(qX+Y)^r.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\sum_{n\ge 0}\frac{B_n(X,Y,Z;t,q)}{(t;q^2)_{n+1}}\frac{u^n}{n!} &= \frac{t-Z}{t-1}\sum_{n\ge 0}\frac{A_n(t,q^2)}{(t;q^2)_{n+1}}\frac{((qX+Y)u)^n}{n!}\\
&\quad + \frac{Z-1}{t-1}\sum_{j\ge 0}\frac{(-qXu)^j}{j!}\sum_{r\ge 0}\frac{A_r(tq^{2j},q^2)}{(tq^{2j},q^2)_{r+1}}\frac{((qX+Y)u)^r}{r!}.
\end{aligned}$$

Now, make use of the classical identity on the Carlitz q-Eulerian polynomials

$$\sum_{n\ge 0}\frac{u^n}{n!}\frac{A_n(t,q)}{(t;q)_{n+1}} = \sum_{s\ge 0}t^s\,\exp(u[s+1]_q),$$

to obtain

$$\sum_{n\ge 0}\frac{B_n(X,Y,Z;t,q)}{(t;q^2)_{n+1}}\frac{u^n}{n!}=\frac{t-Z}{t-1}\sum_{s\ge 0}t^s\,\exp((qX+Y)u\,[s+1]_{q^2}) \tag{6.6}$$

$$+\frac{Z-1}{t-1}\sum_{j\ge 0}\frac{(-qXu)^j}{j!}\sum_{s\ge 0}(tq^{2j})^s\exp((qX+Y)u\,[s+1]_{q^2}).$$

There remains to extract the coefficient of u^n on both sides. This leads to:

$$\frac{B_n(X,Y,Z;t,q)}{n!\,(t;q^2)_{n+1}}=\frac{t-Z}{t-1}\sum_{s\ge 0}t^s\,\big((qX+Y)\,[s+1]_{q^2}\big)^n+\frac{Z-1}{t-1}C, \tag{6.7}$$

where C is the coefficient of u^n in

$$\sum_{j\ge 0}\frac{(-qXu)^j}{j!}\sum_{s\ge 0}(tq^{2j})^s\sum_{m\ge 0}\frac{((qX+Y)u\,[s+1]_{q^2})^m}{m!},$$

that is,

$$\begin{aligned}C&=\sum_{s\ge 0}t^s\sum_{j\ge 0}\frac{(-qX)^j}{j!}q^{2js}\frac{((qX+Y)\,[s+1]_{q^2})^{n-j}}{(n-j)!}\\&=\frac{1}{n!}\sum_{s\ge 0}t^s\sum_{j\ge 0}\binom{n}{j}(-qXq^{2s})^j((qX+Y)\,[s+1]_{q^2})^{n-j}\\&=\frac{1}{n!}\sum_{s\ge 0}t^s\big((qX+Y)[s+1]_{q^2}-Xq^{2s+1}\big)^n\end{aligned}$$

Reporting the last expression in (6.7) yields identity (6.2). ∎

Proof of Theorem 6.2. When $q=1$ in (6.2), we obtain

$$\frac{B_n(X,Y,Z;t)}{(1-t)^{n+1}}=\frac{t-Z}{t-1}\sum_{s\ge 0}t^s\big((X+Y)(s+1)\big)^n \tag{6.8}$$

$$+\frac{Z-1}{t-1}\sum_{s\ge 0}t^s\big((X+Y)(s+1)-X\big)^n.$$

Hence,

$$\begin{aligned}&\sum_{n\ge 0}\frac{u^n}{(1-t)^n}B_n(X,Y,Z;t)\\&\qquad=(Z-t)\sum_{s\ge 0}t^s\sum_{n\ge 0}\frac{\big(u(X+Y)(s+1)\big)^n}{n!}\\&\qquad\qquad+(1-Z)\sum_{s\ge 0}t^s\sum_{n\ge 0}\frac{\big(u(X+Y)(s+1)-X\big)^n}{n!}\end{aligned}$$

$$= (Z-t)\sum_{s\ge 0} t^s \exp(u(X+Y)(s+1))$$
$$+ (1-Z)\sum_{s\ge 0} t^s \sum_{n\ge 0} \exp(u(X+Y)(s+1)-X)$$
$$= \big((Z-t)(\exp(u(X+Y))+(1-Z)\exp(uY)\big)\sum_{s\ge 0} t^s \exp(u(X+Y)s)$$
$$= \frac{(Z-t)\exp(u(X+Y)(1-Z))\exp(uY)}{1-t\exp(u(X+Y))},$$

