Doubloons and q-secant numbers

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Abstract. Based on the evaluation at $t = -1$ of the generating polynomial for the hyper-octahedral group by the number of descents, an observation recently made by Hirzebruch, a new q-secant number is derived by working with the Chow-Gessel q-polynomial involving the flag major index. Using the doubloon combinatorial model we show that this new q-secant number is a polynomial with positive integral coefficients, a property apparently hard to prove by analytical methods.

1. Introduction

This paper, in harmony with our previous two papers on doubloons [11, 12], is motivated by our intention of finding a combinatorial connection between the Eulerian polynomials, on the one hand, and the trigonometric functions, tangent and secant, on the other hand, when the connection is further carried over to a q-analog environment.

Let $(t; q)_n := (1 - t)(1 - tq) \cdots (1 - tq^{n-1})$ if $n \geq 1$ and $(t; q)_0 := 1$ be the traditional q-ascending factorial and $[j]_q := 1 + q + \cdots + q^{j-1}$ be the q-analog of the positive integer j. The q-*analogs* $A_n(t, q)$, introduced by Carlitz [3, 4], of the *Eulerian polynomials*, may be defined by the identity

$$(1) \qquad \frac{A_n(t, q)}{(t; q)_{n+1}} = \sum_{j \geq 0} t^j ([j + 1]_q)^n \quad (n \geq 0).$$

For each $n \geq 0$ the q-analog $A_n(t, q)$ is a polynomial with positive integral coefficients [in short, a PIC *polynomial*], such that $A_n(t, 1)$ is equal to the traditional *Eulerian polynomial* $A_n(t)$ introduced by Euler himself [7], who also derived the exponential generating function:

$$(2) \qquad \sum_{n \geq 0} \frac{u^n}{n!} A_n(t) = \frac{1 - t}{-t + \exp(u(t - 1))}.$$

As $A_n(1, 1) = A_n(1) = n!$, each PIC polynomial $A_n(t)$ (resp. $A_n(t, q)$) has been regarded as a generating function for the symmetric group $\mathfrak{S}_n$ by several integral-valued statistics (resp. pairs of such statistics) [20, 13, 4]. Note that (2) is easily derived from (1).

In the same manner, the next two identities

(3)
$$\frac{B_n(t,q)}{(t;q^2)_{n+1}} = \sum_{j\geq 0} t^j([2j+1]_q)^n \quad (n\geq 0);$$

(4)
$$\sum_{n\geq 0}\frac{u^n}{n!}B_n(t) = \frac{(1-t)\exp(u(t-1))}{-t+\exp(2u(t-1))};$$

may serve to define two families of polynomials $(B_n(t))$, $(B_n(t,q))$ $(n\geq 0)$.

Again, both $B_n(t)$ and $B_n(t,q)$ are PIC polynomials and $B_n(t) = B_n(t,1)$. Moreover, (4) is easily derived from (3). The interpretation of $B_n(t)$ as a generating polynomial for the hyperoctahedral group B_n, together with the derivations of (3) for $q = 1$ and (4), was first obtained by Reiner [19], also by Cohen [6] in the general context of the Coxeter groups of spherical type. Formula (3) was derived and fully interpreted by Chow and Gessel [5].

While studying the signatures of the toric varieties, Hirzebruch [16] is led to calculate the values of both polynomials $A_n(t)$ and $B_n(t)$ at $t = -1$. He first quotes Euler's identities [7]

(5) $A_{2n}(-1) = 0 \ (n\geq 1); \quad (-1)^n A_{2n+1}(-1) = T_{2n+1} \ (n\geq 0),$

where the coefficients T_{2n+1} $(n\geq 0)$ are the *tangent numbers* occurring in the Taylor expansion of $\tan u$:

(6) $\tan u = \sum_{n\geq 0}\frac{u^{2n+1}}{(2n+1)!}T_{2n+1}$

$$= \frac{u}{1!}1 + \frac{u^3}{3!}2 + \frac{u^5}{5!}16 + \frac{u^7}{7!}272 + \frac{u^9}{9!}7936 + \frac{u^{11}}{11!}353792 + \cdots$$

Then, he notes that

(7) $B_{2n+1}(-1) = 0 \ (n\geq 0); \quad (-1)^n B_{2n}(-1) = 2^{2n}E_{2n} \ (n\geq 0),$

where the coefficients E_{2n} $(n\geq 0)$ are the *secant numbers* occurring in the Taylor expansion of $\sec u$

(8) $\sec u = \frac{1}{\cos u} = \sum_{n\geq 0}\frac{u^{2n}}{(2n)!}E_{2n}$

$$= 1 + \frac{u^2}{2!}1 + \frac{u^4}{4!}5 + \frac{u^6}{6!}61 + \frac{u^8}{8!}1385 + \frac{u^{10}}{10!}50521 + \cdots$$

since, by (4),

$$\sum_{n\geq 0}\frac{(iu)^n}{2^n n!}B_n(-1) = \frac{2}{e^{iu}+e^{-iu}} = \sec u = \sum_{n\geq 0}\frac{u^{2n}}{(2n)!}E_{2n}.$$

It so happens that (7) is just the relation needed to construct a new q-analog of the secant number, in parallel with what has been done already for the tangent number.

Theorem 1.1. *Let* $(B_n(t,q))$ $(n \geq 0)$ *be the sequence of polynomials defined by* (3) *and let*

$$(9) \qquad E_{2n}(q) := (-1)^n \, q^{n^2} \, B_{2n}(-q^{-2n}, q) \quad (n \geq 1).$$

Then,
    (a) *each* $E_{2n}(q)$ *is a* PIC *polynomial;*
    (b) *it admits the factorization*

$$(10) \qquad E_{2n}(q) = (1+q^2)(1+q^4) \cdots (1+q^{2n}) F_{2n}(q),$$

where $F_{2n}(q)$ *is a* PIC *polynomial;*
    (c) $E_{2n}(1) = 2^n F_{2n}(1) = 2^{2n} E_{2n}$ (E_{2n} *the secant number);*
    (d) $B_{2n+1}(-q^{-(2n+1)}, q) = 0$ $(n \geq 0)$.

Property (c) follows from (7) and (9). Property (d) is proved in Section 5. As is often the case, it is much harder to derive the factorization shown in (b) and prove that the coefficients of $E_{2n}(q)$ are *positive*. It requires a long *combinatorial* development, given in the next three Sections. We reproduce the first values of the polynomials $B_n(t,q)$ and $E_{2n}(q)$ in Tables 1.1 and 1.2.

$B_1(t,q) = 1 + qt; \qquad B_2(t,q) = 1 + (2q + 2q^2 + 2q^3)t + q^4 t^2;$
$B_3(t,q) = 1 + (3q + 5q^2 + 7q^3 + 5q^4 + 3q^5)t$
$\qquad\qquad + (3q^4 + 5q^5 + 7q^6 + 5q^7 + 3q^8)t^2 + q^9 t^3;$
$B_4(t,q) = 1 + (4q + 9q^2 + 16q^3 + 18q^4 + 16q^5 + 9q^6 + 4q^7)t$
$\qquad + (6q^4 + 16q^5 + 30q^6 + 40q^7 + 46q^8 + 40q^9 + 30q^{10} + 16q^{11} + 6q^{12})t^2$
$\qquad + (4q^9 + 9q^{10} + 16q^{11} + 18q^{12} + 16q^{13} + 9q^{14} + 4q^{15})t^3 + q^{16}t^4.$

<div align="center">Table 1.1. The polynomials $B_n(t,q)$.</div>

$E_2(q) = (1+q^2)2; \qquad E_4(q) = (1+q^2)(1+q^4)(6 + 8q + 6q^2);$
$E_6(q) = (1+q^2)(1+q^4)(1+q^6)(20 + 60q + 104q^2 + 120q^3 + 104q^4$
$\qquad\qquad + 60q^5 + 20q^6);$
$E_8(q) = (1+q^2)(1+q^4)(1+q^6)(1+q^8)(70 + 336q + 910q^2 + 1760q^3$
$\qquad\qquad + 2702q^4 + 3440q^5 + 3724q^6 + 3440q^7 + 2702q^8 + 1760q^9 + 910q^{10}$
$\qquad\qquad + 336q^{11} + 70q^{12}).$

<div align="center">Table 1.2. The polynomials $E_{2n}(q)$.</div>

Following the method developed in [11] and [12], the proof of Theorem 1.1 (a) and (b) will consist of making the polynomial $E_{2n+2}(q)$, defined in (9), appear as a generating function by an appropriate statistic "smaj," combined with a sign "sgn"

$$E_{2n+2}(q) = \sum_{w \in B_{2n+2}} \operatorname{sgn} w \, q^{\operatorname{smaj} w} \qquad (n \geq 1)$$

and constructing a sign-reversing involution on B_{2n+2}, in such a way that after its application the remaining terms in the sum have positive signs. We leave out the banal case: $E_2(q) = 2(1+q^2)$.

The final step is then to prove the identity

$$(11) \qquad E_{2n+2}(q) = (1+q^2)(1+q^4)\cdots(1+q^{2n+2}) \sum_{w \in \mathcal{SN}_{2n+2}} q^{\mathrm{smaj}\, w},$$

where the sum is over a specific class $\mathcal{SN}_{2n+2}$ of signed permutations, called *normalized signed doubloons* (see Section 4).

