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Finite Difference Calculus for Alternating Permutations

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Abstract. The finite difference equation system introduced by Christiane Poupard in the study of tangent trees is reinterpreted in the alternating permutation environment. It makes it possible to make a joint study of both tangent and secant trees and calculate the generating polynomial for alternating permutations by a new statistic, referred to as being the greater neighbor of the maximum.

1. Introduction

Let $f = (f_n(k))$ $(n \ge 1,\ 1 \le k \le 2n-1)$ be a family of rational numbers, displayed in a triangular array of the form

$$
\text{(1.1)} \qquad f = \begin{array}{ccccccccc}
 & & & & f_1(1) & & & & \\
 & & & f_2(1) & f_2(2) & f_2(3) & & & \\
 & & f_3(1) & f_3(2) & f_3(3) & f_3(4) & f_3(5) & & \\
 & f_4(1) & f_4(2) & f_4(3) & f_4(4) & f_4(5) & f_4(6) & f_4(7) & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}
$$

and consider the finite difference equation system

$$\text{(1.2)} \qquad \Delta^2 f_n(k) + 4\, f_{n-1}(k) = 0 \quad (n \ge 2,\ 1 \le k \le 2n-3),$$

where Δ stands for the classical finite difference operator (see, e.g., [Jo39])

$$\text{(1.3)} \qquad \Delta f_n(k) := f_n(k+1) - f_n(k),$$

so that

$$\text{(1.4)} \qquad \Delta^2 f_n(k) = f_n(k+2) - 2f_n(k+1) + f_n(k).$$

If at each step $n \ge 2$ the two entries $f_n(1)$ and $f_n(2)$ are given explicit values, the whole system (1.2) has a *unique* solution, as the equation $\Delta^2 f_n(1) + 4\, f_{n-1}(1) = 0$ yields the value of $f_n(3)$, then $\Delta^2 f_n(2) + 4\, f_{n-1}(2) = 0$ the value of $f_n(4)$, etc.

The same conclusion holds if the two bordered diagonals

$$(f_1(1), f_2(1), f_3(1), f_4(1), \ldots, f_n(1), \ldots),$$
$$(f_1(1), f_2(3), f_3(5), f_4(7), \ldots, f_n(2n-1), \ldots)$$

are taken as initial values. To see this we first note that the equation $f_2(1)-2f_2(2)+f_2(3)+4f_1(1)=0$ determines $f_2(2)$ uniquely. Assuming that the triangle $(f_{n'}(m))$ $(1\le m\le 2n'-1,\ n'\le n)$ has been determined, the system $\Delta^2 f_{n+1}(m)+4\,f_n(m)=0$ $(1\le m\le 2n-1)$ consists of $(2n-1)$ linear equations with $(2n-1)$ unknowns, namely, $f_{n+1}(2)$, $f_{n+1}(3)$, … , $f_{n+1}(2n)$, the underlying matrix being trigonal of the form

$$F_{n+1} := \begin{pmatrix} -2 & 1 & 0 & 0\cdots 0 & 0 & 0\\ 1 & -2 & 1 & 0\cdots 0 & 0 & 0\\ 0 & 1 & -2 & 1\cdots 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots\ddots\vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0\cdots 1 & -2 & 1\\ 0 & 0 & 0 & 0\cdots 0 & 1 & -2 \end{pmatrix}.$$

As $\det F_{n+1}=-2n$ $(n\ge 1)$, the system has a unique solution.

The purpose of this paper is to solve (1.2) in four cases, when the sets of initial values called `[tan1]`, `[tan2]`, `[sec1]`, `[sec2]` are the following:

`[tan1]` $f_1(1)=1$; $f_n(1)=0$ and $f_n(2)=2\sum\limits_k f_{n-1}(k)$ for $n\ge 2$;

`[tan2]` $f_1(1)=1$; $f_n(1)=f_n(2n-1)=0$ for $n\ge 2$;

`[sec1]` $f_1(1)=1$; $f_n(1)=\sum\limits_k f_{n-1}(k)$ and $f_n(2)=3\sum\limits_k f_{n-1}(k)$ for $n\ge 2$;

`[sec2]` $f_1(1)=1$; $f_n(1)=f_n(2n-1)=\sum\limits_k f_{n-1}(k)$ for $n\ge 2$;

What is meant by *solving* is to see whether the integral values found for the $f_n(k)$'s have an interesting combinatorial interpretation and whether their generating function relates to some classical special function. It will be further proved (see Theorem 1.5) that both initial values `[tan1]` and `[tan2]` (resp. `[sec1]` and `[sec2]`) in fact lead to the same solution of the system. To avoid any confusion the solutions of (1.2) will be denoted by $(g_n(k))$ (resp. $(h_n(k))$) when using `[tan1]` (resp. `[sec1]`). The first numerical values of those solutions are displayed in Fig. 1.1.

			$g_1(1)=1$				$T_1=1$
		$g_2(1)=0$	$g_2(2)=2$	$g_2(3)=0$			$T_3=2$
	$g_3(1)=0$	$g_3(2)=4$	$g_3(3)=8$	$g_3(4)=4$	$g_3(5)=0$		$T_5=16$
$g_4(1)=0$	$g_4(2)=32$	$g_4(3)=64$	$g_4(4)=80$	$g_4(5)=64$	$g_4(6)=32$	$g_4(7)=0$	$T_7=272$

			$h_1(1)=1$				$E_2=1$
		$h_2(1)=1$	$h_2(2)=3$	$h_2(3)=1$			$E_4=5$
	$h_3(1)=5$	$h_3(2)=15$	$h_3(3)=21$	$h_3(4)=15$	$h_3(5)=5$		$E_6=61$
$h_4(1)=61$	$h_4(2)=183$	$h_4(3)=285$	$h_4(4)=327$	$h_4(5)=285$	$h_4(6)=183$	$h_4(7)=61$	[E_8=1385

Fig. 1.1. The two triangles of the $g_n(k)$'s and $h_n(k)$'s

To the right of each triangle have been calculated the row sums, which are equal, as stated in the next theorem, to the *tangent numbers* (resp.

the *secant numbers*). Those classical numbers, denoted by T_{2n+1} and E_{2n}, appear in the Taylor expansions of $\tan u$ and $\sec u$:

$$\begin{aligned}\tan u &= \sum_{n\ge 0} \frac{u^{2n+1}}{(2n+1)!} T_{2n+1} \\ &= \frac{u}{1!}1 + \frac{u^3}{3!}2 + \frac{u^5}{5!}16 + \frac{u^7}{7!}272 + \frac{u^9}{9!}7936 + \cdots\end{aligned} \tag{1.5}$$

$$\begin{aligned}\sec u = \frac{1}{\cos u} &= \sum_{n\ge 0} \frac{u^{2n}}{(2n)!} E_{2n} \\ &= 1 + \frac{u^2}{2!}1 + \frac{u^4}{4!}5 + \frac{u^6}{6!}61 + \frac{u^8}{8!}1385 + \frac{u^{10}}{10!}50521 + \cdots\end{aligned} \tag{1.6}$$

(see, e.g., [Ni23, p. 177-178], [Co74, p. 258-259]).