which is identity (6.3) by replacing u by $u(1-t)$. ∎

Next, we derive specializations of Theorems 6.1 and 6.2 when the pair (Ξ, des) is replaced by "des_B" and "fdes" (see (2.3) and (2.4)). We get:

$$\sum_{w\in B_n} X^{\mathrm{neg}\,w} Y^{\mathrm{pos}\,w} t^{\mathrm{des}_B\,w} q^{\mathrm{fmaj}\,w} = B_n(X,Y,t;t,q); \tag{6.9}$$

$$\sum_{w\in B_n} X^{\mathrm{neg}\,w} Y^{\mathrm{pos}\,w} t^{\mathrm{fdes}\,w} q^{\mathrm{fmaj}\,w} = B_n(X,Y,t;t^2,q). \tag{6.10}$$

Also, note that $B_n(0,1,1;t,q)$ is the Carlitz q-Eulerian polynomial $A_n(t,q)$.

First,

$$\frac{B_n(X,Y,t;t,q)}{(t;q^2)_{n+1}} = \sum_{s\ge 0} t^s \big((qX+Y)[s+1]_{q^2} - Xq^{2s+1}\big)^n; \tag{6.11}$$

$$\frac{B_n(1,1,t;t,q)}{(t;q^2)_{n+1}} = \sum_{w\in B_n} t^{\mathrm{des}_B\,w} q^{\mathrm{fmaj}\,w} = \sum_{s\ge 0} t^s \big([2s+1]_q\big)^n. \tag{6.12}$$

Second,

$$\frac{B_n(X,Y,t;t^2,q)}{(t^2;q^2)_{n+1}} = \frac{t^2-t}{t-1}\sum_{s\ge 0} t^{2s}\big((qX+Y)\,[s+1]_{q^2}\big)^n$$
$$+ \frac{t-1}{t^2-1}\sum_{s\ge 0} t^{2s}\big((qX+Y)[s+1]_{q^2} - Xq^{2s+1}\big)^n,$$

so that

$$\frac{(1+t)B_n(X,Y,t;t^2,q)}{(t^2;q^2)_{n+1}} = \sum_{s\ge 0} t^{2s+1}\big((qX+Y)\,[s+1]_{q^2}\big)^n + \sum_{s\ge 0} t^{2s}\big((qX+Y)[s]_{q^2} + Yq^{2s}\big)^n. \tag{6.13}$$

In particular,

$$\frac{(1+t)B_n(1,1,t;t,q)}{(t^2;q^2)_{n+1}} = \sum_{w\in B_n} t^{\mathrm{fdes}\,w} q^{\mathrm{fmaj}\,w} = \sum_{s\ge 0} t^s \big([s+1]_q\big)^n. \tag{6.14}$$

The specializations of (6.12) and (6.14) for $q = 1$ are banal and not reproduced. However, it is worth writing the exponential generating functions for the polynomials $B_n(1,1,t;t)$ and $B_n(1,1,t;t^2)$ directly obtained from (6.3):

$$\sum_{n\ge 0}\frac{u^n}{n!}\,B_n(1,1,t;t)=\sum_{n\ge 0}\frac{u^n}{n!}\sum_{w\in B_n}t^{\operatorname{des}_B w}=\frac{(1-t)\exp(u(t-1))}{-t+\exp(2u(t-1))};\tag{6.15}$$

$$\sum_{n\ge 0}\frac{u^n}{n!}\,B_n(1,1,t;t^2)=\frac{(1-t)(t+\exp(u(t^2-1)))}{-t^2+\exp(2u(t^2-1))};$$

so that

$$\sum_{n\ge 0}\frac{u^n}{n!}\,B_n(1,1,t;t^2)=\sum_{n\ge 0}\frac{u^n}{n!}\sum_{w\in B_n}t^{\operatorname{fdes} w}=\frac{1-t}{-t+\exp(u(t^2-1))}.\tag{6.16}$$

The statistics "fdes" and "fmaj" were introduced by Adin and Roichman [AR01]. Identity (6.14) with their equivalent adaptations were derived by Brenti et al. [ABR01], Haglund et al. [HLR05] and reproved by the authors ([FH06], [FH09]) as specializations of identities involving several-variable statistics. Note that (6.16) implies that $\sum_{w\in B_n}(-1)^{\operatorname{fdes} w}$ is null for every $n\ge 1$. Accordingly, the statistic "fdes" would have been a wrong choice for obtaining a q-extension!

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