More importantly, the generating polynomial for $\mathcal{SN}_{2n+2}$ occurring in (11) will be explicitly calculated by means of the *doubloon polynomials* $(d_{n,j}(q))$ $(n \geq 1, 2 \leq j \leq 2n)$, which are defined by the recurrence

(D1) $d_{0,j}(q) = \delta_{1,j}$ (Kronecker symbol);
(D2) $d_{n,j}(q) = 0$ for $n \geq 1$ and $j \leq 1$ or $j \geq 2n+1$;
(D3) $d_{n,2}(q) = \sum_{j} q^{j-1} d_{n-1,j}(q)$ for $n \geq 1$;
(D4) $d_{n,j}(q) - 2\, d_{n,j-1}(q) + d_{n,j-2}(q)$
$$= -(1-q) \sum_{i=1}^{j-3} q^{n+i+1-j}\, d_{n-1,i}(q)$$
$$\qquad - (1+q^{n-1})\, d_{n-1,j-2}(q) + (1-q) \sum_{i=j-1}^{2n-1} q^{i-j+1}\, d_{n-1,i}(q)$$
for $n \geq 2$ and $3 \leq j \leq 2n$;

the first values being:

$d_{1,2}(q) = 1; \qquad d_{2,2}(q) = q; \quad d_{2,3}(q) = q+1; \quad d_{2,4}(q) = 1;$
$d_{3,2}(q) = 2q^3 + 2q^2; \quad d_{3,3}(q) = 2q^3 + 4q^2 + 2q; \quad d_{3,4}(q) = q^3 + 4q^2 + 4q + 1;$
$d_{3,5}(q) = 2q^2 + 4q + 2; \quad d_{3,6}(q) = 2q + 2;$
$d_{4,2}(q) = 5q^6 + 12q^5 + 12q^4 + 5q^3; \quad d_{4,3}(q) = 5q^6 + 17q^5 + 24q^4 + 17q^3 + 5q^2;$
$d_{4,4}(q) = 3q^6 + 15q^5 + 29q^4 + 29q^3 + 15q^2 + 3q;$
$d_{4,5}(q) = q^6 + 9q^5 + 25q^4 + 34q^3 + 25q^2 + 9q + 1;$
$d_{4,6}(q) = 3q^5 + 15q^4 + 29q^3 + 29q^2 + 15q + 3;$
$d_{4,7}(q) = 5q^4 + 17q^3 + 24q^2 + 17q + 5; \quad d_{4,8}(q) = 5q^3 + 12q^2 + 12q + 5.$

Those polynomials were introduced and used in [12] to evaluate a new q-analog

$$(12) \qquad T_{2n+1}(q) := (-1)^n q^{\binom{n}{2}} A_{2n+1}(-q^{-n}, q)$$

of the *tangent number* based on the Carlitz q-Eulerian polynomial $A_n(t, q)$ defined in (1). It was shown that $T_{2n+1}(q)$ was a PIC polynomial equal to

$$(13) \qquad T_{2n+1}(q) = (1+q)(1+q^2)\cdots(1+q^n) \sum_{k=2}^{2n+2} d_{n,k}(q).$$

The parallel expression for the PIC polynomials $E_{2n+2}(q)$ is next stated.

Theorem 1.2. *For each $n \geq 1$ the polynomial $E_{2n+2}(q)$ has the following expression:*

$$(14) \qquad E_{2n+2}(q) = (1+q^2)(1+q^4)\cdots(1+q^{2n+2}) \sum_{k=2}^{2n} d_{n,k}(q^2) P_{n,k}(q),$$

where the coefficients $P_{n,k}(q)$ ($n \geq 1$, $2 \leq j \leq 2n$) are defined by

$$(15) \qquad P_{n,k}(q) := \sum_{i=0}^{2n+1-k} q^{n-1-2i} \sum_{l=i+1}^{i+k} \binom{2n+2}{l} q^l.$$

The quantities $Q_{n,k}(q) := q^{n+1-k} P_{n,k}(q)$ are PIC polynomials. Their first values are listed in Table 1.3.

$Q_{1,2}(q) = 6 + 8q + 6q^2$;
$Q_{2,2}(q) = 15 + 26q + 30q^2 + 26q^3 + 15q^4$;
$Q_{2,3}(q) = 20 + 30q + 32q^2 + 30q^3 + 20q^4$; $\quad Q_{2,4}(q) = Q_{2,2}(q)$
$Q_{3,2}(q) = 28 + 64q + 98q^2 + 112q^3 + 98q^4 + 64q^5 + 28q^6$;
$Q_{3,3}(q) = 56 + 98q + 120q^2 + 126q^3 + 120q^4 + 98q^5 + 56q^6$;
$Q_{3,4}(q) = 70 + 112q + 126q^2 + 128q^3 + 126q^4 + 112q^5 + 70q^6$;
$Q_{3,5}(q) = Q_{3,3}(q)$; $\quad Q_{3,6}(q) = Q_{3,2}(q)$;
$Q_{4,2}(q) = 45 + 130q + 255q^2 + 372q^3 + 420q^4 + 372q^5 + 255q^6 + 130q^7 + 45q^8$;
$Q_{4,3}(q) = 120 + 255q + 382q^2 + 465q^3 + 492q^4 + 465q^5 + 382q^6 + 255q^7 + 120q^8$;
$Q_{4,4}(q) = 210 + 372q + 465q^2 + 502q^3 + 510q^4 + 502q^5 + 465q^6 + 372q^7 + 210q^8$;
$Q_{4,5}(q) = 252 + 420q + 492q^2 + 510q^3 + 512q^4 + 510q^5 + 492q^6 + 420q^7 + 252q^8$;
$Q_{4,6}(q) = Q_{4,4}(q)$; $\quad Q_{4,7}(q) = Q_{4,3}(q)$; $\quad Q_{4,8}(q) = Q_{4,2}(q)$.

Table 1.3. The polynomials $Q_{n,k}(q)$.

The proofs of Theorems 1.1 and 1.2 are given in Sections 3 and 4. In the last Section we obtain a global expression for the generating polynomial for the group B_n by a five-variable statistic, which takes the two classical *descent* definitions into account.

To end this introduction we point out that the identity

$$(16) \qquad T_{2n+1} = 2^n \sum_{k=2}^{2n} d_{n,k},$$

which is the $q = 1$ version of (13), is originally due to Christiane Poupard [18], who worked out the recurrence for the now called *Poupard triangle* $d_{n,k} := d_{n,k}(1)$ ($n \geq 1$, $2 \leq k \leq 2n$), obtainable from $(D1)$–$(D4)$ for $q = 1$.

We reproduce the first values of the Poupard triangle $(d_{n,k})$, together with the first values of

$$(17) \qquad Q_{n,k} := Q_{n,k}(1) = P_{n,k}(1) = \sum_{i=0}^{2n+1-k} \sum_{l=i+1}^{i+k} \binom{2n+2}{l}.$$

Both $d_{n,k}$ and $Q_{n,k}$ are displayed in triangles ($2 \leq k \leq 2n$, $1 \leq n \leq 4$), as shown in Fig. 1.4.

The $q = 1$ version of identity (14) reads:

$$(18) \qquad 2^{n+1} E_{2n+2} = \sum_{k=2}^{2n} d_{n,k} Q_{n,k}.$$

For instance, (18) for $n = 2$ yields: $2^3 E_6 = 8 \times 61 = 488 = 1 \times 112 + 2 \times 132 + 1 \times 112$. There exists a rich formulary of relations for tangent and secant numbers (see, e.g., the old monograph by Nielsen [17]). Identities (16) and (18) provide a new parametrization of those coefficients by means of the Poupard triangle $(d_{n,k})$.
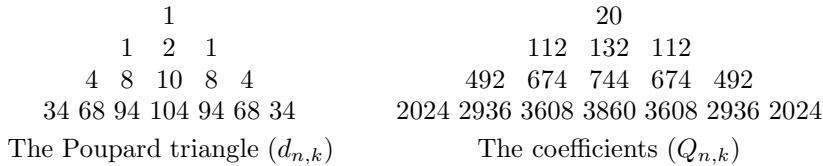
$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & 2 & 1 & & \\
 & 4 & 8 & 10 & 8 & 4 & \\
34 & 68 & 94 & 104 & 94 & 68 & 34
\end{array}
\qquad
\begin{array}{ccccccc}
 & & & 20 & & & \\
 & & 112 & 132 & 112 & & \\
 & 492 & 674 & 744 & 674 & 492 & \\
2024 & 2936 & 3608 & 3860 & 3608 & 2936 & 2024
\end{array}
$$

The Poupard triangle $(d_{n,k})$ \qquad The coefficients $(Q_{n,k})$

Fig. 1.4.

2. STATISTICS ON THE HYPEROCTAHEDRAL GROUP

The elements of the hyperoctahedral group B_n, usually called *signed permutations*, may be viewed as *words* $w = x_1 x_2 \cdots x_n$, where each x_i belongs to the set $\{-n, \ldots, -1, 1, \ldots, n\}$ and $|x_1||x_2| \cdots |x_n|$ is a permutation of $12 \ldots n$. The *set* (resp. the *number*) of *negative* letters among the x_i's is denoted by $\mathrm{Neg}\, w$ (resp. $\mathrm{neg}\, w$). In the same manner, let $\mathrm{Pos}\, w$ (resp. $\mathrm{pos}\, w$) be the set (resp. the number) of all positive letters in w. It is convenient to write $\bar{i} := -i$ for each integer i. There are $2^n n!$ signed permutations of order n. The symmetric group $\mathfrak{S}_n$ may be considered as the subset of all w from B_n such that $\mathrm{Neg}\, w = \varnothing$.