Theorem 1.1. *Let $(g_n(k))$ (resp. $(h_n(k))$ be the unique solution of the finite difference equation system* (1.2) *when using the initial values* `[tan1]` *(resp.* `[sec1]`*). Then, the row sums of the solutions are equal to*

$$\sum_k g_n(k) = T_{2n-1} \quad (n \ge 1); \tag{1.7}$$

$$\sum_k h_n(k) = E_{2n} \quad (n \ge 1). \tag{1.8}$$

It will also be shown that the generating functions for the coefficients $g_n(k)$ and $h_n(k)$ can be evaluated in the following forms.

Theorem 1.2. *Let*

$$Z(x,y) := 1 + \sum_{n\ge 1} \sum_{1\le k\le 2n+1} f_{n+1}(k) \frac{x^{2n+1-k}}{(2n+1-k)!} \frac{y^{k-1}}{(k-1)!}$$

and $Z^{\tan}(x,y)$ (resp. $Z^{\sec}(x,y)$) when $f_n(k) := g_n(k)$ (resp. $f_n(k) := h_n(k)$). Then,

$$Z^{\tan}(x,y) = \sec(x+y)\cos(x-y); \tag{1.9}$$

$$Z^{\sec}(x,y) = \sec^2(x+y)\cos(x-y). \tag{1.10}$$

As $Z^{\tan}(y,x) = Z^{\tan}(x,y)$ and $Z^{\sec}(y,x) = Z^{\sec}(x,y)$, this implies the following Corollary.

Corollary 1.3. *The entries $g_n(k)$ and $h_n(k)$ have the symmetry property:*

$$g_n(k) = g_n(2n-k), \quad h_n(k) = h_n(2n-k) \quad (1 \le k \le 2n-1). \tag{1.11}$$

In view of (1.7) and (1.8), two finite sets $\mathfrak{A}_{2n-1}$ and $\mathfrak{A}_{2n}$, of cardinalities T_{2n-1} and E_{2n}, are to be found, together with a statistic, call it "grn," defined on those sets with the property that

$$\sum_{\sigma\in\mathfrak{A}_{2n-1}} x^{\operatorname{grn}\sigma} = \sum_k g_n(k) x^k; \tag{1.11}$$

$$\sum_{\sigma\in\mathfrak{A}_{2n}} x^{\operatorname{grn}\sigma} = \sum_k h_n(k)x^k. \tag{1.12}$$

We shall use Désiré André's old result [An1879, An1881], who introduced the notion of *alternating* permutation, as being a permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ of $12\cdots n$ with the property that $\sigma(1) > \sigma(2)$, $\sigma(2) < \sigma(3)$, $\sigma(3) > \sigma(4)$, etc. in an alternating way. For each $n \geq 1$ let $\mathfrak{A}_n$ denote the set of all alternating permutations of $12\cdots n$. He proved that $\#\mathfrak{A}_{2n-1} = T_{2n-1}$, $\#\mathfrak{A}_{2n} = E_{2n}$. The desired statistic "grn" is then the following.

Definition. Let $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ be an alternating permutation from $\mathfrak{A}_n$, so that $\sigma(i) = n$ for a certain i ($1 \leq i \leq n$). By convention, let $\sigma(0) = \sigma(n+1) := 0$. Define the **gr***eater* **n***eighbor of* n *in* σ to be

$$\operatorname{grn}(\sigma) := \max\{\sigma(i-1), \sigma(i+1)\}. \tag{1.13}$$

Also, let

$$\mathfrak{A}_{n,k} := \{\sigma \in \mathfrak{A}_n : \operatorname{grn}(\sigma) = k\} \quad (0 \leq k \leq n-1). \tag{1.14}$$

Theorem 1.4. *Under the same assumptions as in Theorem* 1.1 *we have*

$$g_n(k) = \#\mathfrak{A}_{2n-1,k-1} \quad (n \geq 1,\ 1 \leq k \leq 2n-1); \tag{1.15}$$

$$h_n(k) = \#\mathfrak{A}_{2n,k} \quad (n \geq 1,\ 1 \leq k \leq 2n-1). \tag{1.16}$$

Example. There are $T_3 = 2$ alternating permutations of length 3, namely, 213 and 312, and $\operatorname{grn}(213) = \operatorname{grn}(312) = 1$, so that $g_2(1) = \#\mathfrak{A}_{3,0} = 0$, $g_2(2) = \#\mathfrak{A}_{3,1} = 2$, $g_2(3) = \#\mathfrak{A}_{3,2} = 0$; there are $E_4 = 5$ alternating permutations of length 4, namely, 4132, 4231, 3142, 3241, 2143, and $\operatorname{grn}(4132) = 1$, $\operatorname{grn}(4231) = \operatorname{grn}(3142) = \operatorname{grn}(3241) = 2$, $\operatorname{grn}(2143) = 3$, so that $h_2(1) = 1$, $h_2(2) = 3$, $h_2(3) = 1$; in accordance with the numerical values in Fig. 1.1.

As $\#\mathfrak{A}_{2n-1} = T_{2n-1}$, $\#\mathfrak{A}_{2n} = E_{2n}$, following Désiré André's result, we see that Theorem 1.1 is a consequence of Theorem 1.4. Thus, an analytical result is proved by combinatorial methods.

In Proposition 2.1 it will be proved that $\#\mathfrak{A}_{1,0} = 1$, $\#\mathfrak{A}_{2n-1,0} = \#\mathfrak{A}_{2n-1,2n-2} = 0$ ($n \geq 2$) and $\#\mathfrak{A}_{2n-1,1} = 2\,T_{2n-3}$ ($n \geq 2$); also, $\#\mathfrak{A}_{2,1} = 1$, $\#\mathfrak{A}_{2n,1} = \#\mathfrak{A}_{2n,2n-1} = E_{2n-2}$ ($n \geq 2$) and $\#\mathfrak{A}_{2n,2} = 3\,E_{2n-2}$ ($n \geq 2$). In view of Theorem 1.4 this implies that conditions `[tan2]` and `[sec2]` are fulfilled as soon as conditions `[tan1]` and `[sec1]` hold, and then the following theorem.