For each statement A let $\chi(A) = 1$ or 0 depending on whether A is true or not. The usual *number of descents* and *major index* of each word $w = x_1 x_2 \cdots x_n$ are defined by

$$
(19) \qquad \mathrm{des}\, w \quad := \quad \sum_{i=1}^{n-1} \chi(x_i > x_{i+1});
$$

$$
(20) \qquad \mathrm{maj}\, w \quad := \quad \sum_{i=1}^{n-1} i\, \chi(x_i > x_{i+1}).
$$

When B_n is regarded as a Coxeter group, an extra descent is counted, when the first letter x_1 of the signed permutation $w = x_1 x_2 \cdots x_n$ is *negative*. In the literature two definitions are then used:

$$
(21) \qquad \mathrm{des}_B\, w \quad := \quad \chi(x_1 < 0) + \mathrm{des}\, w;
$$

$$
(22) \qquad \mathrm{fdes}\, w \quad := \quad \chi(x_1 < 0) + 2\, \mathrm{des}\, w.
$$

Furthermore, a *flag major index* "fmaj" defined by

$$
(23) \qquad \mathrm{fmaj}\, w := 2\, \mathrm{maj}\, w + \mathrm{neg}\, w,
$$

has been adopted for B_n, because it is equidistributed with the *Coxeter length* "ℓ" for B_n (see, e.g., [1, 8]), a property that extends the corresponding property for the symmetric group $\mathfrak{S}_n$, which says that the major index "maj" and the number of inversions "inv" (the Coxeter length for $\mathfrak{S}_n$) are equidistributed.

Proposition 2.1. *The polynomial $B_n(t,q)$ defined by (3) has the following combinatorial interpretation:*

$$(24) \qquad B_n(t,q) = \sum_{w \in B_n} t^{\operatorname{des}_B w} q^{\operatorname{fmaj} w}.$$

In other words, $B_n(t,q)$ is the generating polynomial for the hyperoctahedral group B_n by the pair $(\operatorname{des}_B, \operatorname{fmaj})$.

The proof of the proposition can be found in [5]. This is also a consequence of Theorem 6.2, that takes both "des_B" and "fdes" into account (see (75) and (76)).

From the definition of the polynomials $E_{2n+2}(q)$ given in (9) and (24) it follows that

$$E_{2n+2}(q) = (-1)^{n+1} q^{(n+1)^2} B_{2n+2}(-q^{-2n+2}, q)$$

may be expressed as

$$(25) \qquad E_{2n+2}(q) = (-1)^{n+1} \sum_{w = x_1 \cdots x_{2n+2} \in B_{2n+2}} (-1)^{\chi(x_1 < 0) + \operatorname{des} w} q^{\operatorname{smaj} w},$$

where "smaj" is a new statistic — call it *signed major index* — defined for each signed permutation $w = x_1 x_2 \cdots x_{2n+2} \in B_{2n+2}$ by

$$(26) \quad \operatorname{smaj} w := (n+1)^2 - 2(n+1)\big(\chi(x_1 < 0) + \operatorname{des} w\big) + 2\operatorname{maj} w + \operatorname{neg} w.$$

A *compressed major index* "cmaj" was defined in [11, 12] on the symmetric group $\mathfrak{S}_n$. Extend its definition to each $w \in B_{2n+2}$, as follows

$$(27) \qquad \operatorname{cmaj} w := \operatorname{maj} w - (n+1)\operatorname{des} w + (n-1)n/2.$$

The next lemma only needs a straightforward calculation.

Lemma 2.2. *For each $w = x_1 x_2 \cdots x_{2n+2} \in B_{2n+2}$ we have:*

$$(28) \qquad \operatorname{smaj} w - 2\operatorname{cmaj} w = 3n + 1 + \operatorname{neg} w - 2(n+1)\chi(x_1 < 0);$$

so that

$$(29) \qquad \operatorname{smaj} w - 2\operatorname{cmaj} w = n + \operatorname{neg} w - 1, \qquad \text{if } x_1 < 0.$$

The *mirror image* of a signed permutation $w = x_1 x_2 \cdots x_{2n+2}$ is defined by $\mathbf{r}\, w := x_{2n+2} \cdots x_2 x_1$. It is easily verified that

$$(30) \qquad \operatorname{des} \mathbf{r}\, w \;=\; (2n+1) - \operatorname{des} w;$$
$$(31) \qquad \operatorname{maj} \mathbf{r}\, w \;=\; (2n+2)(2n+1)/2 - (2n+2)\operatorname{des} w + \operatorname{maj} w.$$

Those two relations suffice to prove the next lemma.

Lemma 2.3. *For each $w = x_1 x_2 \cdots x_{2n+2} \in B_{2n+2}$ we have:*

$$(32) \quad \operatorname{smaj} \mathbf{r}\, w - \operatorname{smaj} w = 2(n+1)\big(\chi(x_1 < 0) - \chi(x_{2n+2} < 0)\big);$$
$$(33) \quad (-1)^{\operatorname{des} \mathbf{r}\, w + \chi(x_{2n+2} < 0)} \times (-1)^{\operatorname{des} w + \chi(x_1 < 0)} = -(-1)^{\chi(x_1 < 0) + \chi(x_{2n+2} < 0)}.$$

The sum displayed in (25) may be decomposed into four subsums:

$$\sum_{w=x_1\cdots x_{2n+2}\in B_{2n+2}} = \sum_{\substack{x_1 x_{2n+2}>0,\\ x_1<x_{2n+2}}} + \sum_{\substack{x_1 x_{2n+2}>0,\\ x_1>x_{2n+2}}} + \sum_{\substack{x_1<0,\\ x_{2n+2}>0}} + \sum_{\substack{x_1>0,\\ x_{2n+2}<0}}$$

$$= \sum_{\substack{x_1 x_{2n+2}>0,\\ x_1<x_{2n+2}}} + \mathbf{r}\sum_{\substack{x_1 x_{2n+2}>0,\\ x_1<x_{2n+2}}} + \sum_{\substack{x_1<0,\\ x_{2n+2}>0}} + \mathbf{r}\sum_{\substack{x_1<0,\\ x_{2n+2}>0}}.$$

It follows from Lemma 2.3 that the sum of the first two subsums vanishes, and the fourth one is equal to the product of the third one by q^{2n+2}. Thus,

$$(34) \qquad E_{2n+2}(q) = (-1)^{n+1}(1+q^{2n+2}) \sum_{\substack{w=x_1\cdots x_{2n+2}\in B_{2n+2},\\ x_1<0<x_{2n+2}}} (-1)^{\operatorname{des} w+1} q^{\operatorname{smaj} w}$$

since $\chi(x_1 < 0) = 1$ for every w occurring in the sum. To pursue the calculation of $E_{2n+2}(q)$ we use the doubloon calculus, as developed in our previous two papers.

3. Doubloons

A *doubloon* of order $(2n + 1)$ is defined to be a permutation of the word $012\cdots(2n + 1)$, represented as a $2 \times (n + 1)$-matrix $\delta = \binom{a_0\ \cdots\ a_n}{b_0\ \cdots\ b_n}$. The word $a_0\cdots a_n b_n\cdots b_0$ is called the *reading* $\rho(\delta)$ of δ. Define $\operatorname{stat}\delta := \operatorname{stat}\rho(\delta)$, whenever "stat" is equal to "des," "maj," "fmaj," "cmaj," or "smaj." Let $F\,\delta := a_0$, $L\,\delta := b_0$. The set of all doubloons of order $(2n + 1)$ is denoted by $\mathcal{D}_{2n+1}$. The subset of all doubloons δ such that $L\,\delta = j$ (resp. $F\,\delta = i$ and $L\,\delta = j$) is denoted by $\mathcal{D}_{2n+1,j}$ (resp. $\mathcal{D}^i_{2n+1,j}$).

Each doubloon $\delta = \binom{a_0\ \cdots\ a_n}{b_0\ \cdots\ b_n}$ from $\mathcal{D}_{2n+1}$ is said to be *interlaced* (resp. *normalized*), if for every $k = 1, 2, \ldots, n$ the sequence $(a_{k-1}, a_k, b_{k-1}, b_k)$ or one of its three *cyclic rearrangements* is monotonic increasing or decreasing (resp. decreasing). Let $\mathcal{I}^i_{2n+1}$ (resp. $\mathcal{I}^i_{2n+1,j}$, resp. $\mathcal{N}^i_{2n+1}$, resp. $\mathcal{N}^i_{2n+1,j}$) denote the set of all doubloons δ from $\mathcal{D}^i_{2n+1}$, which are interlaced (resp. interlaced with $L\,\delta = j$, resp. normalized, resp. normalized with $L\,\delta = j$).

For instance, the doubloon $\delta = \binom{0\,4\,3}{2\,1\,5}$ is normalized, since both sequences $(4, 2, 1, 0)$ and $(5, 4, 3, 1)$, which are cyclic rearrangements of $(0, 4, 2, 1)$ and $(4, 3, 1, 5)$, respectively, are decreasing.

The geometry of interlaced and normalized doubloons has been studied in [11]. The connection between interlaced doubloons and *split-pair arrangements*, introduced by Graham and Zang [14], is explicitly made in [12].