Theorem 1.5. *The entries* $(g_n(k))$ *and* $(h_n(k))$ *given by* (1.15) *and* (1.16) *are also solutions of the finite difference equation system* (1.2) *when the initial values* `[tan2]` *and* `[sec2]` *are used, respectively.*

Theorem 1.4 will be proved in Section 3, once evaluations of some cardinalities such as $\#\mathfrak{A}_{n,k}$ will be made, as done in Section 2. The proof of Theorem 1.2 is given in Section 4, together with further identities on the $g_n(k)$ and $h_n(k)$'s.

There are other combinatorial models which are also counted by tangent and secant numbers, or in a one-to-one correspondence with alternating permutations, in particular, the *labeled, binary, increasing, topological trees*, also called "arbres binaires croissants complets" by Viennot [Vi88, chap. 3, p. 111]. The set of those trees having n labeled nodes is denoted by $\mathfrak{T}_n$. The statistic "pom" (**p***arent* **o***f the* **m***aximum leaf*), introduced by Poupard [Po89] for her strictly ordered, binary trees can be extended to all of $\mathfrak{T}_n$. The usual bijection (see [Vi88]) $\gamma : \mathfrak{T}_n \to \mathfrak{A}_n$, called *projection*, has the property: $\mathrm{pom}(t) = \mathrm{grn}(\gamma(t))$, as proved in Theorem 5.1. We then have another combinatorial interpretation for the polynomials $\sum_k g_n(k)x^k$ and $\sum_k h_n(k)x^k$.

As such, the triangle $(g_n(k))$ $(n \geq 1, 1 \leq k \leq 2n-1)$ does not appear in Sloane's On-Line Encyclopedia of Integer Sequences [Sl06], but the triangle $(g_n(k)/2^{n-1})$ does under reference A008301 and is called Poupard's triangle, after her pioneering work on strictly ordered binary trees [Po89]. It is banal to verify that 2^{n-1} divides $g_n(k)$ when dealing with the combinatorial model $\mathfrak{T}_n$ for n odd.

In contrast to Christiane Poupard [Po89], who showed that the distribution of the strictly ordered binary trees satisfied the finite difference equation system $\Delta^2 f_{2n+1}(k)+2\, f_{2n-1}(k) = 0$, we have used the multiplicative factor "4" in equation (1.2) to make a unified study of the tangent *and* secant cases and deal with objects in one-to-one correspondence with alternating permutations. *Mutatis mutandis*, identities (1.15), as well as (1.11) concerning the tangent numbers, are due to her. She obtains the generating function for her trees in the form: $\sec((x+y)/\sqrt{2})\cos((x-y)/\sqrt{2})$ instead of (1.9). However, the alternating permutation development in Sections 2 and 3, identity (1.10) and the combinatorial properties of the entries $h_n(k)$ are new.

The triangle of the $h_n(k)$'s appears in Sloane's [Sl06] as sequence A125053. It was deposited there by Paul D. Hanna. The entries have been calculated by using a procedure equivalent to (1.2) and the initial condition `[tan2]`. No combinatorial interpretation is given and no generating function calculated.

2. Some special values

The evaluations of $\#\mathfrak{A}_{2n-1,k-1}$ and $\#\mathfrak{A}_{2n,k}$ made in the next proposition for some values of n and k will facilitate the derivation of the proof of Theorem 1.2. They also have their own combinatorial interests.

Proposition 2.1. *The following relations hold:*

(2.1) $$\#\mathfrak{A}_{1,0}=1,\quad \#\mathfrak{A}_{2n-1,0}=\#\mathfrak{A}_{2n-1,2n-2}=0\quad (n\ge 2)$$

(2.2) $$\#\mathfrak{A}_{2n-1,1}=\#\mathfrak{A}_{2n-1,2n-3}=2\,T_{2n-3}\quad (n\ge 2);$$

(2.3) $$\#\mathfrak{A}_{2n-1,2}=\#\mathfrak{A}_{2n-1,2n-4}=4\,T_{2n-3}\quad (n\ge 3);$$

(2.4) $$\#\mathfrak{A}_{2n-1,3}=\#\mathfrak{A}_{2n-1,2n-5}=6\,T_{2n-3}-8\,T_{2n-5}\quad (n\ge 3);$$

(2.5) $$\sum_{k\ge 0}\#\mathfrak{A}_{2n-1,k}=T_{2n-1}\quad (n\ge 1).$$

(2.6) $$\#\mathfrak{A}_{2,1}=1;$$

(2.7) $$\#\mathfrak{A}_{2n,1}=\#\mathfrak{A}_{2n,2n-1}=E_{2n-2}\quad (n\ge 2);$$

(2.8) $$\#\mathfrak{A}_{2n,2}=\#\mathfrak{A}_{2n,2n-2}=3\,E_{2n-2}\quad (n\ge 2);$$

(2.9) $$\#\mathfrak{A}_{2n,3}=\#\mathfrak{A}_{2n,2n-3}=5\,E_{2n-2}-4\,E_{2n-4}\quad (n\ge 2);$$

(2.10) $$\sum_{k\ge 1}\#\mathfrak{A}_{2n,k}=E_{2n}\quad (n\ge 1).$$

Proof. (2.1) The set $\mathfrak{A}_1$ is the singleton 1 and $\mathrm{grn}(1)=0$ by definition, so that $\#\mathfrak{A}_{1,0}=1$. For $n\ge 2$ all alternating permutations from $\mathfrak{A}_{2n-1}$ have a "grn" at least equal to 1. Hence, $\#\mathfrak{A}_{2n-1,0}=0$. Finally, each alternating permutation of length $(2n-1)$ $(n\ge 2)$ contains neither the factor $(2n-2)(2n-1)$, nor $(2n-1)(2n-2)$. Hence, $\#\mathfrak{A}_{2n-1,2n-2}=0$.

(2.2) When $n\ge 2$, each alternating permutation from $\mathfrak{A}_{2n-1,1}$ starts with $(2n-1)\,1$, or ends with $1\,(2n-1)$. After removal of those two letters, there remains an alternating permutation on $\{2,3,\ldots,2n-2\}$. Hence, $\#\mathfrak{A}_{2n-1,1}=2\,T_{2n-3}$. Next, each permutation from $\mathfrak{A}_{2n-1,2n-3}$ must contain, either the three-letter factor $(2n-1)(2n-3)(2n-2)$, or $(2n-2)(2n-3)(2n-1)$. The removal of the factor $(2n-1)(2n-3)$ (resp. $(2n-3)(2n-1)$) yields an alternating permutation of the set $\{1,2,\ldots,(2n-4),(2n-2)\}$, of cardinality $(2n-3)$. This proves relation $\#\mathfrak{A}_{2n-1,2n-3}=2\,T_{2n-3}$.

(2.3) Start with an alternating permutation on $\{1,3,4,\ldots,2n-2\}$, then having $(2n-3)$ elements. There are four possibilities to generate a permutation from $\mathfrak{A}_{2n-1,2}$: (1) insert $(2n-1)\,2$ to the left; (2) insert $2\,(2n-1)$ to the right; (3) insert $2(2n-1)$ just before 1; (4) insert $(2n-1)\,2$ just after 1. For the second identity in (2.3) proceed in the same way: in each alternating permutation on $\{1,2,\ldots,2n-1\}\setminus\{2n-4,2n-1\}$ the two letters $(2n-3)$, $(2n-2)$ are necessarily local maxima. There are four possibilities to obtain a permutation from $\mathfrak{A}_{2n-1,2n-4}$: insert $(2n-1)\,(2n-4)$ just before, either $(2n-3)$, or $(2n-2)$; also insert $(2n-4)\,(2n-1)$ just after, either $(2n-3)$, or $(2n-2)$.

(2.4) Each permutation from $\mathfrak{A}_{2n-1,3}$ containing the factor $2\,1$ (resp. $1\,2$) starts with $2\,1$ (resp. ends with $1\,2$). Dropping the factor $2\,1$ (resp. $1\,2$) and subtracting 2 from the remaining letters yields an alternating permutation from $\mathfrak{A}_{2n-3,1}$. There are then $2(2\,T_{2n-5})$ permutations from $\mathfrak{A}_{2n-1,3}$ containing, either $2\,1$, or $1\,2$.

If a permutation from $\mathfrak{A}_{2n-1,3}$ contains neither one of those two factors, it has one of the *six* properties: it starts with $(2n-1)\,3$, or contains one of the three-letter factor $1\,(2n-1)\,3$, $3\,(2n-1)\,1$, $2\,(2n-1)\,3$, $3\,(2n-1)\,2$, or still ends with $3\,(2n-1)$. After removal of the two-letter factor $(2n-1)\,3$ or $3\,(2n-1)$ there remains an alternating permutation on $\{1,2,4,\ldots,(2n-2)\}$ *not starting with* $2\,1$ and *not ending with* $1\,2$. There are then $6(T_{2n-3}-2\,T_{2n-5})$ such permutations. Altogether, $\#\mathfrak{A}_{2n-1,3} = 4\,T_{2n-5}+6(T_{2n-3}-2\,T_{2n-5}) = 6\,T_{2n-3}-8\,T_{2n-5}$.

The proof of the second identity in (2.4) follows a different pattern. If the letter $(2n-4)$ is a local minimum (i.e., less than its two adjacent letters) in a permutation σ from $\mathfrak{A}_{2n-1,2n-5}$, then σ necessarily contains one of the four five-letter factors $(2n-1)(2n-5)(2n-2)(2n-4)(2n-3)$, $(2n-1)(2n-5)(2n-3)(2n-4)(2n-2)$, $(2n-2)(2n-4)(2n-3)(2n-5)(2n-1)$, $(2n-3)(2n-4)(2n-2)(2n-5)(2n-1)$. Replacing this five-letter factor by $(2n-5)$ yields a permutation from $\mathfrak{A}_{2n-5}$. Thus, there are $4\,T_{2n-5}$ permutations from $\mathfrak{A}_{2n-1,2n-5}$ in which $(2n-4)$ is a local minimum.

In the other permutations from $\mathfrak{A}_{2n-1,2n-5}$ all the four letters $(2n-4)$, $(2n-3)$, $(2n-2)$, $(2n-1)$ are local maxima (i.e., greater than their adjacent letters). Let $\mathfrak{A}'_{2n-1,2n-5}$ be the set of those permutations. When the two-letter factor $(2n-1)(2n-5)$ or $(2n-5)(2n-1)$ is deleted from such a permutation, there remains a permutation on $\{1,2,\ldots,(2n-1)\}\setminus\{(2n-5),(2n-1)\}$ in which the third largest letter $(2n-4)$ is not a local minimum. Let $\mathfrak{A}''_{2n-3}$ be the set of those permutations. But the alternating permutations on the latter set in which $(2n-4)$ is a local minimum necessarily contain the three-letter factor $(2n-3)(2n-4)(2n-2)$ or $(2n-2)(2n-4)(2n-3)$. There are then $2\,T_{2n-5}$ such permutations. Hence, $\#\mathfrak{A}''_{2n-3} = T_{2n-3}-2\,T_{2n-5}$. To obtain a permutation from $\mathfrak{A}'_{2n-1,2n-5}$ it suffices to start from a permutation σ'' from $\mathfrak{A}''_{2n-3}$ and insert $(2n-1)(2n-5)$ (resp. $(2n-5)(2n-1)$) just before (resp. just after) each one of the three letters $(2n-4)$, $(2n-3)$, $(2n-2)$ (which are all local maxima). There are then $6(T_{2n-3}-2\,T_{2n-5})$ such permutations. Altogether, $\#\mathfrak{A}_{2n-1,2n-5} = 4\,T_{2n-5}+6(T_{2n-3}-2\,T_{2n-5}) = 6\,T_{2n-3}-8\,T_{2n-5}$. No comment for (2.5) and (2.6).

(2.7) Simply note that the only alternating permutations from $\mathfrak{A}_{2n,1}$ and $\mathfrak{A}_{2n,2n-1}$ are, respectively, of the form: $(2n)\,1\,\sigma(3)\cdots\sigma(2n)$ and $\sigma(1)\sigma(2)\cdots(2n)\,(2n-1)$.

(2.8) Same proof as for (2.3): start with an alternating permutation on $\{1,3,4,\ldots,(2n-1)\}$. There are exactly three possibilities to generate a permutation from $\mathfrak{A}_{2n,2}$: insert $(2n)\,2$ to the left, or just after the letter 1, or still insert $2\,(2n)$ just before the letter 1. For the second identity start with a permutation on $\{1,2,\ldots,2n\}\setminus\{2n-2,2n\}$ and insert $(2n)\,(2n-2)$ either to the right, or just before $(2n-1)$, or still insert $(2n-2)\,(2n)$ just after $(2n-1)$.