We now recall several properties on doubloons already proved in [11, 12]. For each doubloon $\delta = \binom{a_0\ a_1\ \cdots\ a_n}{b_0\ b_1\ \cdots\ b_n}$ from $\mathcal{D}_{2n+1}$ and each integer h let $\delta + h$ be the doubloon

$$(35) \qquad \delta + h := \begin{pmatrix} a_0 + h & a_1 + h & \cdots & a_n + h \\ b_0 + h & b_1 + h & \cdots & b_n + h \end{pmatrix},$$

where each entry is expressed as a residue $\operatorname{mod}(2n + 2)$.

Property 3.1. *The mapping $\delta \mapsto \delta + h$ is a bijection of $\mathcal{I}^i_{2n+1,j}$ (resp. $\mathcal{N}^i_{2n+1,j}$) onto $\mathcal{I}^{i+h}_{2n+1,j+h}$ (resp. $\mathcal{N}^{i+h}_{2n+1,j+h}$) (superscript and subscript being taken* $\mod(2n+2)$).

See [12], Proposition 2.1.

Property 3.2. *Let $0 \le i < j$ and $\delta = \left(\begin{smallmatrix} a_0\, a_1\, \cdots\, a_n \\ b_0\, b_1\, \cdots\, b_n \end{smallmatrix}\right)$ be a doubloon from $\mathcal{D}^i_{2n+1,j}$, so that $\delta - i = \left(\begin{smallmatrix} 0\ \ a_1-i\, \cdots\, a_n-i \\ j-i\, b_1-i\, \cdots\, b_n-i \end{smallmatrix}\right)$ belongs to $\mathcal{D}^0_{2n+1,j-i}$. Then,*

$$(36) \qquad \operatorname{des}(\delta - i) = \operatorname{des}\delta, \qquad \operatorname{cmaj}(\delta - i) = \operatorname{cmaj}\delta + i.$$

See [12], Lemma 3.2.

Property 3.3. *For each integer k there is a sign-reversing involution on $\mathcal{D}^0_{2n+1,k} \setminus \mathcal{I}^0_{2n+1,k}$ having the property that*

$$(37) \qquad \sum_{\delta \in \mathcal{D}^0_{2n+1,k}} (-1)^{n+\operatorname{des}\delta} q^{\operatorname{cmaj}\delta} = \sum_{\delta \in \mathcal{I}^0_{2n+1,k}} q^{\operatorname{cmaj}\delta}.$$

Moreover,

$$(38) \qquad \sum_{\delta \in \mathcal{I}^0_{2n+1,k}} q^{\operatorname{cmaj}\delta} = (1+q)(1+q^2)\cdots(1+q^n) \sum_{\delta' \in \mathcal{N}^0_{2n+1,k}} q^{\operatorname{cmaj}\delta}.$$

Proof. Refer to the proofs of Theorems 4.2 and 1.6 in [11], and observe that the first column $\binom{0}{k}$ is left invariant under each macro flip. $\qquad\square$

4. Signed doubloons

Now, we extend the notion of doubloon to the group of signed permutations and speak of *signed doubloons*, but only for those signed permutations $w = x_1 x_2 \cdots x_{2n+2} \in B_{2n+2}$ occurring in the summation displayed in (34). They have the property that $Fw := x_1 < 0 < x_{2n+2} =: Lw$. We represent them as $2 \times (n+1)$-matrices $w = \begin{pmatrix} x_1 & x_2 & \cdots & x_{n+1} \\ x_{2n+2} & x_{2n+1} & \cdots & x_{n+2} \end{pmatrix}$. The set of all those signed doubloons will be denoted by $\mathcal{SD}_{2n+2}$.

For each $w = \begin{pmatrix} x_1 & x_2 & \cdots & x_{n+1} \\ x_{2n+2} & x_{2n+1} & \cdots & x_{n+2} \end{pmatrix}$ from $\mathcal{SD}_{2n+2}$ let ϕ_w be the increasing bijection of $\{x_1, x_2, \ldots, x_{2n+2}\}$ onto $\{0, 1, 2, \ldots, 2n+1\}$ and form the (un signed) doubloon $\delta_w := \begin{pmatrix} \phi_w(x_1) & \phi_w(x_2) & \cdots & \phi_w(x_{n+1}) \\ \phi_w(x_{2n+2}) & \phi_w(x_{2n+1}) & \cdots & \phi_w(x_{n+2}) \end{pmatrix}$. The signed doubloon w is characterized by the pair $(\delta_w, -\operatorname{Neg} w)$. Moreover, $\operatorname{stat} w = \operatorname{stat} \delta_w$ whenever "stat" is equal to "des," "maj," "fmaj," "cmaj," or "smaj." The signed doubloon w is said to be *interlaced* (resp. *normalized*), if δ_w is interlaced (resp. normalized).

As $Fw < 0 < Lw$ when w belongs to $\mathcal{SD}_{2n+2}$, the mapping

$$(39) \qquad w \mapsto (\delta_w, -\operatorname{Neg} w)$$

is a bijection of the set $\mathcal{SD}_{2n+2}$ onto the set of pairs (δ, J) such that $\delta \in \mathcal{D}_{2n+1}$, and J a subset of $\{1, 2, \ldots, 2n+2\}$ such that $F\delta + 1 \le \#J \le L\delta$.

For instance, if $\delta = \binom{042}{315} \in \mathcal{D}_5$, then $F\delta + 1 = 1 \leq \#J \leq 3 = L\delta$. Take $J = \{3\}, \{1,3\}, \{2,3,5\}$ for example, the three signed doubloons $w \in \mathcal{SD}_6$ associated with those three subsets J are the following:

$$\begin{aligned}
\binom{\overline{3}52}{416} &\mapsto \left(\binom{042}{315}, \{3\}\right), \\
\binom{\overline{3}52}{4\overline{1}6} &\mapsto \left(\binom{042}{315}, \{1,3\}\right), \\
\binom{\overline{5}4\overline{2}}{1\overline{3}6} &\mapsto \left(\binom{042}{315}, \{2,3,5\}\right).
\end{aligned}$$

If $(\delta_w, -\operatorname{Neg} w) = (\delta, J)$, then (see (29))

$$(40) \qquad \operatorname{des} w = \operatorname{des} \delta; \quad \operatorname{smaj} w = 2\operatorname{cmaj} \delta + \#J + n - 1.$$

We next make the composition product of the two mappings described in (35) and (39).

Theorem 4.1. *For each pair (i,k) of integers such that $1 \leq k \leq 2n$ and $0 \leq i \leq 2n+1-k$ the mapping*

$$(41) \qquad\qquad w \mapsto (\delta_w - i, -\operatorname{Neg} w)$$

is a bijection of the set $\mathcal{SD}^i_{2n+2,i+k}$ of the signed doubloons w satisfying $F\delta_w = i$, $L\delta_w = i+k$ onto the set of pairs (δ, J) such that $\delta \in \mathcal{D}^0_{2n+1,k}$ and $J \subset [1, 2n+2]$ with $i+1 \leq \#J \leq i+k$. Moreover, if w is interlaced (resp. normalized), so is $\delta_w - i$, and conversely. Finally, if $\delta = \delta_w - i$, then

$$(42) \qquad \operatorname{des} w = \operatorname{des} \delta; \quad \operatorname{smaj} w = 2\operatorname{cmaj} \delta - 2i + \#J + n - 1.$$

Proof. The theorem is a consequence of Properties 3.1 and 3.2 and the properties of the bijection $w \mapsto (\delta_w, -\operatorname{Neg} w)$ given in (40). $\qquad\square$

Identity (34) may be rewritten as

$$\begin{aligned}
E_{2n+2}(q) &= (-1)^{n+1}(1 + q^{2n+2}) \sum_{w \in \mathcal{SD}_{2n+2}} (-1)^{\operatorname{des} w + 1} q^{\operatorname{smaj} w} \\
&= (1 + q^{2n+2}) \sum_{k=1}^{2n} \sum_{i=0}^{2n+1-k} \sum_{w \in \mathcal{SD}^i_{2n+2,i+k}} (-1)^{n+\operatorname{des} w} q^{\operatorname{smaj} w}.
\end{aligned}$$

Let

$$(43) \qquad\qquad P_{n,i,k}(q) := q^{n-1-2i} \sum_{l=i+1}^{i+k} \binom{2n+2}{l} q^l.$$

Using the preceding theorem and Property 3.3 we evaluate the third sum as follows.

$$\sum_{w \in \mathcal{SD}_{2n+2,i+k}^i} (-1)^{n+\operatorname{des} w} q^{\operatorname{smaj} w}$$

$$= \sum_{\delta \in \mathcal{D}_{2n+1,k}^0} \sum_{i+1 \leq \#J \leq i+k} (-1)^{n+\operatorname{des} \delta} q^{2\operatorname{cmaj} \delta - 2i + \#J + n - 1}$$

$$= q^{n-1-2i} \sum_{\delta \in \mathcal{D}_{2n+1,k}^0} (-1)^{n+\operatorname{des} \delta} q^{2\operatorname{cmaj} \delta} \sum_{l=i+1}^{i+k} \binom{2n+2}{l} q^l$$

$$= P_{n,i,k}(q) \sum_{\delta \in \mathcal{D}_{2n+1,k}^0} (-1)^{n+\operatorname{des} \delta} q^{2\operatorname{cmaj} \delta}$$

$$= P_{n,i,k}(q) \sum_{\delta \in \mathcal{I}_{2n+1,k}^0} q^{2\operatorname{cmaj} \delta}$$

$$= (1+q^2) \cdots (1+q^{2n}) P_{n,i,k}(q) \sum_{\delta \in \mathcal{N}_{2n+1,k}^0} q^{2\operatorname{cmaj} \delta}$$

$$(44) \qquad = (1+q^2) \cdots (1+q^{2n}) P_{n,i,k}(q) \, d_{n,k}(q^2),$$

where the last equality follows from [12], Theorem 1.2. By multiplying (44) by $(1+q^{2n+2})$ and summing over all pairs (k,i) such that $1 \leq k \leq 2n$ and $0 \leq i \leq 2n+1-k$ we derive identity (14), keeping in mind that $P_{n,k}(q) = \sum_{0 \leq i \leq 2n+1-k} P_{n,i,k}$.