(2.9) Each permutation from $\mathfrak{A}_{2n,3}$ containing the factor $2\,1$ is necessarily of the form $\sigma = 2\,1\,\sigma(3)\cdots\sigma(2n)$, so that the alternating permutation $\sigma' := (\sigma(3)-2)\cdots(\sigma(2n-2)$ belongs to $\mathfrak{A}_{2n-2,1}$. There are then E_{2n-4} such permutations. If a permutation from $\mathfrak{A}_{2n,3}$ does not contain $2\,1$, it has one of the *five* properties: it starts with $(2n)\,3$, or contains one of the three-letter factor $1\,(2n)\,3$, $3\,(2n)\,1$, $2\,(2n)\,3$, $3\,(2n)\,2$. After removal of the two-letter factor $(2n)\,3$ or $3\,(2n)$ there remains an alternating permutation on $\{1,2,4,\ldots,(2n-1)\}$ *not starting with* $2\,1$. There are then $5(E_{2n-2}-E_{2n-4})$ such permutations. Altogether, $\#\mathfrak{A}_{2n,3} = E_{2n-4} + 5(E_{2n-2}-E_{2n-4}) = 5E_{2n-2} - 4E_{2n-4}$.

The proof for the second identity in (2.9) is quite similar. Each permutation from $\mathfrak{A}_{2n,2n-3}$ containing the factor $(2n-1)(2n-2)$ necessarily ends with the four-letter factor $(2n)(2n-3)(2n-1)(2n-2)$. There are then E_{2n-4} such permutations. The other permutations from $\mathfrak{A}_{2n,2n-3}$ contain one of the four three-letter factors $(2n)(2n-3)(2n-2)$, $(2n-2)(2n-3)(2n)$, $(2n)(2n-3)(2n-1)$, $(2n-1)(2n-3)(2n)$, or ends with $(2n)(2n-3)$. After removal of the two-letter factor $(2n)(2n-3)$ or $(2n-3)(2n)$ there remains an alternating permutation on $\{1,2,\ldots,(2n-4),(2n-2),(2n-1)\}$, not ending with the two-letter factor $(2n-1)(2n-2)$. There are $E_{2n-2}-E_{2n-4}$ such permutations. Altogether, $\#\mathfrak{A}_{2n,2n-3} = E_{2n-4} + 5(E_{2n-2}-E_{2n-4})$.

No comment for (2.10). $\square$

3. Proof of Theorem 1.4

Let $a_n(k) := \#\mathfrak{A}_{2n-1,k-1}$ and $b_n(k) := \#\mathfrak{A}_{2n,k}$. From Proposition 2.1 and Theorem 1.5 it follows that the initial conditions `[tan1]` and `[tan2]` hold when $f_n(k) = a_n(k)$, and `[sec1]` and also `[sec2]` when $f_n(k) = b_n(k)$. It remains to prove that in each case (1.2) holds.

By means of identities (2.2)–(2.4) and (2.7)–(2.9) we easily verify that (1.2) holds for both $a_n(k)$ and $b_n(k)$ when $n = 2,3$ and $1 \le k \le 2n-3$. It also holds for $a_n(k)$ when $n \ge 4$ and $k = 1,2,2n-4,2n-3$, and for $b_n(k)$ when $n \ge 4$ and $k = 1, 2n-3$.

What is left to prove is: $\Delta^2 a_n(k) + 4a_{n-1}(k) = 0$, that is, $\Delta^2\mathfrak{A}_{2n-1,k-1} + 4\,\mathfrak{A}_{2n-3,k-1} = 0$ for $n \ge 4$ and $3 \le k \le 2n-5$—by identifying each finite set with its cardinality—and also $\Delta^2 b_n(k) + 4b_{n-1}(k) = 0$, that is, $\Delta^2\mathfrak{A}_{2n,k} + 4\,\mathfrak{A}_{2n-2,k} = 0$ for $n \ge 4$ and $2 \le k \le 2n-4$; altogether,

$$\Delta^2\mathfrak{A}_{n,k} + 4\,\mathfrak{A}_{n-2,k} = 0 \quad \text{for } n \ge 7 \text{ and } \begin{array}{l} 2 \le k \le n-4 \ (n \text{ even}) \\ 2 \le k \le n-5 \ (n \text{ odd}). \end{array} \tag{2.11}$$

Let $v = y_1 \cdots y_m$ be a nonempty word with *distinct* letters from the set $\{0, 1, 2, \ldots, n\}$ and $\widetilde{v} = y_m \cdots y_1$ be its mirror-image. If $m = 1$ and $y_1 = 0$, let $[v] = [0]$ be the empty set. If $m \geq 2$ and $y_1 = 0$ (resp. $y_m = 0$), let $[v]$ be the set of all alternating permutations from $\mathfrak{A}_n$, if any, whose left factors are equal to $y_2 \cdots y_m$, or whose right factors are equal to $y_m \cdots y_2$. When $y_1 \geq 1$, let $[v]$ the set of all alternating permutations from $\mathfrak{A}_n$, if any, containing, either the factor v, or the factor $\widetilde{v}$. Finally, let $[\,\widetilde{v}\,] := [v]$.

Using those notations we get

$$\begin{aligned}
\mathfrak{A}_{n,k} &= \sum_{0 \leq y \leq k-1} [ynk] \\
&= \sum_{0 \leq y \leq k-1} [ynk(k+1)] + \sum_{\substack{0 \leq y \leq k-1 \\ k+2 \leq z \leq n-1}} [ynkz] + \sum_{1 \leq y \leq k-1} [ynk0]; \\
\mathfrak{A}_{n,k+1} &= \sum_{0 \leq y \leq k} [yn(k+1)] \\
&= [kn(k+1)] + \sum_{\substack{0 \leq y \leq k-1 \\ k+2 \leq z \leq n-1}} [yn(k+1)z] + \sum_{1 \leq y \leq k-1} [yn(k+1)0].
\end{aligned}$$

The transposition $(k, k+1)$ maps the set $[ynkz]$ onto the set $[yn(k+1)z]$ for $z \in \{k+2, \ldots, n-1\} \cup \{0\}$, so that we may write

$$\begin{aligned}
\Delta\, \mathfrak{A}_{n,k} = \mathfrak{A}_{n,k+1} - \mathfrak{A}_{n,k} &= [kn(k+1)] - \sum_{0 \leq y \leq k-1} [ynk(k+1)] \\
&= [kn(k+1)] - \sum_{\substack{0 \leq y_1, y_2 \leq k-1 \\ y_1 \neq y_2}} [y_1 nk(k+1) y_2];
\end{aligned}$$

$$\begin{aligned}
&\Delta\, \mathfrak{A}_{n,k+1} = \mathfrak{A}_{n,k+2} - \mathfrak{A}_{n,k+1} \\
&\quad = [(k+1)n(k+2)] - \sum_{0 \leq y \leq k} [yn(k+1)(k+2)] \\
&\quad = [(k+1)n(k+2)] - [kn(k+1)(k+2)] \\
&\qquad - \sum_{0 \leq y \leq k-1} [yn(k+1)(k+2)k] - \sum_{\substack{0 \leq y_1, y_2 \leq k-1 \\ y_1 \neq y_2}} [y_1 n(k+1)(k+2) y_2].
\end{aligned}$$

For $2 \leq k \leq n-4$ the permutation $\left(\begin{smallmatrix} k & k+1 & k+2 \\ k+1 & k+2 & k \end{smallmatrix}\right)$ maps $[y_1 nk(k+1) y_2]$ onto $[y_1 n(k+1)(k+2) y_2]$ in a bijective manner. Hence,

$$\begin{aligned}
\Delta^2 \mathfrak{A}_{n,k} &= \Delta\, \mathfrak{A}_{n,k+1} - \Delta\, \mathfrak{A}_{n,k} \\
&= [(k+1)n(k+2)] - [kn(k+1)(k+2)] \\
&\qquad - \sum_{0 \leq y \leq k-1} [yn(k+1)(k+2)k] - [kn(k+1)].
\end{aligned}$$

But

$$[kn(k+1)] = [(k+2)kn(k+1)] + [kn(k+1)(k+2)] \\ + \sum_{\substack{z_1,z_2\in\{k+3,\ldots,n-1\}\cup\{0\}\\ z_1\neq z_2}} [z_1kn(k+1)z_2].$$

Again, the permutation $\begin{pmatrix} k & k+1 & k+2 \\ k+1 & k+2 & k\end{pmatrix}$ maps the last sum onto the set $[(k+1)n(k+2)]$. Altogether, as $[(k+2)kn(k+1)] = [kn(k+1)(k+2)]$, we have

$$\Delta^2\mathfrak{A}_{n,k} = -3\,[kn(k+1)(k+2)] - \sum_{0\le y\le k-1} [yn(k+1)(k+2)k].$$

When removing the factor $(k+1)(k+2)$ and replacing each integer $z\ge k+2$ by $(z-2)$, in each alternating permutation, both sets $[kn(k+1)(k+2)]$ and $\sum\limits_{0\le y\le k-1}[yn(k+1)(k+2)k]$ are transformed into $\mathfrak{A}_{n-2,k}$; so that $\Delta^2\mathfrak{A}_{n,k} = -4\,\mathfrak{A}_{n-2,k}$.

4. The bivariable generating functions

Let $f = (f_n(k))$ $(n\ge 1,\ 1\le k\le 2n-1)$ be the family of rational numbers, as displayed in (1.1), that satisfies the finite-difference equation system (1.2) under the initial conditions `[tan2]` of `[sec2]`. We know that the system has then a unique solution. With the triangle f associate the infinite matrix

$$\text{(4.1)}\quad \Gamma=(\gamma_{ij})_{(i\ge 0,j\ge 0)} := \begin{pmatrix} f_1(1) & 0 & f_2(3) & 0 & f_3(5) & 0 & f_4(7)\cdots \\ 0 & f_2(2) & 0 & f_3(4) & 0 & f_4(6) & \cdots \\ f_2(1) & 0 & f_3(3) & 0 & f_4(5) & \cdots & \\ 0 & f_3(2) & 0 & f_4(4) & \cdots & & \\ f_3(1) & 0 & f_4(3) & \cdots & & & \\ 0 & f_4(2) & \cdots & & & & \\ f_4(1) & \cdots & & & & & \end{pmatrix}.$$

In other words, define $\gamma_{ij} := 0$ when $i+j$ is odd, and $\gamma_{ij} := f_n(k)$ with $k := j+1$, $2n = 2+i+j$ when $i+j$ is even. For $i+j$ even the mapping $(i,j)\mapsto(n,k)$ is one-to-one, the reverse mapping being for $n\ge 1$, $1\le k\le 2n-1$ given by $i = 2n-1-k$, $j = k-1$.

In terms of the entries γ_{ij} relation (1.2) may be written in the form

$$\text{(4.2)}\qquad \gamma_{i,j} = 2\,\gamma_{i-1,j-1} + \frac{1}{2}(\gamma_{i-1,j+1} + \gamma_{i+1,j-1})\quad (i\ge 1,\ j\ge 1);$$

$$\text{(4.3)}\qquad \gamma_{ij} = 0,\qquad \text{if } i+j \text{ odd}.$$

Furthermore, the full matrix $\Gamma = (\gamma_{i,j})$ $(i \geq 0,\, j \geq 0)$ is completely determined as soon as its *first row* $(\gamma_{0,j})$ $(j \geq 0)$ and *first column* $(\gamma_{i,0})$ $(i \geq 0)$ are known. Let $f \mapsto \Gamma$ denote the above correspondence between those triangles and matrices.

Let $Z(x,y) := \sum_{i\geq 0,\, j\geq 0} \gamma_{i,j} \frac{x^i}{i!} \frac{y^j}{j!}$. It is easily verified that $Z(x,y)$ satisfies the partial differential equation

$$\frac{\partial^2 Z(x,y)}{\partial x\, \partial y} = 2\, Z(x,y) + \frac{1}{2} \frac{\partial^2 Z(x,y)}{\partial x^2} + \frac{1}{2} \frac{\partial^2 Z(x,y)}{\partial y^2}, \tag{4.4}$$

if and only if the coefficients $\gamma_{i,j}$ satisfy relation (4.2). Hence, $Z(x,y)$ is fully determined by (4.4) and by the generating functions $Z(x,0) = \sum_{i\geq 0} \gamma_{i,0}\, x^i/i!$ and $Z(0,y) = \sum_{j\geq 0} \gamma_{0,j}\, y^j/j!$ for the first column and first row of the matrix Γ.

But for any given formal power series in one variable $f(x) = 1 + \sum_{n\geq 1} f_{2n} \frac{x^{2n}}{(2n)!}$ it can be also verified that the bivariable formal power series

$$Z(x,y) = \sum_{i\geq 0,\, j\geq 0} \gamma_{i,j} \frac{x^i}{i!} \frac{y^j}{j!} = f(x+y) \sec(x+y) \cos(x-y) \tag{4.5}$$

satisfies (4.4) and that the generating functions for its first column and first row are given by $f(x)$ and $f(y)$, respectively. This proves the following proposition.