This achieves the proofs of both Theorems 1.1 and 1.2, except part (d).

Let $\mathcal{SN}_{2n+2,i+k}^i$ be the set of the normalized signed doubloons w satisfying $F \delta_w = i$, $L \delta_w = i + k$. It also follows from Theorem 4.1 that

$$(45) \qquad \sum_{w \in \mathcal{SN}_{2n+2,i+k}^i} q^{\operatorname{smaj} w} = P_{n,i,k}(q) \sum_{\delta \in \mathcal{N}_{2n+1,k}^0} q^{2\operatorname{cmaj} \delta}.$$

From (44) it follows that

$$(46) \qquad \sum_{w \in \mathcal{SD}_{2n+2,i+k}^i} (-1)^{n+\operatorname{des} w} q^{\operatorname{smaj} w} = (1+q^2) \cdots (1+q^{2n}) \sum_{w \in \mathcal{SN}_{2n+2,i+k}^i} q^{\operatorname{smaj} w}.$$

By multiplying (46) by $(1+q^{2n+2})$ and summing over all pairs (k,i) such that $1 \leq k \leq 2n$ and $0 \leq i \leq 2n+1-k$ we derive identity (11).

5. Proof of theorem 1.1 (d)

Recall that for each $w = x_1 x_2 \cdots x_{2n+1} \in B_{2n+1}$ we have used the notations $F w := x_1$ and $L w := x_{2n+1}$. As $B_{2n+1}(t,q) = \sum_{w \in B_{2n+1}} t^{\operatorname{des}_B w} q^{\operatorname{fmaj} w}$, we may write

$$(47) \qquad B_{2n+1}(-q^{-(2n+1)}, q) = \sum_{w \in B_{2n+1}} (-1)^{\operatorname{sgn} w} q^{\operatorname{smaj} w},$$

where

(48) $\operatorname{sgn} w \ := \ (-1)^{\operatorname{des} w + \chi(F\, w < 0)};$

(49) $\operatorname{smaj} w \ := \ 2 \operatorname{maj} w + \operatorname{neg} w - (2n+1)(\operatorname{des} w + \chi(F\, w < 0)),$

as there is no ambiguity to adopt this definition of "smaj" for signed permutations from B_{2n+1}.

For proving the identity $A_{2n}(-q^{-n}, q) = 0$ in [11] we had recourse to the classical properties of the dihedral group acting on $\mathfrak{S}_{2n}$. Actually, the mirror image $\mathbf{r}$ provided the sign-reversing involution that was needed. With the group B_{2n+1} the supplementary descent to be counted, when the first letter is negative, makes it necessary to include another dihedral group involution, as well as a sign change operation.

In this section the elements of B_{2n+1} will be regarded as two-row matrices $w = \binom{|w|}{\epsilon} := \binom{|x_1|\,|x_2|\,\cdots\,|x_{2n+1}|}{\epsilon_1\ \ \epsilon_2\ \ \cdots\ \ \epsilon_{2n+1}}$, where $|w| := |x_1||x_2|\cdots|x_{2n+1}|$ becomes an *ordinary* permutation and $\epsilon := \epsilon_1 \epsilon_2 \cdots \epsilon_{2n+1}$ is the sign word defined by $\epsilon_i := 1$ or -1, depending on whether x_i is positive or negative ($1 \le i \le 2n+1$).

Three operations $\mathbf{r}$, $\mathbf{c}$, $\mathbf{s}$ are now introduced and further extended to all of B_{2n+1}: first, the *mirror image*

$$\mathbf{r} : y_1 y_2 \cdots y_{2n+1} \mapsto y_{2n+1} \cdots y_2 y_1,$$

defined for every *arbitrary word*; second, the *complement to* $(2n+2)$, defined for each *permutation* from $\mathfrak{S}_{2n+1}$, by

$$\mathbf{c} : y_1 y_2 \cdots y_{2n+1} \mapsto (2n+2-y_1)(2n+2-y_2)\cdots(2n+2-y_{2n+1});$$

third, the *sign change* $\mathbf{s}$, defined for each *binary word*, such as $\epsilon = \epsilon_1 \epsilon_2 \cdots \epsilon_{2n+1}$, whose letters are equal to $+1$ or -1, by

$$\mathbf{s} : \epsilon_1 \epsilon_2 \cdots \epsilon_{2n+1} \mapsto \overline{\epsilon}_1 \overline{\epsilon}_2 \cdots \overline{\epsilon}_{2n+1}.$$

We use the same symbols for their extensions to B_{2n+1}:

(50) $\mathbf{r} : \binom{|w|}{\epsilon} = \binom{|x_1|\,|x_2|\,\cdots\,|x_{2n+1}|}{\epsilon_1\ \ \epsilon_2\ \ \cdots\ \ \epsilon_{2n+1}} \mapsto \binom{\mathbf{r}\,|w|}{\mathbf{r}\,\epsilon} = \binom{|x_{2n+1}|\,\cdots\,|x_2|\,|x_1|}{\epsilon_{2n+1}\ \ \cdots\ \ \epsilon_2\ \ \epsilon_1};$

(51) $\mathbf{c} : \binom{|w|}{\epsilon} = \binom{|x_1|\,\cdots\,|x_{2n+1}|}{\epsilon_1\ \ \cdots\ \ \epsilon_{2n+1}} \mapsto \binom{\mathbf{c}\,|w|}{\epsilon} = \binom{(2n+2-|x_1|)\cdots(2n+2-|x_{2n+1}|)}{\epsilon_1\ \ \ \ \cdots\ \ \ \ \epsilon_{2n+1}};$

(52) $\mathbf{s} : \binom{|w|}{\epsilon} = \binom{|x_1|\,|x_2|\,\cdots\,|x_{2n+1}|}{\epsilon_1\ \ \epsilon_2\ \ \cdots\ \ \epsilon_{2n+1}} \mapsto \binom{|w|}{\mathbf{s}\,\epsilon} = \binom{|x_1|\,|x_2|\,\cdots\,|x_{2n+1}|}{\overline{\epsilon}_1\ \ \overline{\epsilon}_2\ \ \cdots\ \ \overline{\epsilon}_{2n+1}}.$

Note that the three involutions $\mathbf{r}$, $\mathbf{c}$, $\mathbf{s}$, defined on B_{2n+1} by (50), (51) and (52) *commute*. The composition product $\mathbf{b} := \mathbf{c}\,\mathbf{s}\,\mathbf{r}$ can also be written as

(53) $\mathbf{b} : \binom{|w|}{\epsilon} = \binom{|x_1|\,\cdots\,|x_{2n+1}|}{\epsilon_1\ \ \cdots\ \ \epsilon_{2n+1}} \mapsto \binom{\mathbf{c}\,\mathbf{r}\,|w|}{\mathbf{r}\,\mathbf{s}\,\epsilon} = \binom{(n+2-|x_{2n+1}|)\cdots(n+2-|x_1|)}{\overline{\epsilon}_{2n+1}\ \ \ \ \cdots\ \ \ \ \overline{\epsilon}_1}.$

Theorem 5.1. *The composition product* $\mathbf{b}$ *defined in* (53) *is a sign-reversing involution of* B_{2n+1}, *i.e.*,

(54) $(\operatorname{sgn}, \operatorname{smaj}) \binom{|w|}{\epsilon} = (-\operatorname{sgn}, \operatorname{smaj})\mathbf{b}\binom{|w|}{\epsilon}.$

The proof of the theorem is based on the next three lemmas. The first two ones being easy to verify are given without proofs.

Lemma 5.2. *For each $w = \binom{|w|}{\epsilon} \in B_{2n+1}$ we have*

$$(55) \qquad \operatorname{sgn} \mathbf{r}\, w \;\; = \;\; \operatorname{sgn} w \cdot (-1)^{\chi(L\, w < 0) + \chi(F\, w < 0)};$$
$$(56) \qquad \operatorname{smaj} \mathbf{r}\, w \;\; = \;\; \operatorname{smaj} w + (2n+1)\big(\chi(F\, w < 0) - \chi(L\, w < 0)\big).$$

Lemma 5.3. *For each $w = \binom{|w|}{\epsilon} \in B_{2n+1}$ we have:*

$$(57) \qquad \qquad \operatorname{sgn} \mathbf{s}\, w \;\; = -\operatorname{sgn} w;$$
$$(58) \qquad \qquad \operatorname{smaj} \mathbf{s}\, w \;\; = -\operatorname{smaj} w.$$

The third lemma requires a careful analysis.

Lemma 5.4. *For each $w = \binom{|w|}{\epsilon} = \binom{|x_1|\;|x_2|\;\cdots\;|x_{2n+1}|}{\epsilon_1\;\;\epsilon_2\;\;\cdots\;\;\epsilon_{2n+1}} \in B_{2n+1}$ we have:*

$$(59) \qquad \operatorname{sgn} \mathbf{c}\, w \qquad\qquad = (-1)^{\chi(F\, w < 0) - \chi(L\, w < 0)} \operatorname{sgn} w;$$
$$(60) \qquad \operatorname{smaj} \mathbf{c}\, w \;\; = -\operatorname{smaj} w - (2n+1)\big(\chi(F\, w < 0) - \chi(L\, w < 0)\big).$$

Proof. If $\binom{|x_i|}{\epsilon_i} > \binom{|x_{i+1}|}{\epsilon_{i+1}}$ (resp. $\binom{|x_i|}{\epsilon_i} < \binom{|x_{i+1}|}{\epsilon_{i+1}}$) say that i is an *interior* descent (resp. rise), if $|x_i| > |x_{i+1}|$ (resp. $|x_i| < |x_{i+1}|$) *and* $\epsilon_i = \epsilon_{i+1}$. Denote the set of all descents (resp. rises) of w by $\operatorname{DES} w$ (resp. $\operatorname{RISE} w$), the set of all *interior* descents (resp. rises) being designated by $\operatorname{DES}^i w$ (resp. $\operatorname{RISE}^i w$), so that $\operatorname{DES} w = \operatorname{DES}^i w + \operatorname{DES} \epsilon$ and $\operatorname{RISE} w = \operatorname{RISE}^i w + \operatorname{RISE} \epsilon$.

First, $\operatorname{DES}^i w = \operatorname{RISE}^i \mathbf{c}\, w$ and $\operatorname{RISE}^i w = \operatorname{DES}^i \mathbf{c}\, w$. Hence,

$$
\begin{aligned}
\operatorname{des} w + \operatorname{des} \mathbf{c}\, w \;\; &= \;\; (\#\operatorname{DES} \epsilon + \#\operatorname{DES}^i w) + (\#\operatorname{DES} \epsilon + \#\operatorname{DES}^i \mathbf{c}\, w) \\
&= \;\; (\#\operatorname{DES}^i w + \#\operatorname{DES} \epsilon) + (\#\operatorname{RISE}^i w + \#\operatorname{RISE} \epsilon) \\
&\qquad\qquad\qquad\qquad\qquad + (\#\operatorname{DES} \epsilon - \#\operatorname{RISE} \epsilon) \\
&= \;\; 2n + (\#\operatorname{DES} \epsilon - \#\operatorname{RISE} \epsilon) \\
&:= \;\; 2n + \operatorname{drise} \epsilon.
\end{aligned}
$$

In the same way, let $\operatorname{DRISE} \epsilon := \sum_i i\,\big(\chi(i \in \operatorname{DES} \epsilon) - \chi(i \in \operatorname{RISE} \epsilon)\big)$. Then,

$$
\begin{aligned}
\operatorname{maj} w + \operatorname{maj} \mathbf{c}\, w \;\; &= \;\; \sum_i \big(i\,\chi(i \in \operatorname{DES} \epsilon) + i\,\chi(i \in \operatorname{DES}^i w)\big) \\
&\qquad + \sum_i \big(i\,\chi(i \in \operatorname{DES} \epsilon) + i\,\chi(i \in \operatorname{DES}^i \mathbf{c}\, w)\big) \\
&= \;\; \sum_i \big(i\,\chi(i \in \operatorname{DES} w) + i\,\chi(i \in \operatorname{RISE} w)\big) \\
&\qquad + \sum_i i\,\big(\chi(i \in \operatorname{DES} \epsilon) - \chi(i \in \operatorname{RISE} \epsilon)\big) \\
&= \;\; (1 + 2 + \cdots + 2n) + \operatorname{DRISE} \epsilon \\
&= \;\; n(2n+1) + \operatorname{DRISE} \epsilon.
\end{aligned}
$$

Let $d_1 < d_2 < \cdots$ (resp. $r_1 < r_2 < \cdots$) denote the sequence of the descents (resp. rises) of ϵ, when reading the word ϵ from left to right. Four cases are now considered.

(a) $\epsilon_1 = \epsilon_{2n+1} = -1$; the rises and descents alternate in such a way that $1 \le r_1 < d_1 < r_2 < d_2 < \cdots < r_k < d_k \le 2n$ and $k \ge 0$. Hence, drise $\epsilon = 0$ and DRISE $\epsilon = \sum\limits_{i=1}^{k} (d_i - r_i) = \operatorname{pos} w$.

(b) $\epsilon_1 = +1$, $\epsilon_{2n+1} = -1$; the alternation becomes: $1 \le d_1 < r_1 < d_2 < r_2 < \cdots < d_k < r_k < d_{k+1} \le 2n$ ($k \ge 0$). In this case, drise $\epsilon = 1$ and DRISE $\epsilon = \operatorname{pos} w$.

(c) $\epsilon_1 = \epsilon_{2n+1} = 1$; the sequence is then: $1 \le d_1 < r_1 < d_2 < r_2 < \cdots < d_k < r_k \le 2n$ ($k \ge 0$). Hence, drise $\epsilon = 0$ and DRISE $\epsilon = -\operatorname{neg} w$.

(d) $\epsilon_1 = -1$, $\epsilon_{2n+1} = 1$; then $1 \le r_1 < d_2 < r_2 < \cdots < r_k < d_k < r_{k+1} \le 2n$ ($k \ge 0$). Hence, drise $\epsilon = -1$ and DRISE $\epsilon = -\operatorname{neg} w$.

Thus, $\operatorname{sgn} w + \operatorname{sgn} \mathbf{c}\, w = \operatorname{des} w + \chi(\epsilon_1 < 0) + \operatorname{des} \mathbf{c}\, w + \chi(\epsilon_1 < 0) \equiv \operatorname{drise} \epsilon$ (mod 2), which is 0 when ϵ_1 and ϵ_{2n+1} are of the same sign (cases (a) and (c)), equal to 1 when $\epsilon_1 = 1$, $\epsilon_{2n+1} = -1$ (case (b)) and -1 when $\epsilon_1 = -1$ and $\epsilon_{2n+1} = +1$ (case (d)). Gathering in a common formula: $\operatorname{sgn} w + \operatorname{sgn} \mathbf{c}\, w \equiv \chi(\epsilon_1 = 1) - \chi(\epsilon_{2n+1} = 1)$. This implies (59).

Finally,

$$
\begin{aligned}
\operatorname{smaj} \mathbf{c}\, w + \operatorname{smaj} w \;=\;& 2(\operatorname{maj} \mathbf{c}\, w + \operatorname{maj} w) + \operatorname{neg} \mathbf{c}\, w + \operatorname{neg} w \\
& - (2n+1)(\operatorname{des} \mathbf{c}\, w + \operatorname{des} w + 2\chi(\epsilon_1 < 0)) \\
=\;& 2n(2n+1) + 2\operatorname{DRISE} \epsilon + 2\operatorname{neg} w \\
& - (2n+1)(2n + \operatorname{drise} \epsilon + 2\chi(\epsilon_1 < 0)) \\
=\;& 2\operatorname{DRISE} \epsilon + 2\operatorname{neg} w \\
& - (2n+1)(\operatorname{drise} \epsilon + 2\chi(\epsilon_1 < 0)) \\
=\;& \begin{cases}
2\operatorname{pos} w + 2\operatorname{neg} w - (2n+1)2 = 0, & \text{in case } (a); \\
2\operatorname{pos} w + 2\operatorname{neg} w - (2n+1) = 2n+1, & \text{in case } (b); \\
-2\operatorname{neg} w + 2\operatorname{neg} w - (2n+1)0 = 0, & \text{in case } (c); \\
-2\operatorname{neg} w + 2\operatorname{neg} w - (2n+1) = -(2n+1), & \text{in case } (d).
\end{cases}
\end{aligned}
$$

Altogether, $\operatorname{smaj} \mathbf{c}\, w = -\operatorname{smaj} w - (2n+1)\big(\chi(\epsilon_1 = -1) - \chi(\epsilon_{2n+1} = -1)\big)$. This proves (60) and also Lemma 5.4. $\qquad\square$

Proof of Theorem 5.1. Let $w \in B_{2n+1}$. By the previous three lemmas

$$
\begin{aligned}
\operatorname{sgn} \mathbf{r}\, \mathbf{c}\, w \;&=\; \operatorname{sgn} \mathbf{c}\, w \cdot (-1)^{\chi(L\, \mathbf{c}\, w < 0) + \chi(F\, \mathbf{c}\, w < 0)} \\
&=\; \operatorname{sgn} w (-1)^{\chi(F\, w < 0) - \chi(L\, w < 0)} (-1)^{\chi(L\, w < 0) + \chi(F\, w < 0)} \\
&=\; \operatorname{sgn} w; \\
\operatorname{smaj} \mathbf{r}\, \mathbf{c}\, w \;&=\; \operatorname{smaj} \mathbf{c}\, w + (2n+1)(\chi(F\, \mathbf{c}\, w < 0) - \chi(L\, \mathbf{c}\, w < 0)) \\
&=\; \operatorname{smaj} \mathbf{c}\, w + (2n+1)(\chi(F\, w < 0) - \chi(L\, w < 0)) \\
&=\; -\operatorname{smaj} w; \\
\operatorname{sgn} \mathbf{s}\, \mathbf{r}\, \mathbf{c}\, w \;&=\; -\operatorname{sgn} \mathbf{r}\, \mathbf{c}\, w = -\operatorname{sgn} w; \\
\operatorname{smaj} \mathbf{s}\, \mathbf{r}\, \mathbf{c}\, w \;&=\; -\operatorname{smaj} \mathbf{r}\, \mathbf{c}\, w = \operatorname{smaj} w. \qquad\square
\end{aligned}
$$

6. WHICH DESCENT FOR THE HYPEROCTAHEDRAL GROUP?

The purpose of this Section is to work out a *global* expression for the generating polynomial for B_n by the five-term statistic (neg, pos, Ξ, des, fmaj), where Ξw is equal to 1 or 0, depending on whether the first letter of w is negative or positive, and to derive the specializations when the pair (Ξ, des) is replaced either by "des_B," or by "fdes," defined in (21) and (22). Our main result is the following.