Proposition 4.1. *Let* $f(x) = 1 + \sum_{n\geq 1} f_{2n} \frac{x^{2n}}{(2n)!}$ *be given and* $\Gamma = (\gamma_{ij})$ $(i \geq 0,\, j \geq 0)$ *be an infinite matrix, whose entries satisfy relations* (4.2) *and* (4.3), *on the one hand, and such that* $\gamma_{0,0} = 1$, $\gamma_{2n+1,0} = \gamma_{0,2n+1} = 0$ *for* $n \geq 0$ *and* $\gamma_{2n,0} = \gamma_{0,2n} = f_{2n}$ *for* $n \geq 1$, *on the other hand. Then, identity* (4.5) *holds.*

Using the correspondence $\gamma_{ij} \leftrightarrow f_n(k)$ above mentioned, the series $Z(x,y)$ can be rewritten

$$Z(x,y) = 1 + \sum_{n\geq 1} \sum_{1\leq k\leq 2n+1} f_{n+1}(k) \frac{x^{2n+1-k}}{(2n+1-k)!} \frac{y^{k-1}}{(k-1)!}, \tag{4.6}$$

which is then equal to $f(x+y) \sec(x+y) \cos(x-y)$ under the assumptions of the previous Proposition.

Now, consider the two triangles described in Fig. 1.1 and let $\Gamma^{\tan} = (\gamma_{ij}^{\tan})$ and $\Gamma^{\sec} = (\gamma_{ij}^{\sec})$ be the two Γ-matrices attached to them:

$$\Gamma^{\tan} = (\gamma_{ij}^{\tan}) = \begin{pmatrix} g_1(1) & 0 & g_2(3) & 0 & g_3(5) & 0 & g_4(7) \cdots \\ 0 & g_2(2) & 0 & g_3(4) & 0 & g_4(6) & \cdots \\ g_2(1) & 0 & g_3(3) & 0 & g_4(5) & \cdots \\ 0 & g_3(2) & 0 & g_4(4) & \cdots \\ g_3(1) & 0 & g_4(3) & \cdots \\ 0 & g_4(2) & \cdots \\ g_4(1) & \cdots \end{pmatrix};$$

$$\Gamma^{\sec} = (\gamma_{ij}^{\sec}) = \begin{pmatrix} h_1(1) & 0 & h_2(3) & 0 & h_3(5) & 0 & h_4(7) \cdots \\ 0 & h_2(2) & 0 & h_3(4) & 0 & h_4(6) & \cdots \\ h_2(1) & 0 & h_3(3) & 0 & h_4(5) & \cdots \\ 0 & h_3(2) & 0 & h_4(4) & \cdots \\ h_3(1) & 0 & h_4(3) & \cdots \\ 0 & h_4(2) & \cdots \\ h_4(1) & \cdots \end{pmatrix}.$$

The exponential generating function for the first row and first column of $\Gamma^{\tan}$ is equal to $f(x) = 1$. On the other hand, as $h_n(1) = h_n(2n+1) = \#\mathfrak{A}_{2n,1} = E_{2n-2}$ by `[sec2]`, (1.8) and (1.16), the exponential generating function for the first row and first column of $\Gamma^{\sec}$ is equal to $h_1(1)+h_2(1)x^2/2!+h_3(1)x^4/4!+\cdots = E_0+E_2x/2!+E_4x^4/4+\cdots = \sec(x)$. Theorem 1.2 is then a consequence of the previous Proposition.

When (x,y) is equal to (x,x), then to $(x,-x)$ in (1.9) and (1.10), we obtain: $Z^{\tan}(x,x) = \sec(2x)$; $Z^{\tan}(x,-x) = \cos(2x)$; $Z^{\sec}(x,x) = \sec^2(2x) = 1 + \sum_{n\ge 1} 4^n T_{2n+1} x^{2n}/(2n)!$ and $Z^{\sec}(x,-x) = \cos(2x)$. Looking for the coefficients of $x^{2n}/(2n)!$ on both sides in the first (resp. last) two formulas yields four further identities

$$\sum_{1\le k\le 2n+1} \binom{2n}{k-1} g_{n+1}(k) = 4^n E_{2n} \quad (n \ge 1); \tag{4.7}$$

$$\sum_{1\le k\le 2n+1} (-1)^k \binom{2n}{k-1} g_{n+1}(k) = (-1)^n 4^n \quad (n \ge 1); \tag{4.8}$$

$$\sum_{1\le k\le 2n+1} \binom{2n}{k-1} h_{n+1}(k) = 4^n T_{2n+1} \quad (n \ge 1) \tag{4.9}$$

$$\sum_{1\le k\le 2n+1} (-1)^k \binom{2n}{k-1} h_{n+1}(k) = (-1)^n 4^n \quad (n \ge 1). \tag{4.10}$$

The last two ones are mentioned in Sloane's Encyclopedia [Sl07] (sequence A125053) without proofs.

5. Alternating permutations and binary trees

In this Section the traditional vocabulary on trees, such as node, leaf, child, root, ... is used. In particular, when a node is not a leaf, it is said to be an *interior node.*

Definition. An *n-labeled, binary, increasing, topological tree* is defined by the following axioms:

(1) it is a *labeled* tree with n nodes, labeled $1, 2, \ldots, n$; the node labeled 1 is called the *root*;

(2) each node has no child (then called a *leaf*), or one child, or two children;

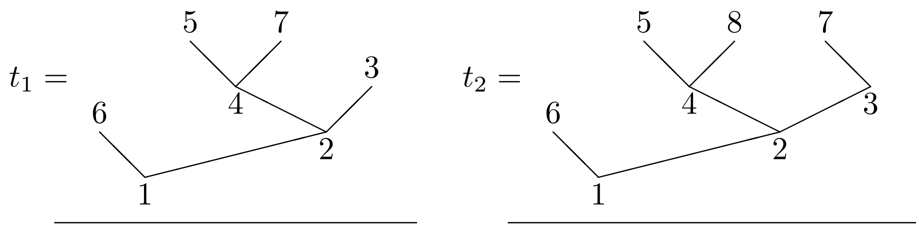
(3) the label of each node is smaller than the label of its children, if any;

(4) the tree is planar and each child of a node is, either on the left (it is then called the *left child*), or on the right (the *right child*);