Theorem 6.1. *Let*

$$(61) \qquad B_n(X, Y, Z; t, q) = \sum_{w \in B_n} X^{\mathrm{neg}\, w} Y^{\mathrm{pos}\, w} Z^{\chi(x_1 < 0)} t^{\mathrm{des}\, w} q^{\mathrm{fmaj}\, w}.$$

Then,

$$(62) \quad \frac{B_n(X, Y, Z; t, q)}{(t; q^2)_{n+1}} \;\; = \;\; \frac{t - Z}{t - 1} \sum_{s \geq 0} t^s \big((qX + Y)[s+1]_{q^2} \big)^n$$

$$+ \quad \frac{Z - 1}{t - 1} \sum_{s \geq 0} t^s \big((qX + Y)[s+1]_{q^2} - X q^{2s+1} \big)^n.$$

When $q = 1$, write $B_n(X, Y, Z; t) := B_n(X, Y, Z; t, 1)$. The exponential generating function for the latter polynomials can be derived in the following form.

Theorem 6.2. *The following identity holds:*

$$(63) \qquad \sum_{n \geq 0} \frac{u^n}{n!} B_n(X, Y, Z; t) = \frac{Z - t + (1 - Z) \exp(uX(t-1))}{-t + \exp(u(X+Y)(t-1))}.$$

Proof of Theorem 6.1. Let $w = x_1 x_2 \cdots x_n$ be a signed permutation from B_n and ϕ be the unique increasing bijection of the set $\{x_1, x_2, \ldots, x_n\}$ onto the interval $[n] := \{1, 2, \ldots, n\}$. The word

$$\sigma = \sigma(1)\sigma(2) \cdots \sigma(n) := \phi(x_1)\phi(x_2) \cdots \phi(x_n)$$

is then an (ordinary) permutation from $\mathfrak{S}_n$ and the map $w \mapsto (\mathrm{Neg}\, w, \sigma)$ a bijection of B_n onto the Cartesian product $2^{[n]} \times \mathfrak{S}_n$ having the following properties:

$$\chi(x_1 < 0) = \chi(\sigma(1) \leq \mathrm{neg}\, w); \quad \mathrm{des}\, w = \mathrm{des}\, \sigma; \quad \mathrm{fmaj}\, w = \mathrm{fmaj}\, \sigma.$$

For convenience, introduce the polynomial

$$A_n^k(Z; t, q) := \sum_\sigma Z^{\chi(\sigma(1) \leq k)} t^{\mathrm{des}\, \sigma} q^{\mathrm{maj}\, \sigma} \quad (\sigma = \sigma(1) \cdots \sigma(n) \in \mathfrak{S}_n)$$

and express $B_n(X, Y, Z; t, q)$ in terms of the latter polynomials, to get:

$$
\begin{aligned}
B_n(X, Y, Z; t, q) &= \sum_{k=0}^{n} \sum_{|E|=k} \sum_{\text{Neg } w = E} (qX)^{\text{neg } w} Y^{\text{pos } w} Z^{\chi(x_1 < 0)} t^{\text{des } w} q^{2 \text{ maj } w} \\
&= \sum_{k=0}^{n} (qX)^k Y^{n-k} \sum_{|E|=k} \sum_{(E, \sigma)} Z^{\chi(\sigma(1) \leq k)} t^{\text{des } \sigma} q^{2 \text{ maj } \sigma} \\
&= \sum_{k=0}^{n} \binom{n}{k} (qX)^k Y^{n-k} A_n^k(Z; t, q^2).
\end{aligned}
$$

Next, with each permutation $\sigma = k \, \sigma(2) \cdots \sigma(n)$ starting with k associate the permutation $\sigma' = \sigma'(1) \cdots \sigma'(n-1) := \psi(\sigma(2)) \cdots \psi(\sigma(n))$, where ψ is the unique increasing bijection of $[n] \setminus \{k\}$ onto $[n-1]$. If $\sigma(2) \leq k - 1$, then $\text{des } \sigma = \text{des } \sigma' + 1$, while $\text{maj } \sigma = \text{maj } \sigma' + \text{des } \sigma' + 1$ and $\sigma'(1) \leq k - 1$. If $\sigma(2) \geq k + 1$, then $\text{des } \sigma = \text{des } \sigma'$, while $\text{maj } \sigma = \text{maj } \sigma' + \text{des } \sigma'$ and $\sigma'(1) \geq k$. Hence,

$$
\begin{aligned}
\sum_{\sigma(1)=k, \sigma(2) \leq k-1} Z^{\chi(\sigma(1) \leq k)} t^{\text{des } \sigma} q^{\text{maj } \sigma} &= Z \sum_{\sigma'(1) \leq k-1} t^{\text{des } \sigma' + 1} q^{\text{maj } \sigma' + \text{des } \sigma' + 1} \\
&= Z \sum_{\sigma'(1) \leq k-1} (tq)^{\chi(\sigma(1) \leq k-1)} (tq)^{\text{des } \sigma'} q^{\text{maj } \sigma'};
\end{aligned}
$$

while

$$
\begin{aligned}
\sum_{\substack{\sigma(1)=k, \\ \sigma(2) \geq k+1}} Z^{\chi(\sigma(1) \leq k)} t^{\text{des } \sigma} q^{\text{maj } \sigma} &= Z \sum_{\sigma'(1) \geq k} t^{\text{des } \sigma'} q^{\text{maj } \sigma' + \text{des } \sigma'} \\
&= Z \sum_{\sigma'(1) \geq k} (tq)^{\chi(\sigma(1) \leq k-1)} (tq)^{\text{des } \sigma'} q^{\text{maj } \sigma'}.
\end{aligned}
$$

Altogether

$$
\sum_{\sigma(1)=k} Z^{\chi(\sigma(1) \leq k)} t^{\text{des } \sigma} q^{\text{maj } \sigma} = Z \, A_{n-1}^{k-1}(tq; tq, q).
$$

In the same manner,

$$
\sum_{\sigma(1)=k} Z^{\chi(\sigma(1) \leq k-1)} t^{\text{des } \sigma} q^{\text{maj } \sigma} = A_{n-1}^{k-1}(tq; tq, q).
$$

Consequently, we have the relation:

(64) $\qquad A_n^k(Z; t, q) = A_n^{k-1}(Z; t, q) + (Z-1) A_{n-1}^{k-1}(tq; tq, q).$

By iteration we are led to:

(65) $\qquad A_n^k(Z; t, q) = A_n^0(Z; t, q)$

$$
+ \frac{Z-1}{t-1} \sum_{j=1}^{k} \binom{k}{j} (t-1)(tq-1) \cdots (tq^{j-1}-1) A_{n-j}^0(tq^j; tq^j, q).
$$

But, the variable Z vanishes from $A_n^k(Z; t, q)$ when $k = 0$ and then $A_n^0(Z; t, q) = A_n(t, q)$, which is the Carlitz q-analog of the Eulerian polynomial ([3, 4]) appearing in (1). Hence,

$$
\begin{aligned}
A_n^k(Z; t, q) &= A_n(t, q) + \frac{Z-1}{t-1} \sum_{j=1}^{k} \binom{k}{j} (-1)^j (t; q)_j A_{n-j}(tq^j, q) \\
&= \frac{t-Z}{t-1} A_n(t, q) + \frac{Z-1}{t-1} \sum_{j=0}^{k} \binom{k}{j} (-1)^j (t; q)_j A_{n-j}(tq^j, q).
\end{aligned}
$$

The next step is to report this new expression of $A_n^k(Z; t, q)$ into the polynomial $B_n(X, Y, Z; t, q)$. We get:

$$
\begin{aligned}
B_n(X, Y, Z; t, q) &= \sum_{k=0}^{n} \binom{n}{k} (qX)^k Y^{n-k} A_n^k(Z; t, q^2) \\
&= \sum_{k=0}^{n} \binom{n}{k} (qX)^k Y^{n-k} \Big(\frac{t-Z}{t-1} A_n(t, q^2) \\
&\quad + \frac{Z-1}{t-1} \sum_{j=0}^{k} \binom{k}{j} (-1)^j (t; q^2)_j A_{n-j}(tq^{2j}, q^2) \Big) \\
&= \frac{t-Z}{t-1} (qX+Y)^n A_n(t, q^2) \\
&\quad + \frac{Z-1}{t-1} \sum_{\substack{j,l,m \geq 0 \\ j+l+m=n}} \frac{n!}{j!\, l!\, m!} (qX)^{j+l} Y^m (-1)^j (t; q^2)_j A_{l+m}(tq^{2j}, q^2),
\end{aligned}
$$

where $k = j + l$.

Next, with $r = l + m$ we get

$$
\begin{aligned}
\frac{B_n(X, Y, Z; t, q)}{(t; q^2)_{n+1}} &= \frac{t-Z}{t-1} (qX+Y)^n \frac{A_n(t, q^2)}{(t; q^2)_{n+1}} \\
&\quad + \frac{Z-1}{t-1} \sum_{j+r=n} \frac{n!}{r!\, j!} (-qX)^j \frac{A_r(tq^{2j}, q^2)}{(tq^{2j}; q^2)_{r+1}} \sum_{l+m=r} \frac{r!}{l!\, m!} (qX)^l Y^m \\
&= \frac{t-Z}{t-1} (qX+Y)^n \frac{A_n(t, q^2)}{(t; q^2)_{n+1}} \\
&\quad + \frac{Z-1}{t-1} \sum_{j+r=n} \frac{n!}{r!\, j!} (-qX)^j \frac{A_r(tq^{2j}, q^2)}{(tq^{2j}; q^2)_{r+1}} (qX+Y)^r.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\sum_{n \geq 0} \frac{B_n(X, Y, Z; t, q)}{(t; q^2)_{n+1}} \frac{u^n}{n!} &= \frac{t-Z}{t-1} \sum_{n \geq 0} \frac{A_n(t, q^2)}{(t; q^2)_{n+1}} \frac{((qX+Y)u)^n}{n!} \\
&\quad + \frac{Z-1}{t-1} \sum_{j \geq 0} \frac{(-qXu)^j}{j!} \sum_{r \geq 0} \frac{A_r(tq^{2j}, q^2)}{(tq^{2j}, q^2)_{r+1}} \frac{((qX+Y)u)^r}{r!}.
\end{aligned}
$$

Now, make use of the classical identity on the Carlitz q-Eulerian polynomials

$$\sum_{n\geq 0}\frac{u^n}{n!}\frac{A_n(t,q)}{(t;q)_{n+1}}=\sum_{s\geq 0}t^s\exp(u[s+1]_q),$$

to obtain

$$(66)\qquad \sum_{n\geq 0}\frac{B_n(X,Y,Z;t,q)}{(t;q^2)_{n+1}}\frac{u^n}{n!}=\frac{t-Z}{t-1}\sum_{s\geq 0}t^s\exp((qX+Y)u\,[s+1]_{q^2})$$

$$+\frac{Z-1}{t-1}\sum_{j\geq 0}\frac{(-qXu)^j}{j!}\sum_{s\geq 0}(tq^{2j})^s\exp((qX+Y)u\,[s+1]_{q^2}).$$

There remains to extract the coefficient of u^n on both sides. This leads to:

$$(67)\qquad \frac{B_n(X,Y,Z;t,q)}{n!\,(t;q^2)_{n+1}}=\frac{t-Z}{t-1}\sum_{s\geq 0}t^s\left((qX+Y)\,[s+1]_{q^2}\right)^n+\frac{Z-1}{t-1}C,$$

where C is the coefficient of u^n in

$$\sum_{j\geq 0}\frac{(-qXu)^j}{j!}\sum_{s\geq 0}(tq^{2j})^s\sum_{m\geq 0}\frac{((qX+Y)u\,[s+1]_{q^2})^m}{m!},$$

that is,

$$\begin{aligned}C &= \sum_{s\geq 0}t^s\sum_{j\geq 0}\frac{(-qX)^j}{j!}q^{2js}\frac{((qX+Y)\,[s+1]_{q^2})^{n-j}}{(n-j)!}\\ &= \frac{1}{n!}\sum_{s\geq 0}t^s\sum_{j\geq 0}\binom{n}{j}(-qXq^{2s})^j((qX+Y)\,[s+1]_{q^2})^{n-j}\\ &= \frac{1}{n!}\sum_{s\geq 0}t^s\left((qX+Y)[s+1]_{q^2}-Xq^{2s+1}\right)^n\end{aligned}$$

Reporting the last expression in (67) yields identity (62). $\square$

Proof of Theorem 6.2. When $q=1$ in (62), we obtain

$$(68)\qquad \frac{B_n(X,Y,Z;t)}{(1-t)^{n+1}}=\frac{t-Z}{t-1}\sum_{s\geq 0}t^s\left((X+Y)(s+1)\right)^n$$

$$+\frac{Z-1}{t-1}\sum_{s\geq 0}t^s\left((X+Y)(s+1)-X\right)^n.$$

Hence,

$$\sum_{n \geq 0} \frac{u^n}{(1-t)^n} B_n(X, Y, Z; t)$$

$$= (Z - t) \sum_{s \geq 0} t^s \sum_{n \geq 0} \frac{\big(u(X+Y)(s+1)\big)^n}{n!}$$

$$+ (1 - Z) \sum_{s \geq 0} t^s \sum_{n \geq 0} \frac{\big(u(X+Y)(s+1) - X\big)^n}{n!}$$

$$= (Z - t) \sum_{s \geq 0} t^s \exp(u(X+Y)(s+1))$$

$$+ (1 - Z) \sum_{s \geq 0} t^s \sum_{n \geq 0} \exp(u(X+Y)(s+1) - X)$$

$$= \big((Z - t)(\exp(u(X+Y)) + (1 - Z) \exp(uY)\big) \sum_{s \geq 0} t^s \exp(u(X+Y)s)$$

$$= \frac{(Z - t) \exp(u(X+Y)(1 - Z)) \exp(uY)}{1 - t \exp(u(X+Y))},$$

which is identity (63) by replacing u by $u(1-t)$. $\qquad\square$

Next, we derive specializations of Theorems 6.1 and 6.2 when the pair (Ξ, des) is replaced by "des_B" and "fdes" (see (21) and (22)). We get:

$$(69) \qquad \sum_{w \in B_n} X^{\mathrm{neg}\, w} Y^{\mathrm{pos}\, w} t^{\mathrm{des}_B\, w} q^{\mathrm{fmaj}\, w} = B_n(X, Y, t; t, q);$$

$$(70) \qquad \sum_{w \in B_n} X^{\mathrm{neg}\, w} Y^{\mathrm{pos}\, w} t^{\mathrm{fdes}\, w} q^{\mathrm{fmaj}\, w} = B_n(X, Y, t; t^2, q).$$

Also, note that $B_n(0, 1, 1; t, q)$ is the Carlitz q-Eulerian polynomial $A_n(t, q)$.

First,

$$(71) \qquad \frac{B_n(X, Y, t; t, q)}{(t; q^2)_{n+1}} = \sum_{s \geq 0} t^s \big((qX + Y)[s+1]_{q^2} - Xq^{2s+1}\big)^n;$$

$$(72) \qquad \frac{B_n(1, 1, t; t, q)}{(t; q^2)_{n+1}} = \sum_{w \in B_n} t^{\mathrm{des}_B\, w} q^{\mathrm{fmaj}\, w} = \sum_{s \geq 0} t^s \big([2s+1]_q\big)^n.$$

Second,

$$\frac{B_n(X, Y, t; t^2, q)}{(t^2; q^2)_{n+1}} = \frac{t^2 - t}{t - 1} \sum_{s \geq 0} t^{2s} \big((qX + Y)\, [s+1]_{q^2}\big)^n$$

$$+ \frac{t - 1}{t^2 - 1} \sum_{s \geq 0} t^{2s} \big((qX + Y)[s+1]_{q^2} - Xq^{2s+1}\big)^n,$$

so that

$$(73) \quad \frac{(1+t)B_n(X,Y,t;t^2,q)}{(t^2;q^2)_{n+1}} \quad = \quad \sum_{s\geq 0} t^{2s+1}\big((qX+Y)\,[s+1]_{q^2}\big)^n$$
$$+ \sum_{s\geq 0} t^{2s}\big((qX+Y)[s]_{q^2}+Yq^{2s}\big)^n.$$

In particular,

$$(74) \quad \frac{(1+t)B_n(1,1,t;t,q)}{(t^2;q^2)_{n+1}} = \sum_{w\in B_n} t^{\mathrm{fdes}\,w} q^{\mathrm{fmaj}\,w} = \sum_{s\geq 0} t^s\big([s+1]_q\big)^n.$$

The specializations of (72) and (74) for $q = 1$ are banal and not reproduced. However, it is worth writing the exponential generating functions for the polynomials $B_n(1,1,t;t)$ and $B_n(1,1,t;t^2)$ directly obtained from (63):

$$(75) \quad \sum_{n\geq 0} \frac{u^n}{n!}\,B_n(1,1,t;t) \quad = \quad \sum_{n\geq 0} \frac{u^n}{n!}\sum_{w\in B_n} t^{\mathrm{des}_B\,w} = \frac{(1-t)\exp(u(t-1))}{-t+\exp(2u(t-1))};$$
$$\sum_{n\geq 0} \frac{u^n}{n!}\,B_n(1,1,t;t^2) \quad = \quad \frac{(1-t)(t+\exp(u(t^2-1)))}{-t^2+\exp(2u(t^2-1))};$$

so that

$$(76) \qquad \sum_{n\geq 0} \frac{u^n}{n!}\,B_n(1,1,t;t^2) = \sum_{n\geq 0} \frac{u^n}{n!}\sum_{w\in B_n} t^{\mathrm{fdes}\,w} = \frac{1-t}{-t+\exp(u(t^2-1))}.$$

The statistics "fdes" and "fmaj" were introduced by Adin and Roichman [2]. Identity (74) with their equivalent adaptations were derived by Brenti et al. [1], Haglund et al. [15] and reproved by the authors ([9, 10]) as specializations of identities involving several-variable statistics. Note that (76) implies that $\sum_{w\in B_n}(-1)^{\mathrm{fdes}\,w}$ is null for every $n \geq 1$. Accordingly, the statistic "fdes" would have been a wrong choice for obtaining a q-extension!

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