(5) when n is odd, each node is, either a leaf, or a node with two children; when n is even, each node is, either a leaf, or a node with two children, except the rightmost node (uniquely defined) which has one *left* child, but no right child. It will be referred to as being the *one-son child.*

Each such binary tree t may be drawn on a Euclidean plane: the root has coordinates $(0, 0)$, the left son of the root $(-1, 1)$ and the right son $(1, 1)$, the grandsons $(-3/2, 2)$, $(-1/2, 2)$, $(1/2, 2)$, $(3/2, 2)$, respectively, the great-grandsons $(-7/4, 3)$, $(-5/4, 3)$, ... , $(7/4, 3)$, etc. With this convention all the nodes have different abscissas. Let t have n nodes and make the orthogonal projections of those nodes on a horizontal axis. Writing the labels of the projected n nodes yields a permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ of $1\,2\,\cdots\,n$. We say that σ is the *projection* of t and t the *spreading out* of σ. Moreover, σ is *alternating.* For instance, the two trees t_1 and t_2 in Fig 1.2 are labeled, binary, increasing, topological trees, with 7 and 8 nodes, respectively. Their projections $\sigma_1 = 6\,1\,5\,4\,7\,2\,3$ and $\sigma_2 = 6\,1\,5\,4\,8\,2\,7\,3$ are alternating.

For each $n \geq 1$ let $\mathfrak{T}_n$ be the set of all n-labeled, binary, increasing, topological trees. Then, $t \mapsto \sigma$ is a bijection of $\mathfrak{T}_n$ onto $\mathfrak{A}_n$, so that we also have: $\#\mathfrak{A}_{2n+1} = T_{2n+1}$, $\#\mathfrak{A}_{2n} = E_{2n}$. Each tree t from $\mathfrak{T}_n$ is said to be *tangent* (resp. *secant*), if n is odd (resp. even).



$\sigma_1 = 6\ \ 1\ \ 5\ \ 4\ \ 7\ \ 2\ \ 3$ $\qquad$ $\sigma_2 = 6\ \ 1\ \ 5\ \ 4\ \ 8\ \ 2\ \ 7\ \ 3$

Fig. 1.2. Tangent, secant trees and alternating permutations.

The two statistics "emc" ("**e**nd **o**f minimal **c**hain") and "pom" ("**p**arent **o**f **m**aximum leaf") we now define on each set $\mathfrak{T}_n$ have been introduced by Christiane Poupard [Po89] in her study of the strictly ordered binary trees and provide two other combinatorial interpretations for the entries $g_n(k)$ and $h_n(k)$. Their definitions are also valid for all binary increasing trees, in particular, for secant and tangent trees. Let $n \geq 2$ and t be a binary increasing tree, with n nodes labeled $1, 2, \ldots, n$. Let a be the label of an interior node. If the node has two children labeled b and c, define $\min a := \min\{b, c\}$; if it has one child b, let $\min a := b$. The *minimal chain* of t is defined to be the sequence $a_1 \to a_2 \to a_3 \to \cdots \to a_{j-1} \to a_j$, with the following properties:

(i) $a_1 = 1$ is the label of the root;

(ii) for $i = 1, 2, \ldots, j-1$ the i-th term a_i is the label of an interior node and $a_{i+1} = \min a_i$;

(iii) a_j is the label of a leaf.

Define the *end of the minimal chain* in t to be: $\mathrm{eoc}(t) := a_j$. As t is increasing, there is a unique leaf with label n. If that leaf is incident to a node labeled k, define the (*parent of the maximum leaf*) in t to be: $\mathrm{pom}(t) := k$. By convention, $\mathrm{eoc}(t) = 1$ and $\mathrm{pom}(t) = 0$ for the unique $t \in \mathfrak{T}_1$.

The minimal chain of the tree t_1 (resp. t_2) displayed in Fig. 1.2 is $1 \to 2 \to 3$ (resp. $1 \to 2 \to 3 \to 7$). Then $\mathrm{eoc}(t_1) = 3$, $\mathrm{eoc}(t_2) = 7$. Also, $\mathrm{pom}(t_1) = 4$ and $\mathrm{pom}(t_2) = 4$.

Theorem 5.1. *Let $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ be the projection of the n-labeled, binary, increasing, topological tree t. Then $\mathrm{pom}(t) = \mathrm{grn}(\sigma)$. In other words, the parent of the maxium leaf in t is the greater neighbor of n in σ.*

Proof. Let $\sigma(i) = n$ with $2 \leq i \leq n-1$. The parent of the node labeled n in t is, either the node labeled $\sigma(i-1)$, or the node labeled $\sigma(i+1)$. Let $\sigma(j)$ be the label of the node of the common ancester of the previous two nodes in t. Then, $\sigma(j) \leq \min\{\sigma(i-1), \sigma(i+1)\}$, $j \neq i$ and $i-1 \leq j \leq i+1$. Hence, either $\sigma(j) = \sigma(i-1) < \sigma(i+1)$, or $\sigma(j) = \sigma(i+1) < \sigma(i-1)$. In the first (resp. the second) case the parent of n is $\sigma(i+1)$ (resp. $\sigma(i-1)$) and $\mathrm{grn}(\sigma) = \max\{\sigma(i-1), \sigma(i+1)\}$. ∎

In [Ha12] an explicit bijection of $\mathfrak{T}_n$ onto itself is constructed that maps the statistic "eoc -1" onto the statistic "pom." For each of the polynomials $\sum_k g_n(k)x^k$, $\sum_k h_n(k)x^k$, we then have *three* combinatorial interpretations one on alternating permutations by the statistic "grn," and two on labeled, binary, increasing, topological trees by "pom" and "eoc."

Recently, there has been a revival of studies on arithmetical and combinatorial properties of both tangent and secant numbers. Ordering the alternating permuations according to their leftmost elements has led to the *Entringer recurrence*, having interesting properties ([En66, FZ71, FZ71a, Po97, GHZ10]). The geometry of those permutations has been fully exploited ([KPP94, St10]), in particular by looking at their quadrant marked mesh patterns in [KR12], or for defining and studying natural q-*analogs* of the tangent and secant numbers ([AG78, AF80, Fo81], [St99, p. 148-149]). Further q-analogs were also introduced, based no longer on alternating permutations, but on the so-called *doubloon* model (see [FH10, FH10a, FH11]). The classical continued fraction expansions of secant and tangent have made possible the discovery of other q-analogs (see [Pr08, Pr00, Fu00, HRZ01, Jos10, SZ10]).

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